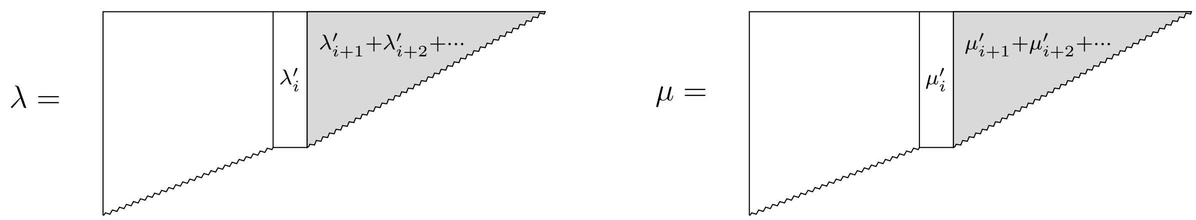
1. Dominance order. Recall that the dominance order ($\geqslant$) on the set $\{\lambda \vdash n\}$ of partitions of size n is defined by $\lambda \geqslant \mu$ if $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all $i \geqslant 1$. This is a total order for $n \leqslant 5$ and a partial order for all $n \geqslant 6$. Show that $\lambda \geqslant \mu$ if and only if $\lambda' \leqslant \mu'$.

Solution. Proceeding by contradiction, assume that $\lambda \geqslant \mu$ but $\lambda' \not\leqslant \mu'$. Then there exists a smallest positive integer i such that $\lambda'_1 + \cdots + \lambda'_i > \mu'_1 + \cdots + \mu'_i$. To pass from this inequality for the area of the first i columns of λ and μ to an equality for the rows, notice that since $|\lambda| = |\mu|$ we also have

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots < \mu'_{i+1} + \mu'_{i+2} + \cdots \,. \tag{1}$$

As is easily seen from the figure



it follows that

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots = \sum_{k=1}^{\lambda'_i} (\lambda_k - i) \qquad \text{and} \qquad \mu'_{i+1} + \mu'_{i+2} + \cdots = \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Hence (1) can also be expressed as

$$\sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

The minimality of i implies that $\lambda'_i > \mu'_i$, so that

$$\sum_{k=1}^{\mu'_i} (\lambda_k - i) < \sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Including the $i \times \mu'_i$ rectangle on both sides, we obtain the following inequality for the area of the first μ'_i rows of λ and μ

$$\sum_{k=1}^{\mu'_i} \lambda_k < \sum_{k=1}^{\mu'_i} \mu_k.$$

This contradicts the fact that $\lambda \geqslant \mu$.

2. The centralizer of the symmetric group. Show that

$$z_\lambda := \prod_{i \geqslant 1} i^{m_i}\, m_i! = |Z_w|,$$

where $m_i = m_i(\lambda)$ is the multiplicity of i (the number of parts equal to i) in λ, $w \in S_n$ has cycle type λ (i.e., w has m_i cycles of length i), and Z_w is the centralizer of w.

Solution. Conjugation of elements of S_n does not change their cycle type (the conjugacy classes of S_n are formed by the permutations of the same cycle type). Specifically, if

$$w = (c_1, \dots, c_{k_1})(c_{k_1+1}, \dots, c_{k_2}) \dots (c_{k_{r-1}+1}, \dots, c_{k_r})$$

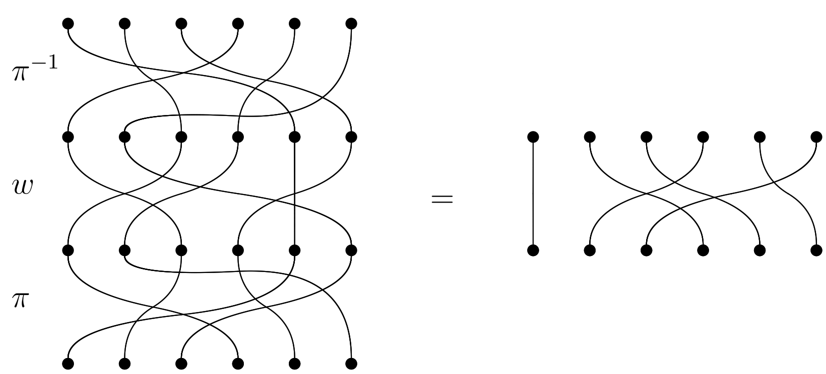
(where $k_r := n$) then

$$\pi w \pi^{-1} = (\pi_{c_1}, \dots, \pi_{c_{k_1}})(\pi_{c_{k_1+1}}, \dots, \pi_{c_{k_2}}) \dots (\pi_{c_{k_{r-1}+1}}, \dots, \pi_{c_{k_r}}).$$

For example, if $w = (13)(264)(5) \in S_6$ (in one-line notation $w = (361254)$) and $\pi = (145)(263)$ (in one-line notation $\pi = (462513)$), then $\pi^{-1} = (154)(236)$ (in one-line notation $\pi^{-1} = (536142)$) and thus

$$\begin{aligned} \pi w \pi^{-1} &= (\pi_1, \pi_3)(\pi_2, \pi_6, \pi_4)(\pi_5) \\ &= (42)(635)(1) \\ &= (1)(24)(356) \end{aligned}$$

(in one-line notation $\pi w \pi^{-1} = (415263)$). Pictorially



For π to be in Z_w we require $\pi w \pi^{-1} = w$. In other words, π can permute the cycles (of w) of fixed length and/or cycle each cycle as in $(abc \dots z) \mapsto (rs \dots zab \dots q)$. If there are m_i cycles of length i in w it thus follows that

$$|Z_w| = \prod_{i \geqslant 1} i^{m_i} m_i!.$$

3. Gaussian polynomials. Let n, m be nonnegative integers. Then the Gaussian polynomials or q-binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\lambda \subseteq (m^n)} q^{|\lambda|}. \tag{2}$$

Here the sum is over all partitions λ that fit in a rectangle of height n and width m, i.e., partitions λ such that $\lambda_1 \leqslant m$ and $l(\lambda) \leqslant n$.

(a) Show that
   (i) $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$ (initial condition);
   (ii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m \\ n \end{smallmatrix}\right]$ (symmetry);
   (iii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + q^n \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right] = q^m \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right]$ (q-Pascal identities).

(b) Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}. \tag{3}$$

Remark. One similarly shows the q-Chu–Vandermonde or Durfee rectangle identity

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\ell=0}^{n} q^{\ell(\ell+k)} \begin{bmatrix} m-k \\ \ell \end{bmatrix} \begin{bmatrix} n+k \\ \ell+k \end{bmatrix},$$

where k is an arbitrary integer, and $k=0$ corresponds to the Durfee square identity. (The Durfee square of a partition is the largest square contained in its diagram.)

(c) Use (a) to construct the first six rows of the q-Pascal triangle and check that $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$ computed this way matches the definition (2).

(d) Let $(a;q)_n := (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denote a q-shifted factorial or q-Pochhammer symbol. Use (a) to show that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1};q)_m}{(q;q)_m}.$$

(e) Let $[n] := (1-q^n)/(1-q) = 1+q+\cdots+q^{n-1}$ and $[n]! := [1][2]\ldots[n]$. Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{[n+m]!}{[n]!\,[m]!} \quad \text{and} \quad \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}.$$

Solution.

(a) (i) The only partition contained in a rectangle of zero width is the empty partition 0.
(ii) Replace the summation index $\lambda \in (n^m)$ by $\lambda' \in (m^n)$ and use $|\lambda| = |\lambda'|$.
(iii) For the first recursion, we dissect the sum (2) according to the length of λ. The term $\left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right]$ corresponds to the generating function of partitions of length at most $n-1$ (those partitions fit in an $(n-1)\times m$ rectangle) and the term $q^n \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right]$

corresponds to the generating function of partitions of length exactly n. Since the diagram of such partitions has a first column of height n (contributing q^n to the generating function), after stripping off this column we are left with a partition that fits in an $n \times (m-1)$ rectangle, which contributes $\left[{n+m-1 \atop m-1}\right]$ to the generating function. The second recursion follows from (ii) or by carrying out a dissection according to $\lambda_1 < m$ and $\lambda_1 = m$.

(b) Note that part (iii) of (a) holds for all integers m if we define $\left[{n+m \atop m}\right] := 0$ for m a negative integer. To obtain (3) we may either iterate the second recursion in (iii), which implies that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m-K-1 \\ m-K-1 \end{bmatrix} + \sum_{k=m-K}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

for all nonnegative integers K. (For $K=-1$ the above is also true but tautological.) Choosing $K=m$ yields (3). Equivalently, conditioning on the value of λ_1 we may write the sum in (2) as

$$\sum_{k=0}^{m} \sum_{\substack{\lambda \subseteq (m^n) \\ \lambda_1 = k}} q^{|\lambda|} = \sum_{k=0}^{m} q^k \sum_{\lambda \subseteq (k^{n-1})} q^{|\lambda|} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}.$$

(For the Durfee rectangle identity fix k and similarly condition on the largest rectangle $((\ell+k)^\ell) \subseteq (m^n)$ that fits in λ.)

(c) Use part (iii) of (a) to construct rows $n+m = 0, 1, \dots, 5$. Let us label the left-pointing arrow by the factor q^n and omit the factor 1 for the arrows going to the right:

$$\begin{tikzcd}[column sep=tiny]
 & & & & & 1 \arrow[dl, "q^0"'] \arrow[dr] & & & & & \\
 & & & & 1 \arrow[dl, "q^0"'] \arrow[dr] & & 1 \arrow[dl, "q^1"'] \arrow[dr] & & & & \\
 & & & 1 \arrow[dl, "q^0"'] \arrow[dr] & & 1+q \arrow[dl, "q^1"'] \arrow[dr] & & 1 \arrow[dl, "q^2"'] \arrow[dr] & & & \\
 & & 1 \arrow[dl, "q^0"'] \arrow[dr] & & 1+q+q^2 \arrow[dl, "q^1"'] \arrow[dr] & & \cdots \arrow[dl, "q^2"'] \arrow[dr] & & 1 \arrow[dl, "q^3"'] \arrow[dr] & & \\
 & 1 \arrow[dl, "q^0"'] \arrow[dr] & & 1+q+q^2+q^3 \arrow[dl, "q^1"'] \arrow[dr] & & 1+q+2q^2+q^3+q^4 \arrow[dl, "q^2"'] \arrow[dr] & & \cdots \arrow[dl, "q^3"'] \arrow[dr] & & 1 \arrow[dl, "q^4"'] \arrow[dr] & \\
1 & & 1+q+q^2+q^3+q^4 & & 1+q+2q^2+2q^3+2q^4+q^5+q^6 & & \cdots & & \cdots & & 1
\end{tikzcd}$$

(d) It suffices to check that the right-hand side with $n \mapsto n+m$ obeys the initial condition (i) and the q-Pascal recurrence (iii) from (a).

(e) This follows in a straightforward manner from (d).

4. Plethystic notation. The aim of this question is to prove the q-binomial theorem

$$\sum_{k\geqslant 0}\frac{(a;q)_k}{(q;q)_k}\,z^k=\frac{(az;q)_\infty}{(z;q)_\infty} \tag{4}$$

using symmetric functions and plethystic notation. There are many alternative proofs, some of which are simpler, but hopefully this demonstrates the power of plethystic manipulations.

(a) To get a better feel for (4) show that, for n a nonnegative integer, it implies

(i) the q-binomial expansion

$$\sum_{k=0}^{n} q^{\binom{k}{2}}\begin{bmatrix} n \\ k\end{bmatrix} w^{n-k}z^k=\prod_{i=0}^{n-1}(w+q^i z);$$

(ii)

$$1+\sum_{k\geqslant 1}\begin{bmatrix} n+k-1 \\ k\end{bmatrix} z^k=\frac{1}{(z;q)_n}.$$

Remark. One can use (i) to prove Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty}(-z)^k q^{\binom{k}{2}}=(z;q)_\infty(q/z;q)_\infty(q;q)_\infty$$

by replacing $n\mapsto 2n$ followed by $k\mapsto k+n$ and using

$$(q^{-n}z;q)_{2n}=q^{-\binom{n+1}{2}}(-z)^n(z;q)_n(q/z;q)_n.$$

The triple product identity plays a key role in the theory of elliptic functions, and is the simplest example of a denominator identity for affine Kac–Moody Lie algebras, corresponding to $\mathrm{A}_1^{(1)}$ (affine $\mathfrak{sl}_2$).

(b) To prepare for the proof of (4) use plethystic notation to show that the generating function $\sigma_z[X]:=\sum_{k\geqslant 0}h_k[X]\,z^k$ satisfies

(i) $\sigma_z[X+Y]=\sigma_z[X]\,\sigma_z[Y]$ and thus $h_k[X+Y]=\sum_{i=0}^{k}h_i[X]\,h_{k-i}[Y]$;

(ii) $\sigma_z[1]=\frac{1}{1-z}$ so $h_n[1]=1$;

(iii) $\sigma_z[\frac{a}{1-q}]=\frac{1}{(az;q)_\infty}$.

(c) Now prove (4) by showing that both sides are equal to $\sigma_z[\frac{1-a}{1-q}]$.
Hint. For the left-hand side of (4) argue that it suffices to check equality with $\sigma_z[\frac{1-a}{1-q}]$ for a number of suitably chosen values of a and use part (b) to manipulate the alphabet on which h_r acts to recognise (3).

Solution.

(a) (i) Replace $(a,z)\mapsto(q^{-n},zq^n/w)$ where n is a nonnegative integer. Multiplying both sides of the resulting identity by w^n, the result follows since

$$\frac{(q^{-n};q)_k}{(q;q)_k}=q^{\binom{k}{2}+nk}\begin{bmatrix}n\\k\end{bmatrix}.$$

(ii) Replace $a\mapsto q^n$ and use that $(q^n;q)_k/(q;q)_k$ is 1 for $k=0$ and $\left[\begin{smallmatrix}k+n-1\\k\end{smallmatrix}\right]$ for $k>0$.

(b) (i) Since $\log\sigma_z[X]=\psi_z[X]:=\sum_{r\geqslant 1}\frac{p_r[X]}{r}\,z^r$ we have

$$\sigma_z[X+Y]=\mathrm{e}^{\psi_z[X+Y]}=\mathrm{e}^{\psi_z[X]+\psi_z[Y]}=\sigma_z[X]\sigma_z[Y].$$

Equating coefficients of z^n the convolution formula for h_k follows.
(ii) Note that for $X=\sum_{i\geqslant 1}x_i$,

$$\sigma_z[X]=\prod_{i\geqslant 1}\frac{1}{1-zx_i}.$$

Hence, if $X=x$ is a single-letter alphabet $\sigma_z[x]=1/(1-zx)$. The special case were this letter is 1 gives (ii).
(iii) Recall that $1/(1-q)=1+q+\cdots$. Hence

$$\sigma_z\Big[\frac{a}{1-q}\Big]=\prod_{i\geqslant 1}\frac{1}{1-azq^{i-1}}=\frac{1}{(az;q)_\infty}.$$

(c) To prove the q-binomial theorem (4) we first note that, since $\sigma_z[X-Y]=\sigma_z[X]/\sigma_z[Y]$,

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sigma_z\Big[\frac{1}{1-q}-\frac{a}{1-q}\Big]=\sigma_z\Big[\frac{1-a}{1-q}\Big].$$

The q-binomial theorem is thus equivalent to

$$\sum_{k\geqslant 0}\frac{(a;q)_k}{(q;q)_k}\,z^k=\sigma_z\Big[\frac{1-a}{1-q}\Big].$$

Equating coefficients of z^k yields

$$\frac{(a;q)_k}{(q;q)_k}=h_k\Big[\frac{1-a}{1-q}\Big],$$

where k is an arbitrary nonnegative integer. Since both sides are polynomials in a of degree k, it suffices to prove the above for $a=q^n$, where n is an arbitrary nonnegative integer. That is, we must prove

$$\frac{(q^n;q)_k}{(q;q)_k}=h_k\Big[\frac{1-q^n}{1-q}\Big]=h_k(1,q,\dots,q^n).$$

For $n = 0$ this is obvious so we may assume that $n \geqslant 1$ in the remainder. Then

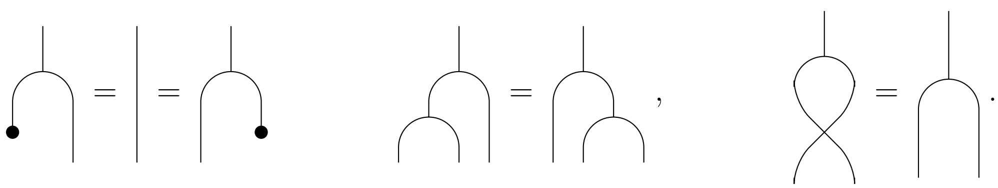
$$\begin{aligned} h_k\left[\frac{1-q^n}{1-q}\right] &= h_k\left[1 + q\,\frac{1-q^{n-1}}{1-q}\right] \\ &= \sum_{i=0}^{k} q^i h_i\left[\frac{1-q^{n-1}}{1-q}\right] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} = \frac{(q^n;q)_k}{(q;q)_k}. \end{aligned}$$

Here we recognised the sum (3) which, together with the initial condition at $k = 0$, characterises the Gaussian polynomials.

5. The Hopf-algebra structure of Λ. In this exercise we examine the algebraic structure of the ring of symmetric functions. All tensor products are over $K := \mathbb{Z}$.

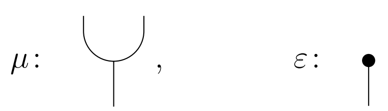
(a) Show that Λ is a commutative (unital associative) K-algebra with the usual product $m\colon \Lambda\otimes\Lambda \longrightarrow \Lambda$, $f[X]\,g[Y] \longmapsto f[X]\,g[X]$ in plethystic notation, and unit $e\colon K \longrightarrow \Lambda$ defined by $1 \longmapsto 1[X] \equiv 1$ extended K-linearly.
This structure can be nicely captured using string diagrams. Think of 'time' as increasing upwards, and depict (note that K is not drawn)

with $\gamma\colon \Lambda\otimes\Lambda \longrightarrow \Lambda\otimes\Lambda$ the permutation $f[X]\,g[Y] \longmapsto f[Y]\,g[X] = g[X]\,f[Y]$. The axioms of a (unital associative) algebra and commutativity then take the form

(5) 

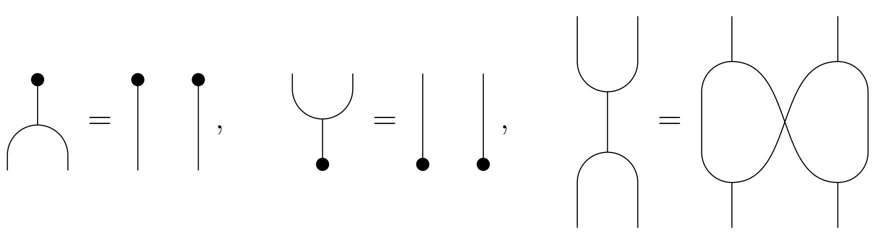
For example, the left-most diagram encodes $\Lambda \cong \Lambda\otimes K \xrightarrow{\mathrm{id}\otimes e} \Lambda\otimes\Lambda \xrightarrow{m} \Lambda$. Such an equality of string diagrams is often alternatively expressed as a commutative diagram.

(b) Show that Λ also is a cocommutative (counital coassociative) coalgebra, with coproduct $\mu\colon \Lambda \longrightarrow \Lambda\otimes\Lambda$ given by $f[X] \longmapsto f[X+Y]$ and counit $\varepsilon\colon \Lambda \longrightarrow K$, $f[X] \longmapsto f[0]$. Here the axioms are given by flipping all diagrams in (5) upside down and interpreting

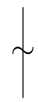
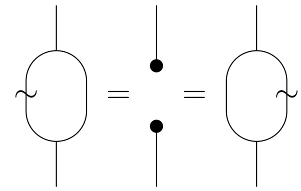


Do this using plethystic notation as well as by explicit computations for the power-sum basis p_λ.
Remark. Elements $f \in \Lambda$ whose coproduct satisfies $\mu(f) = f\otimes 1 + 1\otimes f$, such as the power sums p_r, are called primitive.

(c) Show that the preceding structures are compatible, making Λ a bialgebra: μ and ε are algebra homomorphisms (equivalently, m and e are coalgebra morphisms),



(d) Show that the bialgebra-structure of Λ extends to that of a Hopf algebra, with antipode $S\colon \Lambda \longrightarrow \Lambda$, $f[X] \longmapsto f[-X]$ extended as an (anti)homomorphism. Here the latter, depicted, say, as

S:  , must obey  .

Again do this using plethystic notation as well as by explicit calculation on the power-sum basis p_λ.
Remark. Viewed as equipped with this structure, Λ is commonly denoted by **Symm**.
Remark. Quantum integrability is related to quantum groups, which are sometimes defined as Hopf algebras that are quasitriangular, i.e., cocommutative up to conjugation by an invertible element of the tensor product of the Hopf algebra with itself, called the R-matrix, that behaves in a certain nice way under $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mu$ to guarantee it obeys the Yang–Baxter equation. Cocommutative Hopf algebras, such as Λ, are either viewed as boring examples ($R = 1 \otimes 1$) or excluded from the definition of a quantum group.

(e) Show that Λ is self dual with respect to the scalar product on $\Lambda \otimes \Lambda$ given by $\langle f_1[X]\, g_1[Y], f_2[X]\, g_2[Y]\rangle := \langle f_1[X], f_2[X]\rangle \langle g_1[Y], g_2[Y]\rangle$ where the right-hand side features the Hall scalar product on Λ. That is, the algebra and coalgebra structure of Λ are dual in the sense that $\langle f[X+Y], g[X]\, h[Y]\rangle = \langle f[X], g[X]\, h[X]\rangle$ and $\langle f[0], n\rangle = \langle f[X], n\, 1[X]\rangle$, where the scalar product on K is given by multiplication.

Solution.

(a) These are just the properties of the usual product.

(b) For the counit,

$$(\varepsilon \otimes \mathrm{id})\, f[X+Y] = f[0+Y] = f[Y] \equiv f[X] = f[X+0] = (\mathrm{id} \otimes \varepsilon),$$

where X and Y are arbitrary alphabets.
Coassociativity is clear since $f[X+(Y+Z)] = f[(X+Y)+Z]$.
Cocommutativity is clear since $f[X+Y] = f[Y+X]$.
On the power-sum basis the coproduct acts by sending $p_\lambda[X]$ to

$$\begin{aligned} p_\lambda[X+Y] &= \prod_{i\geqslant 1} p_{\lambda_i}[X+Y] = \prod_{i\geqslant 1} (p_{\lambda_i}[X] + p_{\lambda_i}[Y]) \\ &= \sum_{\substack{\nu^{(1)},\nu^{(2)} \\ \nu^{(1)}\cup\nu^{(2)}=\lambda}} p_{\nu^{(1)}}[X]\, p_{\nu^{(2)}}[Y], \end{aligned}$$

from which cocommutativity follows. Alternatively, from the lectures,

$$\sum_\lambda \frac{p_\lambda[X]}{z_\lambda} = \sigma_1[X],$$

so that

$$\sum_\lambda \frac{p_\lambda[X+Y]}{z_\lambda} = \sigma_1[X+Y] = \sigma_1[X]\sigma_1[Y],$$

from which cocommutativity again follows.
For the counit,

$$p_\lambda[0] = \prod_i p_{\lambda_i}[0] = \delta_{l(\lambda),0} = \delta_{\lambda,0}$$

so that $\varepsilon \otimes \mathrm{id}$ forces $\nu^{(1)} = 0$. Hence $\nu^{(2)} = \lambda$. The second half follows by cocommutativity.
For coassociativity, note that the set of all triples of partitions $\nu^{(1)}, \nu^{(2,1)}, \nu^{(2,2)}$ such that $\nu^{(1)} \cup (\nu^{(2,1)} \cup \nu^{(2,2)}) = \lambda$ is equivalent to the set of all triples $\nu^{(1,1)}, \nu^{(1,2)}, \nu^{(2)}$ such that $(\nu^{(1,1)} \cup \nu^{(1,2)}) \cup \nu^{(2)} = \lambda$.
For $l(\lambda) = 1$, $\lambda = (r)$, say, the coproduct of p_r is $p_r[X+Y] = p_r[X] + p_r[Y]$, so that $\mu(p_r) = p_r \otimes 1 + 1 \otimes p_r$. The power sums p_r are thus primitive as per the above remark.

(c) Clearly

$$f[X]\,g[Y] \overset{\mu}{\longmapsto} f[X]\,g[X] \overset{\varepsilon}{\longmapsto} f[0]\,g[0] = (\varepsilon \otimes \varepsilon)(f[X]\,g[Y]).$$

Next, $\mu(1) = 1 \otimes 1$ as $\lambda = 0$ implies that $\nu^{(1)} = \nu^{(2)} = 0$. Alternatively, $1[X+Y] \equiv 1[X]\,1[Y]$ since $p_0 = 1$ is the constant function.
We check the last axiom on the power-sum basis of Λ. Let us first consider primitive elements. The string diagram on the left gives the coproduct of $p_r\,p_s = p_{(r)\cup(s)}$. This matches the result of the diagram on the right:

$$\begin{aligned}
p_r \otimes p_s &\overset{\mu\otimes\mu}{\longmapsto} (p_r \otimes 1 + 1 \otimes p_r) \otimes (p_s \otimes 1 + 1 \otimes p_s) \\
&= p_r \otimes 1 \otimes p_s \otimes 1 + p_r \otimes 1 \otimes 1 \otimes p_s + 1 \otimes p_r \otimes p_s \otimes 1 + 1 \otimes p_r \otimes 1 \otimes p_s \\
&\overset{\mathrm{id}\otimes\gamma\otimes\mathrm{id}}{\longmapsto} p_r \otimes p_s \otimes 1 \otimes 1 + p_r \otimes 1 \otimes 1 \otimes p_s + 1 \otimes p_s \otimes p_r \otimes 1 + 1 \otimes 1 \otimes p_r \otimes p_s \\
&\overset{m\otimes m}{\longmapsto} p_r\,p_s \otimes 1 + p_r \otimes p_s + p_s \otimes p_r + 1 \otimes p_r\,p_s \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}} \otimes p_{\nu^{(2)}}, \qquad\qquad \lambda := (r) \cup (s).
\end{aligned}$$

In general let $\lambda^{(1)}$ and $\lambda^{(2)}$ be two partitions. The string diagram on the left gives

$$p_{\lambda^{(1)}}[X]\,p_{\lambda^{(2)}}[Y] \overset{m}{\longmapsto} p_{\lambda^{(1)}}[X]\,p_{\lambda^{(2)}}[X] = p_{\lambda^{(1)}\cup\lambda^{(2)}}[X] \overset{\mu}{\longmapsto} p_{\lambda^{(1)}\cup\lambda^{(2)}}[X+Y].$$

This matches the result from the diagram on the right:

$$
\begin{aligned}
p_{\lambda^{(1)}}[X]\, p_{\lambda^{(2)}}[Y] \overset{\mu\otimes\mu}{\longmapsto} & \; p_{\lambda^{(1)}}[X+X']\, p_{\lambda^{(2)}}[Y+Y'] \\
&= \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\, p_{\nu^{(1,2)}}[X']\, p_{\nu^{(2,1)}}[Y]\, p_{\nu^{(2,2)}}[Y'] \\
&\overset{\mathrm{id}\otimes\gamma\otimes\mathrm{id}}{\longmapsto} \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\, p_{\nu^{(2,1)}}[X']\, p_{\nu^{(1,2)}}[Y]\, p_{\nu^{(2,2)}}[Y'] \\
&\overset{m\otimes m}{\longmapsto} \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\, p_{\nu^{(2,1)}}[X]\, p_{\nu^{(1,2)}}[Y]\, p_{\nu^{(2,2)}}[Y] \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}}[X]\, p_{\nu^{(2)}}[Y] \qquad \begin{aligned} \nu^{(i)} &:= \nu^{(1,i)}\cup\nu^{(2,i)} \\ \lambda &:= \lambda^{(1)}\cup\lambda^{(2)}, \end{aligned}
\end{aligned}
$$

where the final equality uses that the set of all subsets $\nu^{(i,1)}$ (which determines $\nu^{(i,2)}$) of λ_i (viewed as a list, not a diagram) for $i=1,2$ separately is the same as the set of all subsets $\nu^{(1)}$ of $\lambda=\lambda^{(1)}\cup\lambda^{(2)}$.

(d) By (co)commutativity it suffices to check the first equality:

$$
f[X] \overset{\mu}{\longmapsto} f[X+Y] \overset{S\otimes\mathrm{id}}{\longmapsto} f[-X+Y] \overset{m}{\longmapsto} f[-X+X] = f[0] = f[0]\,1[X].
$$

Equivalently, on power sums,

$$
S(p_\lambda) = S\Big(\prod_i p_{\lambda_i}\Big) = \prod_i S(p_{\lambda_i}) = (-1)^{l(\lambda)}\, p_\lambda,
$$

where the ordering is irrelevant by commutativity. Thus

$$
\begin{aligned}
p_\lambda &\overset{\mu}{\longmapsto} \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}}\otimes p_{\nu^{(2)}} \\
&\overset{S\otimes\mathrm{id}}{\longmapsto} \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})}\, p_{\nu^{(1)}}\otimes p_{\nu^{(2)}} \\
&\overset{m}{\longmapsto} \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})}\, p_\lambda = \delta_{\lambda,0}\,.
\end{aligned}
$$

Here the last equality uses that there are $\binom{l(\lambda)}{k}$ possible $\nu^{(1)}$ with $l(\nu^{(1)})=k$, whence

$$
\sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})} = \sum_{k=0}^{l(\lambda)} \binom{l(\lambda)}{k} (-1)^k = \big(1+(-1)\big)^{l(\lambda)} = \delta_{l(\lambda),0} = \delta_{\lambda,0}\,.
$$

(e) Again compute with power sums: $\langle p_\lambda[0], 1\rangle = p_\lambda[0] = \delta_{\lambda,0} = \langle p_\lambda[X], 1[X]\rangle$, while

$$\begin{aligned}
\langle p_\lambda[X+Y], p_{\mu^{(1)}}[X]\, p_{\mu^{(2)}}[Y]\rangle &= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} \langle p_{\nu^{(1)}}[X]\, p_{\nu^{(2)}}[Y], p_{\mu^{(1)}}[X]\, p_{\mu^{(2)}}[Y]\rangle \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} \langle p_{\nu^{(1)}}[X], p_{\mu^{(1)}}[X]\rangle\langle\, p_{\nu^{(2)}}[Y], p_{\mu^{(2)}}[Y]\rangle \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} z_{\mu^{(1)}}^{-1}\, \delta_{\nu^{(1)},\,\mu^{(1)}}\, z_{\mu^{(2)}}^{-1}\, \delta_{\nu^{(2)},\,\mu^{(2)}} \\
&= z_\lambda^{-1}\, \delta_{\lambda,\mu} = \langle p_\lambda[X], p_\mu[X]\rangle, \qquad \mu := \mu^{(1)} \cup \mu^{(2)}.
\end{aligned}$$

6. Principal specialisation. Recall that the hook-length of a square $s=(i,j)\in\lambda$ is given by $h(s)=\lambda_i+\lambda_j-i-j+1$. Show that

$$\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)=\prod_{i=1}^{n}(q^{n-i+1};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{j-i}}{1-q^{\lambda_i-\lambda_j+j-i}},\tag{6}$$

where $\mathscr{H}(\lambda)$ denotes the multiset of hook-lengths of λ, n is any integer such that $n\geqslant l(\lambda)$ and $(a;q)_m=(1-a)(1-aq)\cdots(1-aq^{m-1})$. Use (6) to prove that

$$s_\lambda\left[\frac{1-a}{1-q}\right]=q^{n(\lambda)}\frac{\prod_{(i,j)\in\lambda}(1-aq^{j-i})}{\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)}=q^{n(\lambda)}\frac{\prod_{i\geqslant1}(aq^{1-i};q)_{\lambda_i}}{\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)},\tag{7}$$

where $n(\lambda):=\sum_{i\geqslant1}(i-1)\lambda_i$. For $a=q^n$ and $l(\lambda)\leqslant n$ this is known as the principal specialisation formula for Schur functions.

Solution. Let $l:=l(\lambda)$. Then

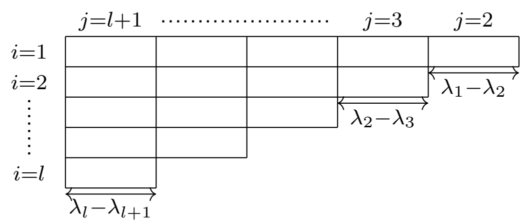
$$\begin{aligned}\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}&=\prod_{1\leqslant i<j\leqslant l}\frac{1-q^{j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}\prod_{i=1}^{l}\prod_{j=l+1}^{n}\frac{1-q^{j-i}}{1-q^{\lambda_i+j-i}}\\&=\prod_{1\leqslant i<j\leqslant l}\frac{1-q^{j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}\prod_{i=1}^{l}\frac{(q^{l-i+1};q)_{n-l}}{(q^{\lambda_i+l-i+1};q)_{n-l}}.\end{aligned}$$

By $(aq^m;q)_k=(a;q)_k(aq^k;q)_m/(a;q)_m$,

$$\frac{(q^{l-i+1};q)_{n-l}}{(q^{\lambda_i+l-i+1};q)_{n-l}}=\frac{(q^{l-i+1};q)_{\lambda_i}}{(q^{n-i+1};q)_{\lambda_i}},$$

which shows that the right-hand side of (6) is independent of n (as long as $n\geqslant l=l(\lambda)$).

Define $\lambda_{l+1}:=0$ and partition the diagram of the partition λ into $\binom{l+1}{2}$ rectangles of height one as follows



where rows are labelled by i ($1\leqslant i\leqslant\ell$ from top and bottom) and the columns by j ($i<j\leqslant\ell+1$ from right to left). The rectangle labelled (i,j) has width $\lambda_{j-1}-\lambda_j$,

$$\sum_{k=i+1}^{j-1}(\lambda_{k-1}-\lambda_k)=\lambda_i-\lambda_{j-1}$$

squares of in the same row to its right and $j-i-1$ rectangles in the same column to the south. Therefore,

$$\begin{aligned}
\prod_{h\in\mathscr{H}(\lambda)}(1-q^h) &= \prod_{1\leqslant i<j\leqslant l+1}(q^{1+(\lambda_i-\lambda_{j-1})+(j-i-1)};q)_{\lambda_{j-1}-\lambda_j} \\
&= \prod_{1\leqslant i<j\leqslant l+1}\frac{(q^{j-i};q)_{\lambda_i-\lambda_j}}{(q^{j-i};q)_{\lambda_i-\lambda_{j-1}}} \\
&= \frac{\prod_{1\leqslant i<j\leqslant l+1}(q^{j-i};q)_{\lambda_i-\lambda_j}}{\prod_{1\leqslant i<j\leqslant l}(q^{j-i+1};q)_{\lambda_i-\lambda_j}} \\
&= \prod_{i=1}^{l}(q^{l-i+1};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant l}\frac{1-q^{j-i}}{1-q^{j-i+\lambda_i-\lambda_j}}.
\end{aligned}$$

Both sides of the specialisation formula (7) are polynomials in a of degree $|\lambda|$. Hence it suffices to show the identity for $a=q^n$ with n an arbitrary nonnegative integer. But $(1-q^n)/(1-q)=1+q+\cdots+q^{n-1}$ so that we only need to prove the principal specialisation formula. If $n\geqslant l(\lambda)$ then both sides trivially vanish, so that without loss of generality we may assume that $l(\lambda)\leqslant n$. Then, by the definition of the Schur function

$$\begin{aligned}
s_\lambda\left[\frac{1-q^n}{1-q}\right] &= \frac{\det_{1\leqslant i,j\leqslant n}(q^{(n-i)(\lambda_j+n-j)})}{\prod_{1\leqslant i<j\leqslant n}(q^{n-i}-q^{n-j})} \\
&= \frac{\det_{1\leqslant i,j\leqslant n}(y_j^{n-i})}{\prod_{1\leqslant i<j\leqslant n}(q^{n-i}-q^{n-j})} \qquad (y_j:=q^{\lambda_j+n-j}) \\
&= \prod_{1\leqslant i<j\leqslant n}\frac{q^{\lambda_i+n-i}-q^{\lambda_j+n-j}}{q^{n-i}-q^{n-j}} \\
&= q^{n(\lambda)}\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}.
\end{aligned}$$

By (6) we arrive at (7) with $a=q^n$.

An alternative proof uses the Jacobi–Trudi identity. Since

$$h_r\left[\frac{1-a}{1-q}\right]=\frac{(a;q)_r}{(q;q)_r}$$

(with the standard convention that $1/(q;q)_r = (q^r;q)_\infty/(q;q)_\infty$ which vanishes for r a negative integer) we have

$$\begin{aligned} s_\lambda\left[\frac{1-a}{1-q}\right] &= \det_{1\leqslant i,j\leqslant l}\left(h_{\lambda_i+j-i}\left[\frac{1-a}{1-q}\right]\right) \\ &= \det_{1\leqslant i,j\leqslant l}\left(\frac{(a;q)_{\lambda_i+j-i}}{(q;q)_{\lambda_i+j-i}}\right) \\ &= \prod_{i=1}^{l}\frac{(a;q)_{\lambda_i+1-i}}{(q;q)_{\lambda_i+l-i}}\det_{1\leqslant i,j\leqslant l}\Big((aq^{\lambda_i+1-i};q)_{j-1}(q^{\lambda_i+j-i+1};q)_{l-j}\Big). \end{aligned}$$

It is not hard to deform the Vandermonde determinant to

$$\det_{1\leqslant i,j\leqslant n}\left(\prod_{k=1}^{j-1}(1-x_ia_k)\prod_{k=j+1}^{n}(1-x_ib_k)\right) = \prod_{1\leqslant i<j\leqslant n}(a_i-b_j)(x_i-x_j).$$

(Both sides are polynomials in each of the variables of degree $n-1$ and vanish when $x_i=x_j$ or $a_i=b_j$ for some $1\leqslant i<j\leqslant n$. Hence they are the same up to a constant, which is readily seen to be 1 by comparing coefficients of $(b_nx_1)^{n-1}(b_{n-1}x_2)^{n-2}\cdots(b_2x_{n-1})^1$.) Replacing n by l and choosing $x_i=q^{\lambda_i+1-i}$, $a_k=aq^{k-1}$ and $b_k=q^{k-1}$ yields

$$\det_{1\leqslant i,j\leqslant l}\Big((aq^{\lambda_i+1-i};q)_{j-1}(q^{\lambda_i+j-i+1};q)_{l-j}\Big) = \prod_{1\leqslant i<j\leqslant l}(aq^{i-1}-q^{j-1})(q^{\lambda_i+1-i}-q^{\lambda_j+1-j})$$

and thus

$$\begin{aligned} s_\lambda\left[\frac{1-a}{1-q}\right] &= \prod_{i=1}^{l}\frac{(a;q)_{\lambda_i+1-i}}{(q;q)_{\lambda_i+l-i}}\prod_{1\leqslant i<j\leqslant l}(aq^{i-1}-q^{j-1})(q^{\lambda_i+1-i}-q^{\lambda_j+1-j}) \\ &= q^{n(\lambda)}\prod_{i=1}^{l}\frac{(a;q)_{\lambda_i+1-i}}{(q;q)_{\lambda_i+l-i}}\prod_{1\leqslant i<j\leqslant l}(1-aq^{i-j})(1-q^{\lambda_i-\lambda_j+j-i}). \end{aligned}$$

Since

$$\prod_{1\leqslant i<j\leqslant l}(1-aq^{i-j})(1-q^{j-i}) = \prod_{i=1}^{l}(aq^{1-i};q)_{i-1}(q;q)_{l-i}$$

and

$$(aq^{1-i};q)_{i-1}(a;q)_{\lambda_i+1-i} = (aq^{1-i};q)_{\lambda_i}, \qquad \frac{(q;q)_{l-i}}{(q;q)_{\lambda_i+l-i}} = \frac{1}{(q^{l-i+1};q)_{\lambda_i}},$$

this is also

$$\begin{aligned} s_\lambda\left[\frac{1-a}{1-q}\right] &= q^{n(\lambda)}\prod_{i=1}^{l}\frac{(aq^{1-i};q)_{\lambda_i}}{(q^{l-i+1};q)_{\lambda_i}}\prod_{1\leqslant i<j\leqslant l}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}} \\ &= q^{n(\lambda)}\frac{\prod_{(i,j)\in\lambda}(1-aq^{j-i})}{\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)}. \end{aligned}$$

7. Inverse branching rule. According to the branching rule for Schur functions,

$$s_\lambda[X+1] = \sum_{\mu \prec \lambda} s_\mu[X]. \tag{8}$$

Prove the combinatorial identity

$$\sum_{\substack{\mu' \prec \lambda' \\ \mu \succ \nu}} (-1)^{|\lambda/\mu|} = \delta_{\lambda\nu}, \tag{9}$$

and use this to show that

$$s_\lambda[X-1] = \sum_{\mu' \prec \lambda'} (-1)^{|\lambda/\mu|} s_\mu[X].$$

For example

$$s_{(3,1)}[X-1] = s_{(3,1)}[X] - s_{(3)}[X] - s_{(2,1)}[X] + s_{(2)}[X].$$

Solution. It is clear that both sides of (9) vanish unless $\nu \subseteq \lambda$ and that the identity trivially holds for $\nu = \lambda$. We may thus assume in the remainder that $\nu \subset \lambda$ (strict inclusion).

Since λ/μ is a vertical strip, there is a smallest admissible μ, say $\mu_{\min}$, given by $\mu'_{\min} = (\lambda'_2, \lambda'_3, \dots)$. Any μ such that $\mu_{\min} \subseteq \mu \subseteq \lambda$ gives a vertical strip $\lambda - \mu$. For example, if $\lambda = (8,8,5,3,2,2,2,1,1)$ then $\lambda' = (9,7,4,3,3,2,2,2)$ so that $\mu'_{\min} = (7,4,3,3,2,2,2)$ and hence $\mu_{\min} = (7,7,4,2,1,1,1)$.

Since μ/ν is a horizontal strip there is a largest possible μ, say $\mu_{\max}$, given by $\mu_{\max} = (\lambda_1, \nu_1, \nu_2, \dots)$. Any μ such that $\nu \subseteq \mu \subseteq \mu_{\max}$ gives a horizontal strip μ/ν. For example, if $\nu = (7,4,3,2,1,1)$ and λ as above then $\mu_{\max} = (8,7,4,3,2,1,1)$.

From the above it follows that we must prove that

$$\sum_{\mu_{\min} \subseteq \mu \subseteq \mu_{\max}} (-z)^{|\mu/\mu_{\min}|}$$

vanishes for all $\nu \subset \lambda$ when $z = 1$. If $\mu_{\min} \not\subseteq \mu_{\max}$ then the sum is empty. This happens when any of the parts of ν are less than that of $\nu_{\min} = \big((\mu_{\min})_2, (\mu_{\min})_3, \dots\big)$. For λ as above $\nu_{\min} = (7,4,2,1,1,1)$. For ν such that $\nu_{\min} \subseteq \nu \subset \lambda$ it follows that $\mu_{\max} - \mu_{\min}$ has at most one box in each row and column, and has et least one box. Thus

$$\sum_{\mu_{\min} \subseteq \mu \subseteq \mu_{\max}} (-z)^{|\mu/\mu_{\min}|} = (1-z)^{|\mu_{\max}/\mu_{\min}|},$$

which has the desired vanishing property.

For λ and ν as in the example the set of admissible partitions μ is

$$\big\{(7,7,4,2,1,1,1), (8,7,4,2,1,1,1), (7,7,4,2,2,1,1), (8,7,4,2,2,1,1),$$
$$(7,7,4,3,1,1,1), (8,7,4,3,1,1,1), (7,7,4,3,2,1,1), (8,7,4,3,2,1,1)\big\}$$

and if we change ν to $\nu_{\min}$ then this reduces to

$$\big\{\mu_{\min}, \mu_{\max}\big\} = \big\{(7,7,4,2,1,1,1), (8,7,4,2,1,1,1)\big\}.$$

Now

$$\begin{aligned}
s_\lambda[X-1] &= \sum_\nu s_\nu[X-1]\delta_{\lambda\nu} \\
&= \sum_\nu s_\nu[X-1] \sum_{\substack{\mu'\prec\lambda' \\ \mu\succ\nu}} (-1)^{|\lambda/\mu|} \\
&= \sum_{\mu'\prec\lambda'} (-1)^{|\lambda/\mu|} \sum_{\nu\prec\mu} s_\nu[X-1] \\
&= \sum_{\mu'\prec\lambda'} (-1)^{|\lambda/\mu|} s_\mu[X],
\end{aligned}$$

where the last equality follows from the branching rule (8).

8. Kostant's multiplicity formula. Kostant's formula is an explicit (computationally not very efficient; Freudenthal's recursion formula is much more practical) expression for the weight multiplicities of irreducible representations of semi-simple Lie algebras, expressing the multiplicities as an alternating sum over what is known as the 'Kostant partition function'. In this question we look at a combinatorial analogue of this formula in the case of $\mathfrak{gl}_n$.
Recall that the Kostka number $K_{\lambda\alpha}$ counts the number of semistandard Young tableaux of shape λ and weight α, i.e., $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.

(a) Show that $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ and thus $h_\lambda = \sum_\mu P_{\lambda\mu} m_\mu$, where $P_{\lambda\mu} := \sum_\omega K_{\omega\lambda} K_{\omega\mu}$ will play the role of Kostant partition function.

(b) Using the RSK correspondence it may be shown that $P_{\alpha\beta}$ (for α and β (weak) compositions) counts the number of matrices with nonnegative integer entries such that the ith row-sum is β_i and the jth column-sum is α_j.
Count $P_{(2,1),(1,1,1)}$ by listing all pairs of semistandard tableaux contributing to the sum and by listing the relevant integer matrices.

(c) Use the Jacobi–Trudi formula to show that for μ a partition of length n,

$$\sum_{w \in S_n} \operatorname{sgn}(w) K_{\lambda, w(\mu+\delta)-\delta} = \delta_{\lambda\mu},$$

where $\delta := (n-1, \dots, 1, 0)$.

(d) For λ a partition of length n, prove the Kostant multiplicity formula

$$K_{\lambda\mu} = \sum_{w \in S_n} \operatorname{sgn}(w) P_{w(\lambda+\delta)-\delta, \mu}$$

and use it to compute $K_{(2,1),(1,1,1)}$.

Solution.

(a) By $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ and the fact that $\{m_\lambda\}$ and $\{h_\mu\}$ are dual bases of Λ, $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$. Since the Schur basis of Λ is self-dual this implies that $h_\mu = \sum_\lambda K_{\lambda\mu} s_\mu$.

(b) The relevant pairs of tableaux are

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ .$$

The corresponding three 3×2 matrices are given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

(c) According to the Jacobi–Trudi formula, for λ a partition such that $l(\lambda) = n$,

$$\begin{aligned} s_\lambda &= \det_{1\leqslant i,j\leqslant n} \big(h_{\lambda_i - i + j}\big) \\ &= \sum_{w\in S_n} \operatorname{sgn}(w) h_{w(\lambda+\delta)-\delta} \\ &= \sum_{w\in S_n} \operatorname{sgn}(w) \sum_\nu K_{\nu, w(\lambda_i+\delta)-\delta} s_\nu. \end{aligned}$$

Equating coefficients of s_λ the result follows.

(d) We have

$$s_\lambda = \sum_{w\in S_n} \operatorname{sgn}(w) h_{w(\lambda+\delta)-\delta} = \sum_{w\in S_n}\sum_\mu P_{w(\lambda+\delta)-\delta,\mu} m_\mu$$

but also $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$. The formula for $K_{\lambda\mu}$ immediately follows. Since

$$\operatorname{SSYT}\big((2,1),(1,1,1)\big) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\},$$

we have $K_{(2,1),(1,1,1)} = 2$. This follows from the Kostant multiplicity formula as

$$K_{(2,1),(1,1,1)} = P_{(2,1),(1,1,1)} - P_{(0,3),(1,1,1)} = 3 - 1 = 2,$$

since the only matrix contributing to $P_{(0,3),(1,1,1)}$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

9. Vertex operators. For n an integer define the linear operator $\alpha_n : \Lambda \to \Lambda$ by

$$\alpha_{-n} s_\mu = \sum_{\substack{\lambda \vdash |\mu|+n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\lambda \tag{10}$$

and

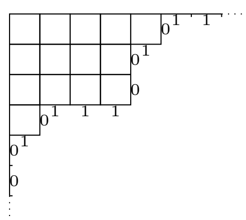
$$\alpha_n s_\lambda = \sum_{\substack{\mu \vdash |\lambda|-n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\mu$$

for $n \geqslant 0$.

(a) Show that α_n and α_{-n} are adjoint with respect to the Hall scalar product on Λ.

(b) Prove that the α_n satisfy the commutation relations of the Heisenberg algebra, i.e., $[\alpha_n, \alpha_m] = n\delta_{n,-m}$.

Hint. Use the representation of a partition in terms of its 0/1-sequence/edge sequence/code/Maya diagram. For example, the 0/1-sequence of the partition $(5, 4, 4, 1)$ is



$\longleftrightarrow \quad \ldots 000101110010111 \ldots$

(c) Prove that the vertex operators

$$\Gamma_\pm(z) := \exp\left(\sum_{n \geqslant 1} \frac{z^n}{n}\, \alpha_{\pm n}\right)$$

obey the commutation relation

$$\Gamma_+(w)\Gamma_-(z) = \frac{1}{1 - zw}\, \Gamma_-(z)\Gamma_+(w).$$

(d) Prove that

$$p_n s_\mu = \sum_{\substack{\lambda \vdash |\mu|+n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\lambda. \tag{11}$$

Remark. For $\lambda, \mu \vdash n$ let $\chi_\lambda(\mu)$ be the character of the irreducible representation of S_n indexed by λ evaluated at (elements of S_n in the conjugacy class indexed by) μ. From (d) and $p_\mu = \sum_\lambda \chi_\lambda(\mu) s_\lambda$ it follows that

$$\chi_\lambda(\mu) = \sum_{T \in \text{BST}(\lambda,\mu)} (-1)^{\text{height}(T)}.$$

where $\text{BST}(\lambda, \mu)$ is the set of borderstrip tableaux of shape λ and weight μ, i.e., the set of tableaux of shape λ and weight μ such that the μ_i boxes filled with the letter i form a borderstrip, and where the height of a borderstrip tableau is the sum of the heights of the individual borderstrips making up the tableau. This is known as the Murnaghan–Nakayama rule.

(e) Use (d) to prove

(i) the 'Pieri rule'

$$\Gamma_-(z)s_\mu[X] = \sigma_z[X]s_\mu[X];$$

(ii) the 'branching rule'

$$\Gamma_+(z)s_\lambda[X] = s_\lambda[X+z];$$

(iii) the skew Schur function identity

$$s_{\lambda/\mu}(z_1,\dots,z_n) = \big\langle \Gamma_+(z_1)\dots\Gamma_+(z_n)s_\lambda, s_\mu\big\rangle.$$

Solution.

(a) By the orthonormality of the Schur functions with respect to the Hall scalar product,

$$\langle s_\lambda, \alpha_{-n}s_\mu\rangle = \langle \alpha_n s_\lambda, s_\mu\rangle = \begin{cases} (-1)^{\mathrm{height}(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a border strip of size } n, \\ 0 & \text{otherwise.} \end{cases}$$

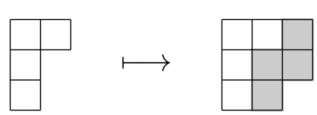
(b) *Remark.* Before sketching the solution using 0/1-sequences, we remark that there are several much simpler (and less combinatorial) methods for showing the Heisenberg commutation relations for the α_n. One uses fermionic Fock space (or semi-infinite wedge space) in which α_n may be realised in terms of fermionic creation and annihilation operators ψ_k and ψ_k^* on $\mathbb{Z}+1/2$ (which satisfy the anti-commutation relations $\psi_k\psi_k^* + \psi_k^*\psi_k = 1$) as follows $\alpha_n = \sum_{k\in\mathbb{Z}+1/2}\psi_{k-n}\psi_k^*$. The other alternative method uses (11) and the fact that the operator $p_n^\perp : \Lambda \to \Lambda$, which act as the adjoint of multiplication by p_n, i.e., $\langle p_n^\perp(f), g\rangle = \langle f, p_n g\rangle$ for $f, g \in \Lambda$ is given by $n\partial/\partial p_n$. Then, for $n \geqslant 1$, $\alpha_n = p_n^\perp$ and $\alpha_{-n} = p_n$. For example, if λ is a partition and μ the partitions obtained from λ by removing a part of size n if it exists and 0 otherwise,

$$\begin{aligned} [p_n^\perp, p_n]p_\lambda &= p_n^\perp(p_n p_\lambda) - p_n p_n^\perp(p_\lambda) \\ &= p_n^\perp(p_{\lambda\cup(n)}) - n m_n(\lambda) p_n p_\mu \\ &= n(m_n(\lambda)+1)p_\lambda - n m_n(\lambda) p_\lambda \\ &= n p_\lambda. \end{aligned}$$

More combinatorially, adding a border strip b of size n to a partition changes its 0/1-sequence by swapping a 0 and a 1 a distance n apart (where the 0 is to the left of the 1):

$$\dots 0w1 \dots \mapsto \dots 1w0 \dots$$

where w represents a word of length $n-1$. Moreover, height(b) is exactly the number of zeros of w. For example, if we go from



this corresponds to

$$\ldots 0010010111\ldots \longmapsto \ldots 0011010011\ldots$$

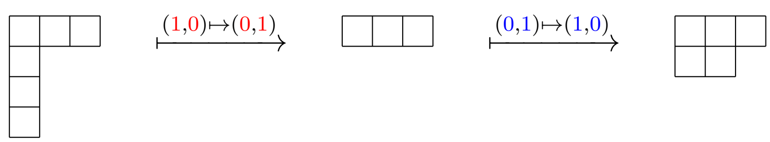
with $w = 010$ so that height$(b) = 2$. Conversely, removing a border strip b of length n changes the $0/1$-sequence by swapping a 1 and a 0 a distance n apart (where the 1 is to the left of the 0:

$$\ldots 1w0\ldots \mapsto \ldots 0w1\ldots.$$

To now see what happens if we act with $[\alpha_n, \alpha_m]$ on a Schur function, we shall assume that $n \leqslant 0$, $m \geqslant 0$ and $m \geqslant -n$. (This also implies the case $n \leqslant 0$, $m \geqslant 0$ and $m \leqslant -n$ by adjointness. The case of equal sign is similar but simpler.) Replacing n by $-n$ we thus need to consider $\alpha_{-n}\alpha_m s_\lambda$ and $\alpha_m\alpha_{-n}s_\lambda$ where n, m are both positive and $m \geqslant n$. By the above description of how adding/removing a border strip affects the $0/1$-sequence, we see that in $\alpha_{-n}\alpha_m s_\lambda$ we first obtain a signed sum of Schur functions indexed by $0/1$-sequences obtained from that of λ by carrying out all possible swaps of $(1, 0)$ pairs at distance m and then we replace each of those Schur functions by a signed sum of Schur functions obtained by all possible swaps of $(0, 1)$ pairs at distance n. For $\alpha_m\alpha_{-n}s_\lambda$ we proceed in the exact opposite order. The upshot is that any of the Schur functions arising in the computation of $\alpha_{-n}\alpha_m s_\lambda$ and $\alpha_m\alpha_{-n}s_\lambda$ is obtained by carrying out two swaps in opposite direction. In the generic case, a pair of swaps commutes and simply involves a $(0, 1)$ and $(1, 0)$ pair that do not interact. As a result, most terms in

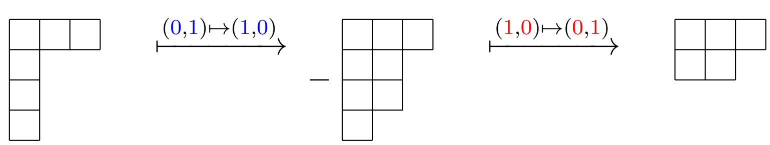
$$\big(\alpha_{-n}\alpha_m - \alpha_m\alpha_{-n}\big)s_\lambda$$

trivially cancel. For example, one of the terms obtained by computing $\alpha_{-2}\alpha_3 s_{(3,1^3)}$ is $s_{(3,2)}$:



$\ldots 010001101\ldots \qquad \ldots 000011101\ldots \qquad \ldots 000110101\ldots$

Exactly the same term is obtained in the computation of $\alpha_3\alpha_{-2}s_{(3,1^3)}$, by first swapping the blue pair and then the red pair:



$\ldots 010001101\ldots \qquad \ldots 010100101\ldots \qquad \ldots 000110101\ldots$

However, it is also possible that the two pairs are not independent and a more careful analysis is required for those. When $m > n$ we can have

$$\ldots 1w1v0\ldots$$

with w a word of length $n-1$ and v a word of length $m-n-1$. Hence the pair $(1,1)$ are a distance n apart and the pair $(1,0)$ are a distance m apart. In the following we write $\operatorname{sgn}(w)$ for the number of zeros in the word w. We can obviously carry out the sequence of two swaps

$$\dots 1w1v0\dots \xmapsto{\alpha_m} \operatorname{sgn}(w)\operatorname{sgn}(v)\dots 0w1v1\dots \xmapsto{\alpha_{-n}} \operatorname{sgn}(v)\dots 1w0v1\dots, \tag{12}$$

where we first swapped red and green and then green and blue. But if we first want to act with α_{-n} and then α_m then it is not clear we can obtain the same 0/1-sequence. It turns out that we need to distinguish two scenarios. The first is

$$\dots 1w1v0u1\dots$$

where u is a word of length $n-1$. The sequence of steps (12) remains the same, the extra $u1$ are just dummies:

$$\dots 1w1v0u1\dots \xmapsto{\alpha_m} \operatorname{sgn}(w)\operatorname{sgn}(v)\dots 0w1v1u1\dots \xmapsto{\alpha_{-n}} \operatorname{sgn}(v)\dots 1w0v1u1\dots.$$

But now we also have

$$\dots 1w1v0u1\dots \xmapsto{\alpha_{-n}} \operatorname{sgn}(u)\dots 1w1v1u0\dots \xmapsto{\alpha_m} \operatorname{sgn}(v)\dots 1w0v1u1\dots.$$

Although the colour coding is different, the final two sequences are the same and hence we again have cancellation. The second scenario is

$$\dots 1w1v0u0\dots$$

where again u is a word of length $n-1$. Once more the sequence of steps (12) remains the same:

$$\dots 1w1v0u0\dots \xmapsto{\alpha_m} \operatorname{sgn}(w)\operatorname{sgn}(v)\dots 0w1v1u0\dots \xmapsto{\alpha_{-n}} \operatorname{sgn}(v)\dots 1w0v1u0\dots.$$

But now we also have

$$\dots 1w1v0u0\dots \xmapsto{\alpha_{-n}} \operatorname{sgn}(u)\operatorname{sgn}(v)\dots 1w0v0u1\dots \xmapsto{\alpha_m} \operatorname{sgn}(v)\dots 1w0v1u0\dots,$$

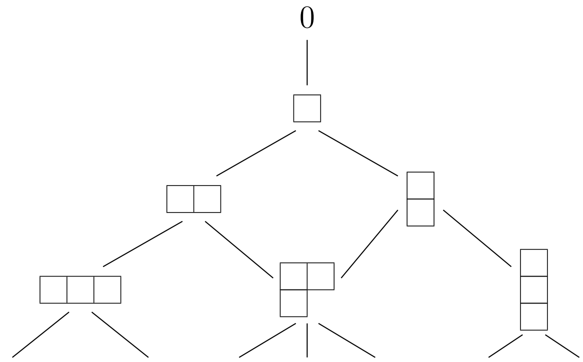
so that cancellation is again guaranteed. There is one case however, when no cancellation occurs, namely when $n=m$. Everything proceeds as before, leading to cancellation, except for pairs of swaps that are each other's inverse, as in

$$\dots 0w1 \xmapsto{\alpha_{-n}} \operatorname{sgn}(w)\dots 1w0 \xmapsto{\alpha_n} \dots 0w1 \qquad \text{(type I)}$$

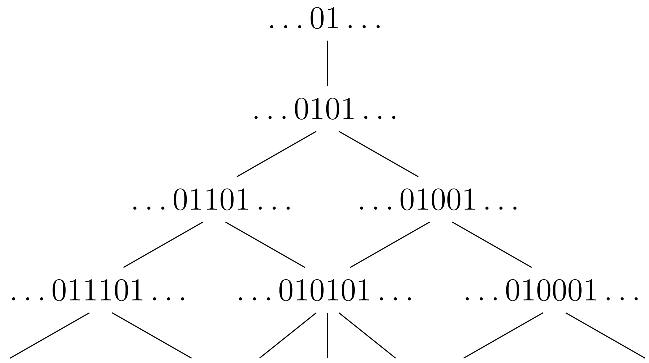
or

$$\dots 1w0 \xmapsto{\alpha_n} \operatorname{sgn}(w)\dots 0w1 \xmapsto{\alpha_{-n}} \dots 1w0 \qquad \text{(type II)}.$$

For any partition λ the number of pairs of swaps of type I exceeds the number of swaps of type II by n. Intuitively the excess of swaps of type I over type II is clear. First moving a 1 to the left is 'easier' as we are moving in the direction of the infinite sea of 0s, whereas moving a 1 to the right is 'harder' as we are moving in the direction of the infinite sea of 1s. It is perhaps surprising that the excess only depends on n and not on λ. This can however be seen as follows. Young's lattice, formed by all integer partitions:



can be reproduced starting from the word $\ldots 01 \ldots$ for 0 and then, to increase the rank, change a single $(0,1)$ pair a distance 1 apart to a $(1,0)$ pair:



Now fix n. It is clear that the word representing 0 has an excess of swaps of type I over type II of exactly n, the former contributing n and the latter 0. One now easily checks that every time one goes one step deeper into Young's lattice this excess remains the same. Indeed increasing the rank by one we only need to consider the change in type I and type II swaps when going from

$$abw01vcd \longmapsto abw10vcd,$$

where w and v are words of length $n-2$ and $a,b,c,d \in \{0,1\}$ are single letters. For example, in the case

$$10w01v00 \longmapsto 10w10v00$$

we see that in the $0/1$ sequence on the left we can carry out a type I swap: $(0,1) \mapsto (1,0) \mapsto (0,1)$ and two type II swaps: $(1,0) \mapsto (0,1) \mapsto (1,0)$ and $(1,0) \mapsto (0,1) \mapsto (1,0)$. In the word on the right we can carry out no type I swaps and one type II swap: $(1,0) \mapsto (0,1) \mapsto (1,0)$. By symmetry it in fact suffices to only analyse the changes in the number of type I and type II moves in

$$abw01 \longmapsto abw10$$

for $a,b \in \{0,1\}$.

(c) More generally, consider

$$\Gamma_\pm(\boldsymbol{z}) := \exp\biggl(\sum_{n\geqslant 1} z_n\,\alpha_{\pm n}\biggr),$$

where $\boldsymbol{z} = (z_1, z_2, \dots)$. Then

$$\Gamma_-(\boldsymbol{z})\Gamma_+(\boldsymbol{w}) = \exp\biggl(\sum_{n\geqslant 1} n z_n w_n\biggr)\Gamma_+(\boldsymbol{w})\Gamma_-(\boldsymbol{z}).$$

To prove this, define $E_\pm(\boldsymbol{z}) := \sum_{n\geqslant 1} z_n\alpha_{\pm n}$ and $F(\boldsymbol{z},\boldsymbol{w}) := \sum_{n\geqslant 1} n z_n w_n$, so that we must show that

$$\mathrm{e}^{E_-(\boldsymbol{z})}\,\mathrm{e}^{E_+(\boldsymbol{w})} = \mathrm{e}^{F(\boldsymbol{z},\boldsymbol{w})}\,\mathrm{e}^{E_+(\boldsymbol{w})}\,\mathrm{e}^{E_-(\boldsymbol{z})}\,. \tag{13}$$

One way to proceed is using the Baker–Campbell–Hausdorff formula but this is overkill, and we can proceed using only elementary means. From $[\alpha_n, \alpha_{-m}] = n\delta_{n,m}$,

$$\begin{aligned}E_+(\boldsymbol{z})E_-(\boldsymbol{w}) &= \sum_{n,m\geqslant 1} z_n w_m \alpha_n\alpha_{-m}\\ &= \sum_{n,m\geqslant 1} z_n w_m\bigl(\alpha_{-m}\alpha_n + n\delta_{n,m}\bigr) = E_-(\boldsymbol{w})E_+(\boldsymbol{z}) + F(\boldsymbol{z},\boldsymbol{w}).\end{aligned}$$

Hence, by induction on k,

$$E_+^k(\boldsymbol{z})E_-(\boldsymbol{w}) = E_-(\boldsymbol{w})E_+^k(\boldsymbol{z}) + kF(\boldsymbol{z},\boldsymbol{w})E_+^{k-1}(\boldsymbol{z})$$

for arbitrary positive integer k. By another round of induction this may be lifted to

$$E_+^k(\boldsymbol{z})E_-^\ell(\boldsymbol{w}) = \sum_{i=0}^{\min\{k,\ell\}} i!\binom{k}{i}\binom{\ell}{i} F^i(\boldsymbol{z},\boldsymbol{w})E_-^{\ell-i}(\boldsymbol{w})E_+^{k-i}(\boldsymbol{z}), \tag{14}$$

for arbitrary positive integers k, ℓ. Since for $k = 0$ or $\ell = 0$ the above identity is trivially true we may extend the range of k and ℓ to all nonnegative integers. Dividing both sides of (14) by $k!\ell!$ and then summing the resulting identity over k and ℓ the claim (13) follows.

(d) We choose n sufficiently large so that $l(\mu) + r \leqslant n$, and consider

$$p_r(x_1,\dots,x_n)s_\mu(x_1,\dots,x_n) = \frac{1}{\Delta(x_1,\dots,x_n)}\sum_{k=1}^n x_k^r \sum_{w\in S_n}\operatorname{sgn}(w)\prod_{i=1}^n x_{w_i}^{\lambda_i+n-i}.$$

Interchanging the order of the two sums and using that

$$\sum_{w\in S_n} f_w \sum_{k=1}^n x_k^r = \sum_{w\in S_n} f_w \sum_{k=1}^n\prod_{i=1}^n x_{w_i}^{r\delta_{k,i}} = \sum_{k=1}^n\sum_{w\in S_n} f_w\prod_{i=1}^n x_{w_i}^{r\delta_{k,i}},$$

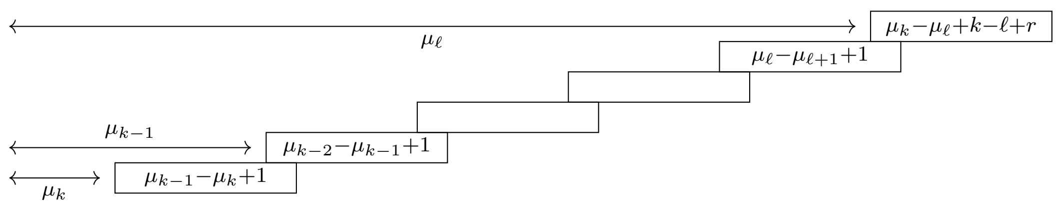
it follows that

$$p_r(x_1,\dots,x_n)s_\mu(x_1,\dots,x_n) = \sum_{k=1}^n s_{\mu+r\varepsilon_k}(x_1,\dots,x_n),$$

where ε_i is the ith unit vector in $\mathbb{Z}^n$. We now want to rectify the Schur functions $s_{\mu+r\varepsilon_k}$ for all k. If the augmented kth column in the determinant is equal to the jth column for some $1 \leqslant j < k$, i.e., if $\mu_k = \mu_j - j + k - r$, the Schur function simply vanishes. For example, if $\mu = (3,1,0,0)$ and $r = 2$ then, for $k = 3$, $\mu_3 = \mu_2 - 2 + 3 - 2$ so that $s_{(3,2,0,0)+2(0,0,1,0)} = 0$. Otherwise, by moving the kth column of the determinant representing $s_{\mu+r\varepsilon_k}$ to the left we can rearrange the determinant so that it corresponds to that of a Schur function s_λ indexed by a partition λ. In this case there exists an ℓ with $1 \leqslant \ell \leqslant k$ such that

$$\lambda = \big(\underbrace{\mu_1 + n - 1, \dots, \mu_{\ell-1} + n - \ell + 1}_{\ell-1 \text{ terms}}, \mu_k + n - k + r,$$
$$\underbrace{\mu_\ell + n - \ell, \dots, \mu_{k-1} + n - k + 1}_{k-\ell \text{ terms}}, \underbrace{\mu_{k+1} + n - k - 1, \dots, \mu_n}_{n-k \text{ terms}}\big).$$

This implies that λ/μ has no boxes in the first $\ell - 1$ rows and last $n - k$ rows, $\mu_k - \mu_\ell - k + \ell + r \geqslant 1$ boxes in the ℓth row and $\mu_{i-1} - \mu_i + 1 \geqslant 1$ boxes in each row for $\ell + 1 \leqslant i \leqslant k$. Moreover, row i and $i + 1$ for $\ell \leqslant i \leqslant k - \ell$ overlap in exactly one column (with column coordinate $j = \mu_i + 1$). Pictorially,



which is a borderstrip of height $k - \ell$ and size r. Since we have moved the kth row exactly $k - \ell$ positions to the left,

$$s_{\mu+r\varepsilon_k} = (-1)^{k-\ell} s_\lambda = (-1)^{\text{height}(\lambda/\mu)} s_\lambda,$$

and thus

$$\sum_{k=1}^{n} s_{\mu+r\varepsilon_k}(x_1, \dots, x_n) = \sum_{\substack{\lambda \vdash |\mu|+r \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\lambda(x_1, \dots, x_n).$$

(e) Comparing (11) with (10) it follows that α_{-n} act on symmetric functions by multiplication by p_n. Hence

$$\Gamma_-(z) = \exp\left(\sum_{n\geqslant 1} \frac{z^n}{n} \alpha_{-n}\right) = \exp\left(\sum_{n\geqslant 1} \frac{z^n}{n} p_n\right) = \exp\big(\psi_z(\cdot)\big) = \sigma_z(\cdot)$$

and thus

$$\Gamma_-(z) s_\mu[X] = \sigma_z[X] s_\mu[X],$$

proving (i). For (ii), since $\Gamma_+(z)$ is the adjoint of $\Gamma_-(z)$,

$$\left\langle \Gamma_+(z)s_\lambda, s_\mu \right\rangle = \left\langle s_\lambda, \Gamma_-(z)s_\mu \right\rangle = \sum_{\nu \succ \mu} z^{|\nu/\mu|} \langle s_\lambda, s_\nu \rangle = \begin{cases} z^{|\lambda/\mu|} & \text{if } \lambda \succ \mu \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with

$$\left\langle s_\lambda[\cdot + z], s_\mu \right\rangle = \sum_{\nu \prec \lambda} z^{|\lambda/\nu|} \langle s_\nu, s_\mu \rangle = \begin{cases} z^{|\lambda/\mu|} & \text{if } \mu \prec \lambda \\ 0 & \text{otherwise,} \end{cases}$$

completes the proof. Finally, (iii) is just an iterated version of (ii), which says that

$$\Gamma_+(z)s_\lambda = \sum_\nu s_{\lambda/\nu}(z)s_\nu$$

Iterating this n times, using that

$$\sum_\mu s_{\lambda/\mu}[X]s_{\mu/\nu}[Y] = s_{\lambda/\nu}[X+Y],$$

yields

$$\Gamma_+(z_1)\dots\Gamma_+(z_n)s_\lambda = \sum_\nu s_{\mu/\nu}(z_1, \dots, z_n)s_\nu.$$

By taking the Hall scalar product with s_μ this gives

$$\left\langle \Gamma_+(z_1)\dots\Gamma_+(z_n)s_\lambda, s_\mu \right\rangle = s_{\lambda/\mu}(z_1, \dots, z_n).$$