A Note on the Trinomial Analogue of Bailey’s Lemma

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Communicated by the Managing Editors
Received April 4, 1997

Recently, Andrews and Berkovich introduced a trinomial version of Bailey’s lemma. In this note we show that each ordinary Bailey pair gives rise to a trinomial Bailey pair. This largely widens the applicability of the trinomial Bailey lemma and proves some of the identities proposed by Andrews and Berkovich.

In a recent paper, Andrews and Berkovich (AB) proposed a trinomial analogue of Bailey’s lemma [3]. As a starting point AB took the following definitions of the $q$-trinomial coefficients:

\[
\binom{L; B; q}{A} \equiv \sum_{j=0}^{\infty} \frac{q^{j(j+B)}(q)_L}{(q)_j(q)_j(A)(q)_{L-2j-A}}
\]

and

\[
T_n(L, A, q) = q^{nL(L-n-A)(A-n)}(L; A-n; q^{-1})_A
\]

Here $(a)_\infty = \prod_{n=0}^\infty (1-aq^n)$ and $(a)_n = (a)_\infty/(aq^n)_\infty$, $n \in \mathbb{Z}$. To simplify equations it will also be convenient to introduce the notation

\[
Q_n(L, A, q) = T_n(L, A, q)/(q)_L.
\]

We note that the $q$-trinomial coefficients are non-zero for $-L \leq A \leq L$ only.

A pair of sequences $\mathbf{\tilde{s}} = \{\mathbf{\tilde{s}}_L\}_{L \geq 0}$ and $\mathbf{\tilde{p}} = \{\mathbf{\tilde{p}}_L\}_{L \geq 0}$ is said to form a trinomial Bailey pair relative to $n$ if

\[
\mathbf{\tilde{p}}_L = \sum_{r=0}^{L} Q_n(L, r, q) \mathbf{\tilde{s}}_r.
\]

The trinomial analogue of the Bailey lemma is stated as follows [3].

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**Lemma 1.** If \((\tilde{x}, \tilde{y})\) is a trinomial Bailey pair relative to 0, then

\[
\sum_{L=0}^{M} (-L)_L q^{L^2/2} \tilde{p}_L = (-1)_M \sum_{L=0}^{M} \tilde{q}_L q^{L^2/2 + L/2} Q_1(M, L, q). \tag{5}
\]

Similarly, if \((\hat{x}, \hat{y})\) is a trinomial Bailey pair relative to 1, then

\[
\sum_{L=0}^{M} (-q^{-1})_L q^{L^2/2} \hat{p}_L = (-1)_M \sum_{L=0}^{M} \hat{q}_L q^{L^2/2 + L/2} \left\{ Q_1(M, L, q) - \frac{Q_1(M - 1, L + 1, q)}{1 + q^{L+1}} \right\}. \tag{6}
\]

As a corollary of their lemma, AB obtain the identities

\[
\frac{1}{2} \sum_{L=0}^{\infty} (-1)_L q^{L^2/2} \tilde{p}_L = \left( \frac{q}{q+1} \right)^{L/4} \sum_{L=0}^{\infty} \tilde{q}_L q^{L^2/2 + L/2} \tag{7}
\]

for a trinomial Bailey pair relative to 0, and

\[
\frac{1}{2} \sum_{L=0}^{\infty} (-q^{-1})_L q^{L^2/2} \hat{p}_L = \left( \frac{q}{q+1} \right)^{L/4} \sum_{L=0}^{\infty} \hat{q}_L \left\{ \frac{1}{1 + q^{L+1}} - \frac{1}{1 + q^{L-1}} \right\} \tag{8}
\]

for a trinomial Bailey pair relative to 1.

In Ref. [3], the Eqs. (7) and (8) are used to derive several new \(q\)-series identities. As input AB take trinomial Bailey pairs obtained from polynomial identities which on one side involve \(q\)-trinomial coefficients. Among these identities is an identity by the author which was stated in Ref. [7] without proof, and therefore AB conclude “We have checked that his conjecture implies” followed by their Eq. (3.21), which is an identity for the characters of the \(N=2\) superconformal models \(SM(2p, (p-1)/2)\).

We now point out that Eq. (3.21) is a simple consequence of Lemma 2 stated below. First we recall the definition of the ordinary (i.e., binomial) Bailey pair. A pair of sequences \((x, y)\) such that

\[
\beta_x = \sum_{r=0}^{L} \frac{x_r}{(q)_{L-r, (aq)_{L+r}}} \tag{9}
\]

is said to form a Bailey pair relative to \(a\).
Lemma 2. Let $(\alpha, \beta)$ form a Bailey pair relative to $a = q^\ell$, where $\ell$ is a non-negative integer. For $n = 0, 1$, the following identity holds:

$$\sum_{s = L + \ell (\mod 2)}^{L - \ell} \frac{q^{s(n - n)/2}}{(q)_s} \beta_{(L - s - \ell)/2} = \sum_{r = 0}^{\infty} Q_n(L, 2r + \ell, q) \alpha_r. \quad (10)$$

For $\ell > L$ the above of course trivializes to $0 = 0$.

Before proving Lemma 2 we note an immediate consequence.

Corollary 1. Let $(\alpha, \beta)$ form a Bailey pair relative to $a = q^\ell$ with non-negative integer $\ell$. Then $(\tilde{\alpha}, \tilde{\beta})$ is defined as

$$\tilde{\alpha}_0, ..., \tilde{\alpha}_{L - 1} = 0, \quad \tilde{\alpha}_{2L + \ell} = \alpha_L, \quad \tilde{\alpha}_{2L + \ell + 1} = 0, \quad L \geq 0$$

$$\tilde{\beta}_0, ..., \tilde{\beta}_{L - 1} = 0, \quad \tilde{\beta}_{L + \ell} = \frac{q^{s(n - n)/2}}{(q)_r} \beta_{(L - s - \ell)/2}, \quad L \geq 0 \quad (11)$$

forms a trinomial Bailey pair relative to $n = 0, 1$.

Proof. The proof is trivial once one adopts the representation of the $q$-trinomial coefficients as given by Eqs. (2.58) and (2.59) of Ref. [2],

$$Q_n(L, A, q) = \frac{T_n(L, A, q)}{(q)_L} = \sum_{s = 0}^{L} \frac{q^{s(n - n)/2}}{(q)_s} (q)_{(L - A - s)/2} (q)_{(L + A - s)/2} \alpha_s, \quad n = 0, 1. \quad (12)$$

Now take the defining relation (9) of a Bailey pair with $a = q^\ell$ and make the replacement $L \rightarrow (L - s - \ell)/2$ where $s$ is an integer $0 \leq s \leq L - \ell$ such that $s \equiv L + \ell (\mod 2)$. After multiplication by $q^{s(n - n)/2}/(q)_s$, this becomes

$$\frac{q^{s(n - n)/2}}{(q)_s} \beta_{(L - s - \ell)/2} = (q)_r \sum_{s = 0}^{\infty} \alpha_s q^{s(n - n)/2} (q)_{(L - s - \ell)/2} (q)_{(L - s + \ell)/2 + r} (q)_r. \quad (13)$$

Summing over $s$ yields Eq. (10). □

Returning to AB's paper, we note that their Eq. (3.21) simply follows from Corollary 1 and the "$M(p - 1, p)$ Bailey pairs" which arise from the $M(p - 1, p)$ polynomial identities proved in Refs. [4, 7]. Of course, an equivalent statement is that the "conjecture of Ref. [7]" is proved using Lemma 2 and the $M(p - 1, p)$ Bailey pairs. To make this somewhat more explicit we consider the special case $p = 3$. Then the $M(2, 3)$ Bailey pairs
are nothing but the entries $A(1)$ and $A(2)$ of Slater's list [6]. Specifically, $A(1)$ contains the following Bailey pair relative to $L$:

$$
\alpha_L = \begin{cases} 
q^{b_j^{j-1}}, & L = 3j \geq 0 \\
q^{b_j^{j+1}}, & L = 3j > 0 \\
-q^{b_j^{j-1}+1}, & L = 3j - 1 > 0 \\
-q^{b_j^{j+1}+1}, & L = 3j + 1 > 0 
\end{cases}
$$

By application of Corollary 1 this gives the trinomial Bailey pair

$$
\tilde{\alpha}_L = \begin{cases} 
q^{b_j^{j-1}}, & L = 6j \geq 0 \\
q^{b_j^{j+1}}, & L = 6j > 0 \\
-q^{b_j^{j-1}+1}, & L = 6j - 2 > 0 \\
-q^{b_j^{j+1}+1}, & L = 6j + 2 > 0 
\end{cases}
$$

and

$$
\tilde{\beta}_L = \sum_{s=0}^{L} \frac{q^{(s-n)/2}}{(q)_s (q)_{L-s}}.
$$

Likewise, using entry $A(2)$, we get

$$
\tilde{\alpha}_L = \begin{cases} 
q^{b_j^{j-1}}, & L = 6j - 1 > 0 \\
q^{b_j^{j+1}}, & L = 6j + 1 > 0 \\
-q^{b_j^{j-1}+1}, & L = 6j - 3 > 0 \\
-q^{b_j^{j+1}+1}, & L = 6j + 3 > 0 
\end{cases}
$$

and

$$
\tilde{\beta}_L = \sum_{s=0}^{L} \frac{q^{(s-n)/2}}{(q)_s (q)_{L-s}}.
$$

Setting $n = 0$ and summing up both trinomial Bailey pairs, we arrive at the trinomial Bailey pair of Eqs. (3.18) and (3.19) of [3]. (Unlike the case $p \geq 4$, this trinomial Bailey pair was actually proven by AB, using Theorem 5.1 of Ref. [1].) Very similar results can be obtained through application of Slater's $A(3)$ and $A(4)$, $A(5)$ and $A(6)$, and $A(7)$ and $A(8)$.

We conclude this note with several remarks. First, it is of course not true that each trinomial Bailey pair is a consequence of an ordinary Bailey pair. The pairs given by Eqs. (3.13) and (3.14) of Ref. [3] are examples of irreducible trinomial Bailey pairs. Second, it is intriguing to observe that if one replaces $Q_n(L, r, q)$ by its $q$-multinomial analogue [5, 8] and
takes that as the definition of a $q$-multinomial Bailey pair, it is, beyond trinomials, no longer possible to construct multinomial Bailey pairs out of ordinary ones. This signals a special relation between $q$-binomial and $q$-trinomial coefficients. Finally, it is worth noting that the Bailey flow from the minimal model $M(p, p+1)$ to the $N=2$ superconformal model $SM(2p, (p-1)/2)$ as concluded by AB could now be replaced by $M(p-1, p) \to N=2 SM(2p, (p-1)/2)$. Perhaps it would be better though to write $M(p-1, p) \to M(p, p+1) \to N=2 SM(2p, (p-1)/2)$, where the first arrow indicates the flow induced by Corollary 1 and the second arrow the flow induced by (7) and (8).

ACKNOWLEDGMENTS

I thank Alexander Berkovich for helpful comments. This work is supported by the Australian Research Council.

REFERENCES