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NOTE

A Note on the Trinomial Analogue of Bailey's Lemma

S. Ole Warnaar*

Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

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Recently, Andrews and Berkovich introduced a trinomial version of Bailey's lemma. In this note we show that each ordinary Bailey pair gives rise to a trinomial Bailey pair. This largely widens the applicability of the trinomial Bailey lemma and proves some of the identities proposed by Andrews and Berkovich. © 1998 Academic Press

In a recent paper, Andrews and Berkovich (AB) proposed a trinomial analogue of Bailey's lemma [3]. As a starting point AB took the following definitions of the q-trinomial coefficients:

$$\binom{L;\,B;\,q}{A}_2=\sum_{j=0}^{\infty}\frac{q^{j(j+B)}(q)_L}{(q)_j\,(q)_{j+A}\,(q)_{L-2j-A}} \tag{1}$$

and

$$T_n(L,A,q)=q^{(L(L-n)-A(A-n))/2}\binom{L;\;A-n;\;q^{-1}}{A}_2. \tag{2}$$

Here $(a)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$ and $(a)_n=(a)_\infty/(aq^n)_\infty$, $n\in\mathbb{Z}$. To simplify equations it will also be convenient to introduce the notation

$$Q_n(L,A,q)=T_n(L,A,q)/(q)_L. \tag{3}$$

We note that the q-trinomial coefficients are non-zero for $-L\leqslant A\leqslant L$ only.

A pair of sequences $\tilde{\alpha}=\{\tilde{\alpha}_L\}_{L\geqslant 0}$ and $\tilde{\beta}=\{\tilde{\beta}_L\}_{L\geqslant 0}$ is said to form a trinomial Bailey pair relative to n if

$$\tilde{\beta}_L=\sum_{r=0}^{L}Q_n(L,r,q)\,\tilde{\alpha}_r. \tag{4}$$

The trinomial analogue of the Bailey lemma is stated as follows [3].

* Present address: Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands.

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LEMMA 1. *If* $(\tilde{\alpha},\tilde{\beta})$ *is a trinomial Bailey pair relative to* 0, *then*

$$\sum_{L=0}^{M} (-L)_L\, q^{L/2} \tilde{\beta}_L = (-1)_{M+1} \sum_{L=0}^{M} \frac{\tilde{\alpha}_L}{q^{L/2}+q^{-L/2}}\, Q_1(M, L, q). \qquad (5)$$

Similarly, if $(\tilde{\alpha}, \tilde{\beta})$ *is a trinomial Bailey pair relative to* 1, *then*

$$\sum_{L=0}^{M} (-q^{-1})_L\, q^L \tilde{\beta}_L = (-1)_M \sum_{L=0}^{M} \alpha_L \left\{ Q_1(M, L, q) - \frac{Q_1(M-1, L+1, q)}{1+q^{-L-1}} - \frac{Q_1(M-1, L-1, q)}{1+q^{L-1}} \right\}. \qquad (6)$$

As a corollary of their lemma, AB obtain the identities

$$\frac{1}{2} \sum_{L=0}^{\infty} (-1)_L\, q^{L/2} \tilde{\beta}_L = \frac{(-q)_\infty^2}{(q)_\infty^2} \sum_{L=0}^{\infty} \frac{\tilde{\alpha}_L}{q^{L/2}+q^{-L/2}} \qquad (7)$$

for a trinomial Bailey pair relative to 0, and

$$\frac{1}{2} \sum_{L=0}^{\infty} (-q^{-1})_L\, q^L \tilde{\beta}_L = \frac{(-q)_\infty^2}{(q)_\infty^2} \sum_{L=0}^{\infty} \tilde{\alpha}_L \left\{ \frac{1}{1+q^{L+1}} - \frac{1}{1+q^{L-1}} \right\} \qquad (8)$$

for a trinomial Bailey pair relative to 1.

In Ref. [3], the Eqs. (7) and (8) are used to derive several new q-series identities. As input AB take trinomial Bailey pairs obtained from polynomial identities which on one side involve q-trinomial coefficients. Among these identities is an identity by the author which was stated in Ref. [7] without proof, and therefore AB conclude "We have checked that his conjecture implies" followed by their Eq. (3.21), which is an identity for the characters of the $N=2$ superconformal models $SM(2p, (p-1)/2)$.

We now point out that Eq. (3.21) is a simple consequence of Lemma 2 stated below. First we recall the definition of the ordinary (i.e., binomial) Bailey pair. A pair of sequences (α, β) such that

$$\beta_L = \sum_{r=0}^{L} \frac{\alpha_r}{(q)_{L-r}\,(aq)_{L+r}} \qquad (9)$$

is said to form a Bailey pair relative to a.

LEMMA 2. *Let* (α, β) *form a Bailey pair relative to* $a = q^{\ell}$, *where* ℓ *is a non-negative integer. For* $n = 0, 1$, *the following identity holds*:

$$\sum_{\substack{s=0 \\ s \equiv L+\ell \ (\mathrm{mod}\ 2)}}^{L-\ell} \frac{q^{s(s-n)/2}}{(q)_{\ell}\,(q)_s}\,\beta_{(L-s-\ell)/2} = \sum_{r=0}^{\infty} Q_n(L, 2r+\ell, q)\,\alpha_r. \tag{10}$$

For $\ell > L$ the above of course trivializes to $0 = 0$.

Before proving Lemma 2 we note an immediate consequence.

COROLLARY 1. *Let* (α, β) *form a Bailey pair relative to* $a = q^{\ell}$ *with non-negative integer* ℓ. *Then* $(\tilde{\alpha}, \tilde{\beta})$ *is defined as*

$$\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{\ell-1} = 0, \qquad \tilde{\alpha}_{2L+\ell} = \alpha_L, \qquad \tilde{\alpha}_{2L+\ell+1} = 0, \qquad L \geqslant 0$$

$$\tilde{\beta}_0, \ldots, \tilde{\beta}_{\ell-1} = 0, \qquad \tilde{\beta}_{L+\ell} = \sum_{\substack{s=0 \\ s \equiv L \ (\mathrm{mod}\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_{\ell}\,(q)_s}\,\beta_{(L-s)/2}, \qquad L \geqslant 0 \tag{11}$$

forms a trinomial Bailey pair relative to $n = 0, 1$.

Proof. The proof is trivial once one adopts the representation of the q-trinomial coefficients as given by Eqs. (2.58) and (2.59) of Ref. [2],

$$Q_n(L, A, q) = \frac{T_n(L, A, q)}{(q)_L} = \sum_{\substack{s=0 \\ s \equiv L+A \ (\mathrm{mod}\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_{(L-A-s)/2}\,(q)_{(L+A-s)/2}\,(q)_s},$$

$$n = 0, 1. \tag{12}$$

Now take the defining relation (9) of a Bailey pair with $a = q^{\ell}$ and make the replacement $L \to (L - s - \ell)/2$ where s is an integer $0 \leqslant s \leqslant L - \ell$ such that $s \equiv L + \ell \ (\mathrm{mod}\ 2)$. After multiplication by $q^{s(s-n)/2}/(q)_s$ this becomes

$$\frac{q^{s(s-n)/2}}{(q)_s}\,\beta_{(L-s-\ell)/2} = (q)_{\ell} \sum_{r=0}^{\infty} \frac{\alpha_r\, q^{s(s-n)/2}}{(q)_{(L-s-\ell)/2-r}\,(q)_{(L-s+\ell)/2+r}\,(q)_s}. \tag{13}$$

Summing over s yields Eq. (10). ∎

Returning to AB's paper, we note that their Eq. (3.21) simply follows from Corollary 1 and the "$M(p-1, p)$ Bailey pairs" which arise from the $M(p-1, p)$ polynomial identities proved in Refs. [4, 7]. Of course, an equivalent statement is that the "conjecture of Ref. [7]" is proved using Lemma 2 and the $M(p-1, p)$ Bailey pairs. To make this somewhat more explicit we consider the special case $p = 3$. Then the $M(2, 3)$ Bailey pairs are nothing but the entries A(1) and A(2) of Slater's list [6]. Specifically, A(1) contains the following Bailey pair relative to 1:

$$\alpha_L = \begin{cases} q^{6j^2-j}, & L=3j \geqslant 0 \\ q^{6j^2+j}, & L=3j>0 \\ -q^{6j^2-5j+1}, & L=3j-1>0 \\ -q^{6j^2+5j+1}, & L=3j+1>0 \end{cases} \quad \text{and} \quad \beta_L = \frac{1}{(q)_{2L}}. \tag{14}$$

By application of Corollary 1 this gives the trinomial Bailey pair

$$\tilde{\alpha}_L = \begin{cases} q^{6j^2-j}, & L=6j \geqslant 0 \\ q^{6j^2+j}, & L=6j>0 \\ -q^{6j^2-5j+1}, & L=6j-2>0 \\ -q^{6j^2+5j+1}, & L=6j+2>0 \end{cases}$$

and (15)

$$\tilde{\beta}_L = \sum_{\substack{s=0 \\ s \equiv L \ (\mathrm{mod}\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_s \, (q)_{L-s}}.$$

Likewise, using entry A(2), we get

$$\tilde{\alpha}_L = \begin{cases} q^{6j^2-j}, & L=6j-1>0 \\ q^{6j^2+j}, & L=6j+1>0 \\ -q^{6j^2-5j+1}, & L=6j-3>0 \\ -q^{6j^2+5j+1}, & L=6j+3>0 \end{cases}$$

and (16)

$$\tilde{\beta}_L = \sum_{\substack{s=0 \\ s \not\equiv L \ (\mathrm{mod}\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_s \, (q)_{L-s}}.$$

Setting $n=0$ and summing up both trinomial Bailey pairs, we arrive at the trinomial Bailey pair of Eqs. (3.18) and (3.19) of [3]. (Unlike the case $p \geqslant 4$, this trinomial Bailey pair was actually proven by AB, using Theorem 5.1 of Ref. [1].) Very similar results can be obtained through application of Slater's A(3) and A(4), A(5) and A(6), and A(7) and A(8).

We conclude this note with several remarks. First, it is of course not true that each trinomial Bailey pair is a consequence of an ordinary Bailey pair. The pairs given by Eqs. (3.13) and (3.14) of Ref. [3] are examples of irreducible trinomial Bailey pairs. Second, it is intriguing to observe that if one replaces $Q_n(L, r, q)$ by its q-multinomial analogue [5, 8] and takes that as the definition of a q-multinomial Bailey pair, it is, beyond trinomials, no longer possible to construct multinomial Bailey pairs out of ordinary ones. This signals a special relation between q-binomial and q-trinomial coefficients. Finally, it is worth noting that the Bailey flow from the minimal model $M(p, p+1)$ to the $N=2$ superconformal model $SM(2p, (p-1)/2)$ as concluded by AB could now be replaced by $M(p-1, p) \to N=2$ $SM(2p, (p-1)/2)$. Perhaps it would be better though to write $M(p-1, p) \to M(p, p+1) \to N=2$ $SM(2p, (p-1)/2)$, where the first arrow indicates the flow induced by Corollary 1 and the second arrow the flow induced by (7) and (8).

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