

# A proof of polynomial identities of type $\widehat{sl}(n)_1 \otimes \widehat{sl}(n)_1 / \widehat{sl}(n)_2$

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We present a proof of polynomial identities related to finite analogs of the branching functions of the coset  $\widehat{sl}(n)_1 \otimes \widehat{sl}(n)_1 / \widehat{sl}(n)_2$ . © 1996 American Institute of Physics. [S0022-2488(96)02102-5]

This paper is dedicated to the memory of Claude Itzykson

## I. INTRODUCTION

Consider the affine algebra  $\widehat{sl}(n)_\ell$ , where  $(n-1)$  is the rank and  $\ell$  is the level.<sup>1-3</sup> Following Refs. 4 and 5, the branching functions of the coset

$$\mathcal{C}_{n,\ell_1,\ell_2} = \widehat{sl}(n)_{\ell_1} \otimes \widehat{sl}(n)_{\ell_2} / \widehat{sl}(n)_{\ell_1+\ell_2} \quad (1)$$

are characters of the highest weight modules (HWMs) of  $W_n$  algebras,<sup>6</sup> where  $W_2$  is the Virasoro algebra.<sup>7</sup> We are interested in computing these branching functions.

### A. $q$ -series identities

An important observation, made independently in Ref. 8 in the context of affine algebras and in Ref. 9 in the context of branching functions, is that different approaches to computing the characters lead to completely different expressions for them. Equating different expressions of the same character leads to generalizations of the Rogers–Ramanujan identities. In the present work, we are interested in the identities related to the branching functions.

#### 1. Boson–fermion identities

Because one side of these identities is generated using operators that obey bosonic commutation relations, while the other is generated using operators that obey fermionlike exclusion principles, these identities are also known as boson–fermion identities.

In Ref. 5, the branching functions of the coset  $\mathcal{C}_{n,\ell_1,\ell_2}$  were obtained by counting certain configurations, known as *weighted paths*. These paths appear naturally in using the corner transfer matrix method to solve statistical mechanical models.<sup>10</sup> The expressions obtained are of the bosonic type. In the present work, we restrict our attention to the coset  $\mathcal{C}_{n,1,1}$ , and obtain expressions for the branching functions by counting the Ferrers graphs that appear in the crystal base description of the HWM's of  $\widehat{sl}(n)_1$ . The expressions obtained are of the fermionic type, and finitize the Lepowsky and Primc character formulas.<sup>8</sup>

## 2. Polynomial identities

In fact, we do not work directly in terms of the characters, which are formal infinite series. Instead, we work in terms of polynomials which depend on a parameter  $L$ , and reduce to the characters in the limit  $L \rightarrow \infty$ . In that sense, the identities we obtain are stronger than identities between characters.

Equating the expressions of Ref. 5, and those obtained in the present work, we obtain polynomial identities, one for each branching function of  $\mathcal{E}_{n,1,1}$ . For fixed  $n$ , there are  $\mathcal{O}(n^2)$  such functions, and corresponding identities. These polynomial identities are generalizations of those considered by Schur in his approach to proving  $q$ -series identities.<sup>11</sup>

### B. Two ways to count

Though the Ferrers graphs that we count are in one-to-one correspondence with the weighted paths, the expressions that we obtain are different from those of Ref. 5 because our approach to counting these objects is inherently different. We wish to outline the usual method of counting, in order to emphasize the contrast to ours.

#### 1. Indirect counting: Sieving

In Ref. 5, the counting was achieved using a *sieving method* to obtain recurrence relations which can be solved. The main idea of the sieving approach can be summarized as follows:<sup>12</sup>

Suppose one wishes to count the number of objects in a certain class  $P_0$  which satisfy certain conditions.<sup>13</sup> This is typically a difficult problem, since the conditions satisfied by  $P_0$  can be quite complicated. However, one can approach it *indirectly* as follows:

As a first step, one considers a larger class of objects  $Q_0$ , that includes  $P_0$ , but satisfies weaker conditions, and hence is easier to evaluate. Suppose one manages to do that. The next step would be to evaluate the difference  $P_1 = Q_0 - P_0$ , and subtract it to obtain  $P_0 = Q_0 - P_1$  (hence the name *sieving*). However, evaluating  $P_1$  *directly* is once again typically just as hard as the initial problem of evaluating  $P_0$ . Hence, it should also be evaluated in two steps: We consider a larger class of objects  $Q_1$  that is easier to evaluate, and subtract that of the difference  $P_2 = Q_1 - P_1$ . We obtain  $P_0 = Q_0 - Q_1 + P_2$ . It is easy to see how the above procedure generalizes to give  $P_0 = Q_0 - Q_1 + \dots + Q_{\text{even}} - Q_{\text{odd}} + \dots$ .

The objects we are interested in—Ferrers graphs and paths—have dimensions. For larger  $i$ ,  $P_i$  typically contains larger objects. If there are no restrictions on the dimensions of the objects being counted, then the above sieving procedure continues indefinitely. If there are such restrictions, then for sufficiently large  $i$ , the procedure terminates. Either way, the procedure amounts to writing a recurrence relation for the set  $\{P_0, P_1, \dots\}$  and solving it.

#### 2. Direct counting: Sectoring

In contrast to the above, the approach used in this paper relies on a direct counting of the objects of interest. The main idea is to divide the set of all objects into sectors, each of which is easier to compute, and then to sum over all sectors. An outline of this approach is given below.

### C. Outline of proof

- (1) Given the set of graphs we wish to count, we propose to distinguish a certain subset to be called *parent graphs*. The remaining graphs are called *non-parents*.
- (2) We propose a set of rules which reduces *any* non-parent graph uniquely to a parent graph by removing nodes from it. Using these rules we can decompose any non-parent graph into a parent graph plus a set of objects called *g-components*. The rules are such that a parent graph cannot be further reduced to another parent graph.

- (3) We show that the above set of rules is invertible. Each non-parent can be uniquely obtained from a parent by attaching  $g$ -components. Consequently, the set of non-parents which reduce to a given parent may be regarded as the *descendants* of that parent.
- (4) From the above, we classify the set of all graphs into sectors. Each sector contains precisely one parent plus its descendants.
- (5) We show that, given a parent graph, the set of all its descendants is generated by a product over Gaussian polynomials.
- (6) Since we know the explicit expression for the Gaussian polynomials in each sector, summing over all sectors, with the proper weighting which follows from the weight of the parent graph, we obtain the desired generating function of the graphs.

**D. Plan of paper**

In Sec. II, we outline a number of technical details related to weighted paths on the set of dominant integral weights of  $\widehat{sl(n)}_2$ , and recall the bosonic generating function as evaluated in Ref. 5. In Sec. III, we introduce the main objects of this paper, K-graphs, and discuss their properties. In Sec. IV, we describe the special set of K-graphs called parents. In Sec. V, we describe the graph components to be added to a parent to generate more general K-graphs, called descendants. In Sec. VI, we describe how the descendants are obtained from their parent, and why each graph is either a parent, or descends from a uniquely defined one. In Sec. VII, we evaluate the number of descendants of a certain parent. In Sec. VIII, we obtain fermionic expressions for the finite analogs of all branching functions of the coset  $\mathcal{E}_{n,1,1}$ . In Sec. IX we summarize our results to obtain the main theorem of this paper: polynomial identities for the finite analogs of the branching functions. This section also contains a discussion of our results.

**II. PATHS**

In this section, we consider weighted paths on the set of level-2 dominant integral weights of  $\widehat{sl(n)}$ , and recall their generating function as computed in Ref. 5.

**A. Roots and weights**

We start with some definitions from the theory of affine algebras.<sup>1</sup> Let  $\Lambda_i, \alpha_i$  ( $i=0, \dots, n-1$ ), and  $\delta$  be the fundamental weights, the simple roots, and the null root of the affine Lie algebra  $\widehat{sl(n)}$ , respectively. The subscript  $i$  of  $\Lambda_i$  can be extended to  $i \in \mathbb{Z}$  by setting  $\Lambda_i = \Lambda_{i'}$  for  $i \equiv i' \pmod{n}$ . Let  $\hat{i} = \Lambda_{i+1} - \Lambda_i$  ( $i=0, \dots, n-1$ ) be the weights of the vector representation of  $sl(n)$ , and  $\rho = \sum_{i=0}^{n-1} \Lambda_i$  be the Weyl vector.

*Remark 1:* For the rest of this work, we will simply use  $a \equiv b$  to indicate  $a \equiv b \pmod{n}$ .

Let  $P = \mathbb{Z}\Lambda_0 \oplus \dots \oplus \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta$  be the weight lattice.<sup>1,2</sup> There is an invariant bilinear form  $(\cdot | \cdot)$  on  $P$  defined by

$$(\Lambda_i | \Lambda_j) = \min(i, j) - \frac{ij}{n}, \quad (\Lambda_i | \delta) = 1, \quad (\delta | \delta) = 0, \tag{2}$$

for  $0 \leq i, j \leq n-1$ .

We are not interested in the full weight lattice, but in certain restrictions of it:

*Definition 1:*  $(P_2^+)$   $P_2^+$  is the set of level-2 dominant integral weights, i.e.,  $P_2^+ = \{\Lambda_i + \Lambda_j | 0 \leq i \leq j \leq n-1\}$ .

Examples of  $P_2^+$  in the case of  $n=2$ , and 3 are shown in Fig. 1.

We can define paths on  $P_2^+$  as follows:

*Definition 2 (paths):* For  $L \in \mathbb{Z}_{\geq 0}$ , we define a path  $p$  as  $p = (\lambda_0, \dots, \lambda_L)$  with all  $\lambda_i \in P_2^+$  and  $\lambda_{i+1} - \lambda_i \in \{\hat{0}, \hat{1}, \dots, \hat{n-1}\}$ .

We are interested in particular sets of paths of length  $L$  defined by the following.

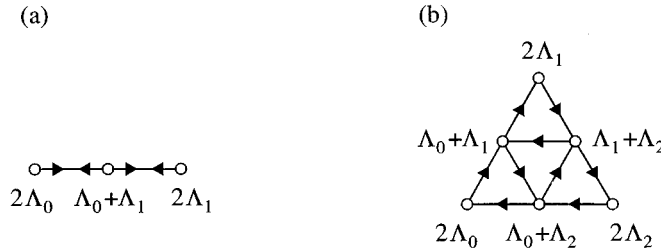


FIG. 1. Examples of the set  $P_2^+$ . A directed bond from  $\lambda$  to  $\lambda'$  ( $\lambda, \lambda' \in P_2^+$ ) indicates that a path can go from  $\lambda$  to  $\lambda'$ . (a)  $n=2$ , (b)  $n=3$ .

*Definition 3:*  $[\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)]$ .

$$\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k) = \{p = (\lambda_0, \dots, \lambda_L) \mid \lambda_0 = \Lambda_i + \Lambda_j, \lambda_L = \Lambda_k + \Lambda_{i+j-k+L}\}. \tag{3}$$

For a path  $p \in \mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  we call  $\Lambda_i + \Lambda_j, \Lambda_k$ , and  $L$  its initial point, boundary, and length, respectively.

We note that  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  is a finite analog (length  $L$ ) of the set of  $(\Lambda_k, \Lambda_{i+j-k})$ -restricted paths of Ref. 14 and 15.

With the paths in  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  we associate a special path  $\bar{p}$  called the ground-state path, as follows:

*Definition 4 (ground-state path  $\bar{p}$ ):*

$$\bar{p} = (\Lambda_k + \Lambda_{i+j-k}, \Lambda_k + \Lambda_{i+j-k+1}, \dots, \Lambda_k + \Lambda_{i+j-k+L}) \in \mathcal{P}_L(\Lambda_k + \Lambda_{i+j-k}, \Lambda_k).$$

Note that the initial point of the ground-state path may be different from that of the paths in  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ .

We can encode a path in terms of a sequence of integers as follows:

*Definition 5 (sequence of integers):* For a path  $p = (\lambda_0, \dots, \lambda_L) \in \mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  we define a sequence of integers  $\iota(p) = (\mu_0, \dots, \mu_L)$ , where  $\hat{\mu}_\ell = \lambda_{\ell+1} - \lambda_\ell$ , and where we have used  $\lambda_{L+1} = \Lambda_k + \Lambda_{L+i+j-k+1}$ . We denote the element  $\mu_\ell$  of  $\iota(p)$  by  $\iota(p)_\ell$ . Note that  $\iota(\bar{p})$  of  $\bar{p}$  in Definition 4 is given by  $\iota(\bar{p})_\ell \equiv i + j - k + \ell$ .

*Example 1:* The ground state path  $\bar{p}$  associated to  $\mathcal{P}_6(\Lambda_i + \Lambda_{-i}, \Lambda_0)$  for  $n=3$ :

$$\bar{p} = (2\Lambda_0, \Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_2, 2\Lambda_0, \Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_2, 2\Lambda_0),$$

$$\iota(\bar{p}) = (0, 1, 2, 0, 1, 2, 0).$$

*Example 2:* A path in  $p^{(1)} \in \mathcal{P}_6(2\Lambda_0, \Lambda_1)$  for  $n=3$ :

$$p^{(1)} = (2\Lambda_0, \Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_2, \Lambda_1 + \Lambda_2, \Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_2, \Lambda_1 + \Lambda_2),$$

$$\iota(p^{(1)}) = (0, 1, 0, 2, 1, 0, 2).$$

*Example 3:* A path in  $p^{(2)} \in \mathcal{P}_6(2\Lambda_0, \Lambda_0)$  for  $n=4$ :

$$p^{(2)} = (2\Lambda_0, \Lambda_0 + \Lambda_1, 2\Lambda_1, \Lambda_1 + \Lambda_2, 2\Lambda_2, \Lambda_2 + \Lambda_3, \Lambda_0 + \Lambda_2),$$

$$\iota(p^{(2)}) = (0, 0, 1, 1, 2, 3, 2).$$

**B. Weighted paths**

Let  $p$  be a path and  $\bar{p}$  be the ground-state path associated to  $p$ , with integer sequences  $\iota(p) = (\mu_0, \dots, \mu_L)$  and  $\iota(\bar{p}) = (\bar{\mu}_0, \dots, \bar{\mu}_L)$ , respectively. We define an energy function  $E$  by the following.

*Definition 6 (energy of a path):*

$$E(p) = \sum_{\ell=1}^L \ell (\theta(\mu_{\ell-1} - \mu_{\ell}) - \theta(\bar{\mu}_{\ell-1} - \bar{\mu}_{\ell})), \tag{4}$$

with  $\theta$  the step function given by

$$\theta(\mu) = \begin{cases} 0 & (\mu < 0) \\ 1 & (\mu \geq 0). \end{cases} \tag{5}$$

**1. Connection with cosets of affine algebras**

Consider the coset  $\mathcal{C}_{n,1,1}$ . The branching functions corresponding to this coset can be defined as follows. Let  $V(\Lambda)$  be an  $s\widehat{\mathfrak{sl}}(n)$  HWM with highest weight  $\Lambda$ , and let  $|\Lambda\rangle$  be its highest weight vector. Consider the tensor product decomposition

$$V(\Lambda_k) \otimes V(\Lambda_{i+j-k}) = \sum_{\Lambda \in P_2^+} \Omega_{\Lambda_k, \Lambda_{i+j-k}, \Lambda} \otimes V(\Lambda). \tag{6}$$

Among all vectors in the tensor product on the left-hand side,  $\Omega_{\Lambda_k, \Lambda_{i+j-k}, \Lambda}$  is the space of highest weight vectors whose weights are equal to  $\Lambda \pmod{\mathbb{Z}\delta}$ . The connection between  $\Omega_{\Lambda_k, \Lambda_{i+j-k}, \Lambda}$ , and  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  is as follows: It has been shown in Ref. 14 that in the limit of  $L \rightarrow \infty$ , there is a bijection between the set of base vectors in  $\Omega_{\Lambda_k, \Lambda_{i+j-k}, \Lambda_i + \Lambda_j}$ , and the set of paths in  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ . This implies that the paths of  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  are characterized by weights. Under this bijection the ground-state path associated to  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  is identified with  $|\Lambda_k\rangle \otimes |\Lambda_{i+j-k}\rangle \in \Omega_{\Lambda_k, \Lambda_{i+j-k}, \Lambda_k + \Lambda_{i+j-k}}$ .

It turns out that the weight of a path can be expressed in terms of its energy function as

*Definition 7:* [weight of a path  $p \in \mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ ]

$$wt(p) = \Lambda_i + \Lambda_j - E(p)\delta. \tag{7}$$

**2. Finite analogs of branching functions**

Given the above considerations, we define finite analogs of the branching functions  $B_L$  for the coset  $\mathcal{C}_{n,1,1}$  as the generating function of the weighted paths in  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ :

$$B_L(\Lambda_i + \Lambda_j, \Lambda_k) = \sum_{p \in \mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)} q^{E(p)}. \tag{8}$$

**C. Bosonic expressions**

We are interested in expressions for the generating function  $B_L(\Lambda_i + \Lambda_j, \Lambda_k)$ . In Ref. 5, the following *bosonic* expression for  $B_L(\Lambda_i + \Lambda_j, \Lambda_k)$  was obtained using recurrence relations based on the sieving method explained in Sec. I:

**Theorem 1:** Let  $\lambda = \sum_{i=0}^{n-1} \lambda_i \hat{2}_i + \mathbb{Z}\delta \in P$ , with all  $\lambda_i \geq 0$  and  $\sum_{i=0}^{n-1} \lambda_i = N$ . For such  $\lambda$  set

$$\left[ \begin{matrix} N \\ \lambda \end{matrix} \right]_q = \frac{(q)_N}{(q)_{\lambda_0} \cdots (q)_{\lambda_{n-1}}}, \tag{9}$$

with  $(q)_m = \prod_{k=1}^m (1 - q^k)$  ( $m \geq 1$ ) and  $(q)_0 = 1$ . Also, let  $\mathscr{W}$  denote the Weyl group of  $\widehat{sl}(n)$  (see, e.g., Ref. 5, p. 91). Then

$$B_L(\Lambda_i + \Lambda_j, \Lambda_k) = q^{-|\Lambda_{i+j-k}|^2/2} \sum_{w \in \mathscr{W}} (\det w) b_{L,i+j-k}(\Lambda_k + \Lambda_{i+j-k+L} + \rho - w(\Lambda_i + \Lambda_j + \rho)), \tag{10}$$

where

$$b_{L,i}(\lambda) = q^{|\lambda - \Lambda_{i+L}|^2/2} \left[ \begin{matrix} L \\ \lambda \end{matrix} \right]_q. \tag{11}$$

For proof we refer the reader to Ref. 5.

### III. K-GRAPHS

Using matrices as intermediate structures, we give an alternative representation of the weighted paths on  $P_2^+$  in terms of Ferrers graphs (or, equivalently, Young diagrams) which satisfy certain restrictions. We refer to these Ferrers graphs, which were introduced and extensively studied by the Kyoto school (see Refs. 14 and 16 and references therein), as K-graphs.

#### A. Interpolating matrices

In this subsection, we associate a matrix  $M(p)$  with two rows to each path  $p \in \mathscr{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ .

*Definition 8 (domain wall):* Let  $\iota(p) = (\mu_0, \dots, \mu_L)$  be the integer sequence of  $p \in \mathscr{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ . If  $\mu_\ell - \mu_{\ell-1} \equiv h_\ell + 1$  ( $0 < h_\ell < n$ ), we say that there is a domain wall in the sequence  $\iota(p)$  of height  $h_\ell$  at position  $\ell$ .

Given a path  $p$  with  $N$  domain walls of heights  $h_1, \dots, h_N$  at the positions  $x_1, \dots, x_N$ , respectively, we define the interpolating matrix  $M(p)$  as follows:

*Definition 9 (interpolating matrix):*

$$M(p) = \begin{pmatrix} x_1 & (x_2 - x_1) & \cdots & (x_N - x_{N-1}) \\ h_1 & h_2 & \cdots & h_N \end{pmatrix}. \tag{12}$$

*Example 4:* The interpolating matrix of  $p^{(2)}$  in Example 3 is

$$M(p^{(2)}) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

#### B. K-graph representation of a path

Let  $p$  be a path and  $M(p)$  its interpolating matrix of the form

$$M(p) = \begin{pmatrix} w_1 & w_2 & \cdots & w_N \\ h_1 & h_2 & \cdots & h_N \end{pmatrix}. \tag{13}$$

Consider a two-dimensional square lattice with an  $(x, y)$ -coordinate system. Set  $W = w_1 + \dots + w_N$ ,  $H = h_1 + \dots + h_N$ . Starting from  $(0, -H)$ , we draw a polygon by moving  $w_1$  steps to the right, then  $h_1$  steps up, then  $w_2$  steps to the right, etc., until we reach the point  $(W, 0)$ . Connecting  $(0, -H)$  and  $(W, 0)$  with the origin by straight line-segments, the resulting graph is the Ferrers graph or Young diagram corresponding to the original path [see Fig. 2(a)].

*Definition 10 (K-graph):* A Ferrers graph obtained from a path  $p$  on  $P_2^+$ , as described above, is called a K-graph.

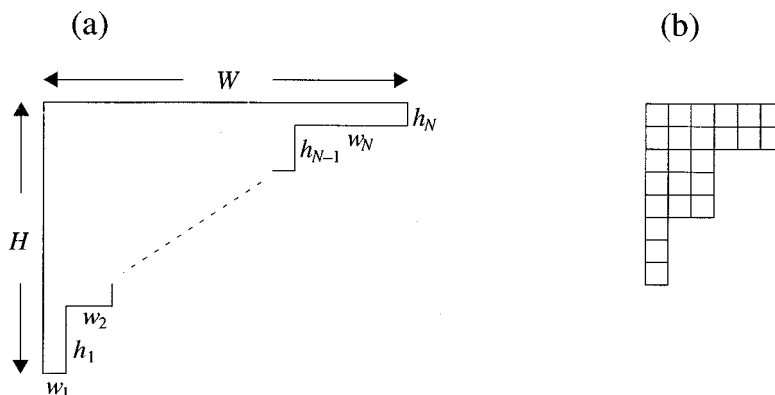


FIG. 2. (a) The general form of a K-graph. (b) A K-graph in  $\mathcal{S}_6(2\Lambda_0, \Lambda_0)$  for  $n=4$ .

*Definition 11:*  $(\mathcal{S}_L(\Lambda_i + \Lambda_j, \Lambda_k))$   $\mathcal{S}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  is defined as the set of K-graphs corresponding to the set of path  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$ .

*Definition 12 (profile of a graph):* The set of horizontal and vertical line segments used to construct a K-graph form the profile of a graph.

*Example 5:* The K-graph corresponding to the interpolating matrix of Example 4 is shown in Fig. 2(b).

*Definition 13 (concave corner):* A corner of the form  $\lrcorner$ .

*Definition 14 (convex corner):* A corner of the form  $\llcorner$ .

*Definition 15 (plain of width  $w$ ):* A horizontal line segment of  $w$  nodes (or boxes) marked by a concave corner to its left and convex corner to its right.

*Definition 16 (cliff of height  $h$ ):* A vertical line segment of  $h$  nodes (or boxes) marked by a convex corner at its bottom and a concave corner at its top.

Notice that a cliff on a K-graph corresponds to a domain wall in the corresponding integer sequence.

*Remark 2:* From now on, we concentrate on K-graphs in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , unless otherwise stated.

**1. From a graph to its sequence of integers**

For a graph  $G \in \mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , we can recover the corresponding integer sequence  $u(p)$  as follows. Let

$$M = \begin{pmatrix} w_1 & w_2 & \cdots & w_N \\ h_1 & h_2 & \cdots & h_N \end{pmatrix} \tag{14}$$

be the interpolating matrix corresponding to  $G$ . Set  $H = h_1 + \cdots + h_N$ , and take the integer sequence  $(0, 1, 2, \dots, n-1, 0, 1, \dots, n-1, 0, 1, \dots)$  of length  $H+L+1$ . Starting from the left moving to the right, we now keep the first  $w_1$  integers, then remove the next  $h_1$  integers, then keep the next  $w_2$  integers, remove the next  $h_2$  integers, etc. The remaining sequence of exactly  $L+1$  integers corresponds to  $u(p)$ .

**2. From a graph to a path**

To go from a graph  $G \in \mathcal{S}_L(2\Lambda_0, \Lambda_k)$  to its corresponding path  $p = (\lambda_0, \dots, \lambda_L)$  on  $P_2^+$  we simply first construct the sequence of integers  $u(p) = (\mu_0, \dots, \mu_L)$  as described above. We then compute  $\lambda_{\ell+1} = \lambda_{\ell} + \hat{\mu}_{\ell}$  using  $\lambda_0 = 2\Lambda_0$ .

**3. Conditions on  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$**

Among all K-graphs, those in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$  are characterized by the following conditions:

- K1:**  $W \leq L$ , with  $W$  the number of nodes in the first row.
- K2:**  $H + k \equiv 0$ , with  $H$  the number of nodes in the first column.
- K3:**  $h_{i-1} + w_i + h_i \equiv 0$  and  $0 < h_i < n$  for  $1 \leq i \leq N$ , with  $h_0 = n$ .

**K1** is obvious. **K2** is obtained by considering the  $L$ th component of  $u(p)$  and the boundary condition. To obtain **K3**, suppose the  $i$ th cliff occurs at the  $r$ th position. We can assume  $\lambda_r = \lambda_a + \Lambda_b, \lambda_{r-w_i} = \lambda_a + \Lambda_{b-w_i}$  for some  $a, b$ . Now we have  $u(p)_{r-1} \equiv b - 1, u(p)_{r-w_i} \equiv b - w_i$ . Since there are cliffs at the  $r$ th and  $(r - w_i)$ -th position, we should have  $u(p)_r \equiv a$  and  $u(p)_{r-w_i} - 1 \equiv a - 1$ . Thus we get  $h_i \equiv a - b, h_{i-1} \equiv b - w_i - a$ , which gives **K3**.

**C. Fermionic expressions**

We now wish to calculate the following sum.

$$F_L(\Lambda_i + \Lambda_j, \Lambda_k) = \sum_{G \in \mathcal{S}_L(\Lambda_i + \Lambda_j, \Lambda_k)} q^{|G|/n}, \tag{15}$$

where  $|G|$  denotes the number of nodes in  $G$ . Regarding the above, we have the following theorem.<sup>14,16</sup>

**Theorem 2:** Let  $p$  be a path in  $\mathcal{P}_L(\Lambda_i + \Lambda_j, \Lambda_k)$  and  $G(p)$  be the corresponding K-graph. The number of nodes of  $G(p)$  is given by

$$|G| = \sum_{\ell=0}^{n-1} m_\ell, \tag{16}$$

where  $m_\ell$  is determined from

$$(\Lambda_k + \Lambda_{i+j-k}) - wt(p) = \sum_{\ell=0}^{n-1} m_\ell \alpha_\ell. \tag{17}$$

Using

$$(\Lambda_i | \alpha_j) = \delta_{ij} \quad (i, j = 0, \dots, n-1),$$

we obtain

$$m_l = (\Lambda_l | (\Lambda_k + \Lambda_{i+j-k}) - wt(p)) \quad (l = 0, \dots, n-1).$$

Since we define the sum (15) in the ‘‘principal picture,’’ i.e., each node has equal weight  $1/n$ , it is invariant under the Dynkin diagram automorphisms. Thus we can reduce the calculation of (15) to that of  $F_L(\Lambda_0 + \Lambda_j, \Lambda_k)$ . From now on, we hence assume  $i=0$ .

Setting  $(\Lambda_k + \Lambda_{j-k}) - (\Lambda_0 + \Lambda_j) = \sum_{\ell=0}^{n-1} \bar{m}_\ell \alpha_\ell$ , we have  $\sum_{\ell=0}^{n-1} \bar{m}_\ell = k(j - k)$  for  $j \geq k, = (k - j)(n - k)$  for  $j < k$ . Calculating  $|\Lambda_k|^2 + |\Lambda_{j-k}|^2 - |\Lambda_j|^2$  and comparing (15) with the bosonic expression, we obtain

$$F_L(\Lambda_0 + \Lambda_j, \Lambda_k) = q^{(|\Lambda_k|^2 + |\Lambda_{j-k}|^2 - |\Lambda_j|^2)/2} B_L(\Lambda_0 + \Lambda_j, \Lambda_k). \tag{18}$$

In the remainder of this paper we will compute a fermionic type of expression for  $F_L$ . Given (18) and the bosonic expression (10) for  $B_L$ , this gives rise to polynomial identities for the finite analogs of the branching functions of the coset  $\mathcal{E}_{n,1,1}$ .



#### IV. PARENTS

From the set of all K-graphs in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$  we select a subset of graphs to be called parent graphs, or simply parents. Let  $\vec{m}^t = (m_1, \dots, m_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$ , such that

$$k + \sum_{i=1}^{n-1} im_i \equiv 0, \tag{19}$$

and let  $M$  be the interpolating matrix of a graph  $G \in \mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , with entries

$$M(p) = \begin{pmatrix} w_1 & w_2 & \cdots & w_N \\ h_1 & h_2 & \cdots & h_N \end{pmatrix}. \tag{20}$$

*Definition 17 (parent associated to  $\vec{m}$ ):*  $G$  is the parent associated to  $\vec{m}$  if

$$\begin{aligned} h_1, \dots, h_{m_{n-1}} &= n-1 \\ h_{m_{n-1}+1}, \dots, h_{m_{n-1}+m_{n-2}} &= n-2 \\ &\vdots \\ h_{N-m_1+1}, \dots, h_N &= 1, \end{aligned} \tag{21}$$

with  $N = \sum_{i=1}^{n-1} m_i$ , and

$$h_{i-1} + w_i + h_i = 2n, \quad 1 \leq i \leq N, \tag{22}$$

where we recall that  $h_0 = n$ .

*Example 6:* The K-graph of  $p^{(2)}$  shown in Fig. 2(b) is the parent associated to  $\vec{m}^t = (0, 1, 2)$ .

#### A. The number of nodes of a parent graph

Let  $C$  be the Cartan matrix of  $sl(n)$ , i.e.,  $C_{ij} = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}$  ( $i, j = 1, \dots, n-1$ ). The inverse of  $C$  is then given by the following formula:

$$(C^{-1})_{ij} = \begin{cases} \frac{i(n-j)}{n} & (i \leq j) \\ \frac{j(n-i)}{n} & (i > j). \end{cases}, \tag{23}$$

With this definition we have the following lemma:

*Lemma 1:* The number of nodes of the parent associated to  $\vec{m}$  is given by  $n\vec{m}^t C^{-1} \vec{m}$ .

Though the proof of this statement is rather elementary, we need to take some care as some of the entries of  $\vec{m}$  can actually be zero. In the following we use the notation  $\langle i \rangle$  to denote  $\sum_{j=1}^i m_{n-j}$ . Clearly,  $\langle i \rangle - \langle i-1 \rangle = m_{n-i}$ . We now compute the number of nodes of a parent  $N(\vec{m})$  as follows:

$$N(\vec{m}) = \sum_{i=1}^N \sum_{j=1}^i h_i w_j = \sum_{k=1}^{n-1} \sum_{i=1+\langle k-1 \rangle}^{\langle k \rangle} \left( \sum_{\ell=1}^{k-1} \sum_{j=1+\langle \ell-1 \rangle}^{\langle \ell \rangle} + \sum_{j=1+\langle k-1 \rangle}^i \right) h_i w_j. \tag{24}$$

Now use (22) and  $h_{1+\langle i-1 \rangle} = \dots = h_{\langle i \rangle} = n-i$  to get

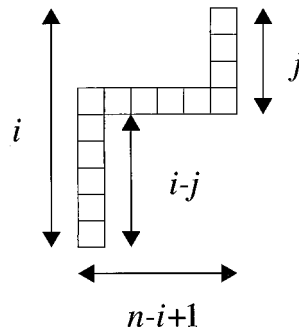


FIG. 3. An  $(i, j)$ -component.

$$\begin{aligned}
 N(\vec{m}) = & \sum_{k=1}^{n-1} \sum_{i=1+\langle k-1 \rangle}^{\langle k \rangle} (n-k) \left( \sum_{\ell=1}^{k-1} \sum_{j=1+\langle \ell-1 \rangle}^{\langle \ell \rangle} \{2\ell - (h_{\langle \ell-1 \rangle} - (n-\ell)) \delta_{j, \langle \ell-1 \rangle + 1}\} \right. \\
 & \left. + \sum_{j=1+\langle k-1 \rangle}^i \{2k - (h_{\langle k-1 \rangle} - (n-k)) \delta_{j, \langle k-1 \rangle + 1}\} \right). \tag{25}
 \end{aligned}$$

Finally, after some changes of variables, we obtain

$$\begin{aligned}
 N(\vec{m}) = & 2 \sum_{k=1}^{n-1} \sum_{\ell=1}^{k-1} \ell (n-k) m_k m_\ell + \sum_{k=1}^{n-1} (n-k) \sum_{i=1}^{m_{n-k}} \left( 2ik - \sum_{\ell=1}^k \sum_{j=1}^{m_{n-\ell}} (h_{\langle \ell-1 \rangle} - (n-\ell)) \delta_{j,1} \right) \\
 = & 2 \sum_{k=1}^{n-1} \sum_{\ell=1}^{k-1} \ell (n-k) m_k m_\ell + \sum_{k=1}^{n-1} \sum_{i=1}^{m_{n-k}} k(n-k)(2i-1). \tag{26}
 \end{aligned}$$

Summing over  $j$  and  $i$  this results in  $n\vec{m}'C^{-1}\vec{m}$ .

### V. $g$ -COMPONENTS

Now that we have distinguished a subset of all K-graphs as parents, we wish to describe the minimal connected configuration of nodes that can be removed or added to a K-graph in  $\mathcal{G}_L(2\Lambda_0, \Lambda_k)$  to obtain another K-graph in  $\mathcal{G}_L(2\Lambda_0, \Lambda_k)$ . Since, as we will see in Sec. VII, these configurations are generated by Gaussian polynomials, we call them  $g$ -components. Eventually, we will show that those graphs which are related by addition and removal of  $g$ -components belong to the same sector, and we will use this observation to relate any non-parent graph to a parent graph.

*Definition 18:* [ $(i, j)$ -component] For all  $i = 1, \dots, n-1$ , and all  $j = 1, \dots, i$ , we define an  $(i, j)$ -component as a connected configuration of  $n$  nodes, as shown in Fig. 3.

Some important characteristics of an  $(i, j)$ -component are the following.

- G1 It consists of  $n$  nodes.
- G2 It has total height  $i$ .
- G3 It has total width  $n-i+1$ .
- G4 It has (at most) two cliffs, one (the lower) of height  $i-j$  and one (the upper) of height  $j$ .

We further note that for an  $(i, i)$ -component the lower cliff vanishes, resulting in a configuration with a single cliff.

*Definition 19 ( $i$ -component):* An  $(i, j)$ -component for arbitrary  $j$  is called an  $i$ -component.

*Definition 20 ( $g$ -component):* An  $(i, j)$ -component for arbitrary  $i$  and  $j$  is called a  $g$ -component.

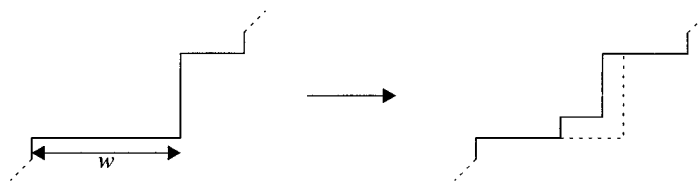


FIG. 4. Removing an  $(i,i)$ -component is only allowed when  $w > 2(n-i)$ . The extra dotted lines in the resulting graph are to indicate the nodes which are removed.

We are now interested in the addition/removal of  $g$ -components to/from a  $K$ -graph. Clearly, in adding or removing a  $g$ -component to or from a  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , we demand that the resulting graph is again a graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ . However, on top of this we impose one additional condition, which basically defines our sectors.

**A. Removing an  $i$ -component**

The removal of an  $(i,j)$ -component from a  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$  is allowed provided the following two conditions are satisfied:

- R1 The resulting graph is again a  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ .
- R2 If  $j=i$  as in Fig. 4, we demand that  $w > 2(n-i)$ .

*Definition 21 (i-candidate):* An  $i$ -component one is allowed to remove from a  $K$ -graph is called an  $i$ -candidate.

Since for any  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$  we have  $h_{j-1} + w_j + h_j = 0$ , three kinds of candidates can occur.

- (1)  $b_{j-1} + w_j + h_j = n$  and  $w_{j-1} > 1$ . In this case we can remove an  $(h_{j-1} + h_j, h_j)$ -component.
- (2)  $h_{j-1} + w_j + h_j = 2n$  and  $w_j > 2(n - h_j)$ . In this case we can remove an  $(h_j, h_j)$ -component.
- (3)  $h_{j-1} + w_j + h_j \geq 3n$ . In this case we can remove an  $(h_j, h_j)$ -component.

Scanning the profile of a non-parent graph, several  $i$ -candidates may occur.

*Definition 22 (leading i-candidate):* The leading  $i$ -candidate is the down- and left-most  $i$ -candidate. (see Fig. 4.)

**B. Attaching an  $i$ -component to a graph**

Attaching an  $i$ -component to a  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$  is allowed provided the following conditions are satisfied:

- A1 The resulting graph is again a  $K$ -graph in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ .
- A2 We do not generate an  $i'$ -candidate, with  $i' > i$ .

*Definition 23 (i-vacancy):* An  $i$ -vacancy is a position on the profile such that one is allowed to attach an  $i$ -component.

An important statement about  $i$ -vacancies is the following. Given a sequence {cliff,plain,cliff} of dimensions  $h_{j-1}, w_j, h_j$  such that  $h_{j-1} + w_j + h_j = n$  and  $h_{j-1} + h_j = i$ , then the following holds: *Lemma 2:* If  $w_{j+1} = n - i$ , the above sequence is not an  $i$ -vacancy.

To prove this, assume the above sequence is an  $i$ -vacancy. Hence we can attach an  $(i, h_j + 1)$ -component as shown in Fig. 5(a). Note that in doing so the height of the  $j$ th cliff increases by 1 to  $h'_j = h_j + 1$ , and the width of the  $(j+1)$ -th plain decreases by 1 to  $w'_{j+1} = w_{j+1} - 1 = n - i - 1$ . Thus we compute

$$0 < h'_j + w'_{j+1} + h_{j+1} = h_j + n - i + h_{j+1} \leq i + (n - i) + (n - 1) = 2n - 1. \tag{27}$$

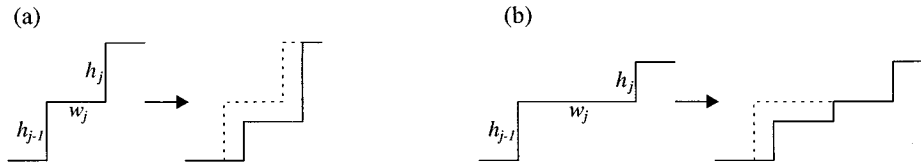


FIG. 5. (a) Attaching an  $(h_{j-1} + h_j, h_j + 1)$ -component with  $h_j \neq 1$ . (b) Attaching an  $(h_{j-1}, 1)$ -component. The extra dotted lines in the resulting graphs are to indicate the profile before attaching the g-component.

Since  $h'_j + w'_{j+1} + h_{j+1} \equiv n$ , we conclude that  $h'_j + w'_{j+1} + h_{j+1} = n$ , and  $h'_j + h_{j+1} = i + 1$ . However, these are the characteristics of an  $(i + 1)$ -candidate. By the second condition for attaching  $i$ -components this contradicts our assumption that the initial sequence was an  $i$ -vacancy.

With the above lemma we note that two kinds of vacancies may occur. The first occurs if we have a sequence {cliff, plain, cliff} of dimensions  $h_{j-1}, w_j, h_j$  such that  $h_{j-1} + w_j + h_j = n$  and  $w_{j+1} > n - h_j - h_{j+1}$ . In this case we can always attach an  $(h_{j-1} + h_j, h_j + 1)$ -component, as shown in Fig. 5(a). The second occurs if we have a sequence {cliff, plain, cliff} of dimensions  $h_{j-1}, w_j, h_j$  such that  $h_{j-1} + w_j + h_j \geq 2n$ . In this case we can always attach an  $(h_{j-1}, 1)$ -component [see Fig. 5(b)].

### VI. DESCENDANTS

In previous sections, we classified all admissible K-graphs into parents and non-parents. We need to show that each non-parent is a descendant of a unique parent. More precisely, we show the following.

- (1) Given a non-parent graph, there is a reduction procedure, such that one can reduce it to a unique parent graph.
- (2) The reduction procedure is reversible: given a parent graph, there is a composition procedure to recover the original non-parent graph.

Because the reduction procedure is reversible, any non-parent graph is a descendant of a unique parent graph. Thus the set of all admissible K-graphs can be divided into nonoverlapping sectors. Each sector contains and is labelled by a parent graph. Any admissible K-graph belongs to one and only one sector.

#### A. Reducing non-parent graphs

Given a non-parent K-graph, we can reduce it to a parent graph as follows.

Red0 Set  $i = n - 1$ .

Red1 Search for the leading  $i$ -candidate and, if it exists, remove it.

Red2 Repeat the above step until no more  $i$ -candidates are found.

Red3 Set  $i \rightarrow i - 1$  and, if  $i \geq 1$ , repeat Red1–Red3.

To prove that a reduced graph is indeed a parent, we proceed as follows: Consider a profile with a sequence {cliff, plain, cliff} of dimensions  $h_{j-1}, w_j, h_j$ , respectively. Suppose that the part of the profile below the above sequence belongs to a parent, i.e.,  $h_{k-1} + w_k + h_k = 2n$  and  $h_{k-1} \geq h_k$  for  $k = 1, \dots, j - 1$ . We wish to show that if the above sequence does not represent a candidate, it belongs to a parent. From Sec. V A we see that unless  $h_{j-1} + w_j + h_j = 2n$  and  $w_j \leq 2(n - h_j)$  or  $h_{j-1} + w_j + h_j = n$  and  $w_{j-1} = 1$ , we always have a candidate.

In the first case we get  $2n = h_{j-1} + w_j + h_j \leq h_{j-1} - h_j + 2n$  and thus  $h_{j-1} \geq h_j$ . This is precisely the right sequence for a parent and we get  $h_{k-1} + w_k + h_k = 2n$  and  $h_{k-1} \geq h_k$  for  $k = 1, \dots, j$ . The second case can, in fact, never occur. Since  $h_{j-2} + w_{j-1} + h_{j-1} = 2n$  and  $h_{j-2}, h_{j-1} < n$  we find that  $w_{j-1} > 1$ .

**B. Generating descendants from parents**

Given the parent associated to  $\vec{m}$ , each cliff of height  $i$  plus the plain immediately to the right of this cliff forms an  $i$ -vacancy. Hence we have  $m_i$   $i$ -vacancies. To obtain an arbitrary descendant of the parent under consideration, we proceed as follows.

Gen0 Set  $i = 1$ .

Gen1 Set  $j = 1$ .

Gen2 Attach  $k_j^{(i)}$   $i$ -components to the  $j$ th  $i$ -vacancy counted from the right.

Gen3 Set  $j \rightarrow j + 1$ . If  $j \leq m_i$ , go to Gen2. If  $j = m_i + 1$ , set  $i \rightarrow i + 1$ , and if  $i \leq n - 1$ , go to Gen1.

To properly interpret these rules, some important remarks need to be made. First, when we say “attach  $k_j^{(i)}$   $i$ -components to the  $j$ th  $i$ -vacancy,” this should be understood as follows. Attaching an  $i$ -component to an  $i$ -vacancy has the effect of moving the vacancy to the right. Hence attaching the  $k$ th  $i$ -component means attaching an  $i$ -component to the image of the  $i$ -vacancy after attaching the  $(k - 1)$ -th  $i$ -component. Second, it may occur that attaching an  $i$ -component to an  $i$ -vacancy does not have the effect of moving the  $i$ -vacancy to the right, but annihilates the vacancy. Hence, there are bounds on the numbers  $k_j^{(i)}$ . In the next section we will show that these bounds are as follows:

$$0 \leq k_{m_i}^{(i)} \leq \dots \leq k_2^{(i)} \leq k_1^{(i)} \leq \ell_i, \tag{28}$$

with  $\ell_i$  fixed by (31).

**C. Reversibility**

There remains the proof that our rules for attaching and removing  $g$ -components are reversible. This is true by construction.

**VII. PROOF OF GAUSSIANS**

In this section we prove that for the case of  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , the generating function for attaching the  $i$ -components to the parent graph associated to  $\vec{m}$  is given by the Gaussian polynomial

$$\begin{bmatrix} \ell_i + m_i \\ m_i \end{bmatrix}_q, \tag{29}$$

where

$$\begin{bmatrix} N \\ m \end{bmatrix}_q = \begin{cases} \frac{(q)_N}{(q)_m (q)_{N-m}}, & 0 \leq m \leq N, \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

and

$$\vec{\ell} = C^{-1}(L\vec{e}_{n-1} + \vec{e}_r - 2\vec{m}), \tag{31}$$

and  $\vec{e}_i$  the  $(n - 1)$ -dimensional unit vector with entries  $(\vec{e}_i)_j = \delta_{i,j}$  and with  $0 < r \leq n$  fixed by

$$L - 2k \equiv r. \tag{32}$$

**A. The  $r = n$  case**

To prove the above result we first treat the simpler case of  $r = n$ . In the next subsection we then show how to modify this to obtain (31) for general  $r$ .

We start with the following important fact, used extensively throughout this section:

*Lemma 3:* For  $G \in \mathcal{S}_L(2\Lambda_0, \Lambda_k)$ , let  $W$  be the number of nodes in the first row and  $h_N$  be the height of the  $N$ th (uppermost) cliff [see Fig. 2(a)]. Then

$$W - h_N \equiv 2k. \tag{33}$$

We prove this by implementing the conditions K2 and K3 of Sec. III B 3, defining the K-graphs in  $\mathcal{S}_L(2\Lambda_0, \Lambda_k)$ . Recalling that  $W$  is the number of nodes in the first row of a K-graph, we have

$$W = \sum_{i=1}^N w_i \equiv - \sum_{i=1}^N (h_{i-1} + h_i) \equiv h_N - 2 \sum_{i=1}^N h_i = h_N - 2H \equiv h_N + 2k, \tag{34}$$

which proves our claim.

### 1. *i*-strips

We are interested in the placement of the *i*-components. From the rules for placing the latter, it is natural to define  $m_i$  *i*-strips as follows:

S1 We define the *i*-strip as consisting of two regions: a principal region and a tail. The principal region is defined in terms of a top segment, a bottom segment, and a left and a right segment. The left and right segments will be called left and right terminal. We start by defining the principal region.

Consider the profile  $P_i$  of the K-graph after attaching all  $i'$ -components, with  $i' = 1, 2, \dots, i-1$ , but before attaching any component of height  $i$  or higher.

If the  $N$ th (highest) cliff of  $P_i$  has height  $h_N$ , extend the *ceiling* of  $P_i$  by drawing a horizontal line of width  $i - h_N$  starting from the top-right corner of the rightmost node of the top row of the graph, and extending to the right. This line will serve as a ceiling to the *i*-components that we will add shortly.

Consider the segment of  $P_i$   $2(n-i)$  columns to the right of the right-most cliff of height  $i$ . This will be the top segment of the *i*-strip. Let us denote this segment by  $P_i^0$ .

Now we proceed to define the bottom segment of the *i*-strip. Move  $P_i^0$  to the left by  $2(n-i)$  columns, and downward by  $i$  rows. Denote this shifted profile by  $P_i^1$ . This is the bottom segment that we are looking for.

Finally, close the figure formed by the top and bottom segments as follows: draw a plain of width  $2(n-i)$  followed by a cliff of height  $i$  to the right (left) of  $P_i^0/P_i^1$ , called the right (left) terminal, respectively. As a result, we now have a region enclosed by  $P_i^0, P_i^1$  and the left and right terminals. This defines the *principal region* of the first *i*-strip.

Next, we define the tail of the first *i*-strip as follows: Compute  $M \equiv 0$  to be the largest integer such that  $W + i - h_N + M \leq L$ . The tail of the *i*-strip is a rectangle, of width  $M$  and height  $i$ , that we place to the right of the principal region in the first row.

The principal region plus the tail define the complete first *i*-strip. An example of the first 3-strip in a typical K-graph for  $sl(\widehat{4})$  is shown in Fig. 6(a).

S2 We draw the second *i*-strip by simply shifting the first *i*-strip to the left and down by  $2(n-i)$  columns and  $i$  rows, respectively.

S3 We repeat the step S2 ( $m_i - 1$ ) times. That is, we define the  $(j+1)$ -th *i*-strip by translating the  $j$ th *i*-strip to the left by  $2(n-i)$  columns and downwards by  $i$  rows. In Fig. 6(b) we have shown the construction of the 3-strips for a typical example of a K-graph for  $sl(\widehat{4})$ .

By construction, adding the *i*-components corresponding to the  $j$ th *i*-vacancy (counted from the right) corresponds to filling the  $j$ th *i*-strip from left to right. In constructing an arbitrary descendant, we will not necessarily fill the complete  $j$ th *i*-strip. Furthermore, we will show below that the filling of the  $(j+1)$ -th strip is bound by the degree of filling of the  $j$ th strip. In particular, we will show that if the  $j$ th strip is filled with  $k_j^{(i)}$  *i*-components, then the  $(j+1)$ -th strip cannot be

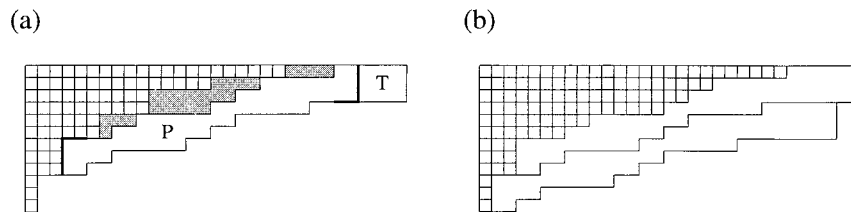


FIG. 6. (a) The construction of the first “3”-strip for a typical K-graph  $G$ . The part of  $G$  corresponding to the parent graph is drawn with open nodes/boxes and the (already) placed  $i'$ -components ( $i' = 1, 2$ ) are shown in grey. The bold segments are the left–right terminals of the principal region (marked P). The tail of the strip is marked with T. (b) The two “3”-strips for the K-graph in (a).

filled with more than  $k_j^{(i)}$   $i$ -components. Since each  $i$ -component contains  $n$  nodes, and thus contributes a single factor  $q$ , we obtain the following expression for the generating function attaching the  $i$ -components:

$$G_i(q) = \sum_{k_1^{(i)}=0}^{\ell_i} \sum_{k_2^{(i)}=0}^{k_1^{(i)}} \cdots \sum_{k_{m_i}^{(i)}=0}^{k_{m_i-1}^{(i)}} q^{k_1^{(i)}+k_2^{(i)}+\cdots+k_{m_i}^{(i)}} \tag{35}$$

Here the number  $n\ell_i$  is the area (=number of nodes) of the first  $i$ -strip.

As defined above,  $G_i$  can be interpreted as the generating function of all partitions with largest part  $\leq \ell_i$  and number of parts  $\leq m_i$ . Therefore

$$G_i(q) = \left[ \begin{matrix} \ell_i + m_i \\ m_i \end{matrix} \right]_q \tag{36}$$

Before ending this subsection, let us return to Lemma 3. We have stated above that attaching  $i$ -components corresponding to the right-most  $i$ -vacancy corresponds to filling the first  $i$ -strip. However, some caution needs to be taken, since in constructing the principal region of the first  $i$ -strip we have extended the profile of the K-graph by drawing a plain of width  $i - h_N$  in the first row to the right of the  $N$ th cliff. This clearly can only be done for all  $i = 1, \dots, n - 1$ , if  $L - W \geq n - 1 - h_N$ . If  $L_s$  is the smallest possible value of  $L$  for which a K-graph of width  $W$  is possible, i.e.,

$$L_s = W + x, \quad 0 \leq x \leq n - 1, \tag{37}$$

with  $x$  fixed by (32), we have

$$x = L_s - W = (L_s - 2k) - (W - 2k) \equiv r - h_N, \tag{38}$$

where we have used Lemma 3 and the definition (32) of  $r$ . Since we require  $x$  to be at least  $n - 1 - h_N$  we should thus have that  $r = n$  or  $n - 1$ . For simplicity we now assume  $r = n$ .

### 2. Calculation of $\ell_i$

To calculate the area of the first  $i$ -strip, we use the simple property that the area remains unchanged by deforming the strip by removing nodes from below and adding them from above.

We now choose to deform the  $i$ -strip such that its upper side corresponds to the profile of its parent graph, being labelled by  $\vec{m}$ . For the example of Fig. 6 this is shown in Fig. 7.

From this particular choice of deformation we can simply compute the area over  $n$  as

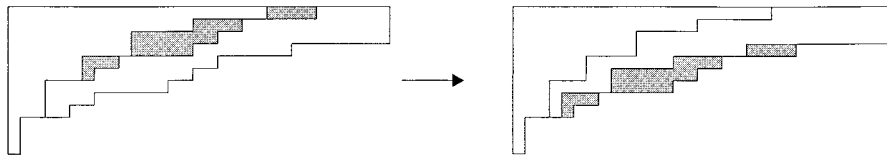


FIG. 7. The deformation of the first “3”-strip of the K-graph of Fig. 6, yielding the first “3”-strip of its parent graph.

$$\ell_i(r=n) = 2 \sum_{j=1}^{i-1} jm_{i-j} + i \left\lfloor \left( L + n - i - 2 \sum_{j=1}^{n-1} jm_{n-j} \right) / n \right\rfloor, \tag{39}$$

with  $\lfloor x \rfloor$  denoting the integer part of  $x$ . Here the first term corresponds to the area of the principal region of the *deformed*  $i$ -strip and the second term to the area of the tail of the deformed strip. In particular, to compute the former we use the fact that it takes  $(j+1)$   $i$ -components to move an  $i$ -vacancy [of the type shown in Fig. 5(b)] upwards across a plain of width  $2n - 2i + j$  ( $j \geq 1$ ). To compute the latter, we compute  $L - (W + i - h_N)$ , using the result (33).

Recasting the definition (23) of the inverse Cartan matrix as

$$(C^{-1})_{i,j} = \frac{i(n-j)}{n} - \sum_{p=1}^{i-1} (i-p) \delta_{j,p} \tag{40}$$

and using the mod properties (19) and (32) we thus obtain

$$\ell_i(r=n) = \frac{iL}{n} - 2 \left( \sum_{j=1}^{n-1} \frac{i(n-j)}{n} - \sum_{j=1}^{i-1} (i-j) \right) m_j = L(C^{-1})_{n-1,i} - 2(C^{-1}\vec{m})_i. \tag{41}$$

This proves (31) for  $r=n$ .

### 3. Proof of Gaussian form

It remains for us to prove that the filling of the  $(j+1)$ -th  $i$ -strip is bound by that of the  $j$ th  $i$ -strip.

Let us assume that we have placed  $k_j^{(i)}$   $i$ -components in the  $j$ th  $i$ -strip, and that the  $k_j^{(i)}$ -th component is an  $i$ -component of the form depicted in Fig. 8(a). Let us further assume that we have already filled the  $(j+1)$ -th strip with  $k_j^{(i)}$   $i$ -components (this is of course always possible). Since

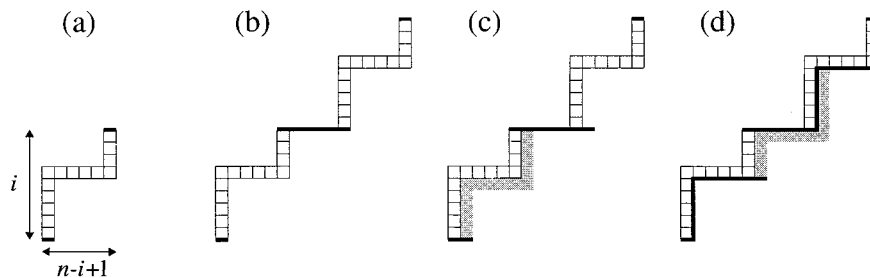


FIG. 8. (a) A typical  $i$ -component. The bold lines indicate part of the boundary of the  $i$ -strip. (b) The  $k_j^{(i)}$ -th  $i$ -component in both the  $j$ th and the  $(j+1)$ -th  $i$ -strip. (c) and (d) The two forbidden placements of an additional  $(k_j^{(i)}+1)$ -th  $i$ -component (shown in grey) in the  $(j+1)$ -th strip.



the  $(j+1)$ -th strip has identical shape as the  $j$ th strip, but is translated to the left and down by  $2(n-i)$  columns and  $i$  rows, we have the configuration shown in Fig. 8(b).

Our claim is now that upon attaching the  $k_j^{(i)}$ -th  $i$ -component in the  $(j+1)$ -th strip we have annihilated the corresponding  $i$ -vacancy. To see this we consider two cases. Either the boundary separating the strips extends at least one more entry to the right, see Fig. 8(c), or the boundary progresses upwards as in Fig. 8(d). In the first case the lowest sequence {cliff,plain,cliff} could be an  $i$ -vacancy, but since the plain immediately above has width  $n-i$ , this is not the case thanks to Lemma 2. Hence placing the  $i$ -component in Fig. 8(c) as shown in grey is not allowed. In the second case, the middle sequence {cliff,plain,cliff} could be an  $i$ -vacancy, but again the plain immediately above has width  $n-i$  and we can once more apply Lemma 2. Hence also the placement as shown in Fig. 8(d) is forbidden.

**B. The general  $r$  case**

As remarked at the end of Sec. VII A 1, only for  $r=n$  and  $n-1$  can we always draw a plain of width  $i-h_N$  to the right of the  $N$ th cliff without violating the condition  $W+i-h_N \leq L$ , for any  $i$ . If  $L-2k \equiv r$ , we can still do so for all  $i \leq r$ . Hence for these cases the principal region of the  $i$ -strips can still be defined as in Sec. VII A 1. However, for  $i=r+a$  ( $a>0$ ), we have to reduce the principal region  $P$  by removing the part of  $P$  which would be occupied by the last  $a$  components to be attached, if  $P$  were to be completely filled from left to right. Of, course, in this case the tail no longer is a rectangle, but has a profile of two plains and two cliffs. An example of this reduction is shown in Fig. 9.

The above considerations lead to the following simple modification of (39):

$$\begin{aligned} \mathcal{L}_i &= 2 \sum_{j=1}^{i-1} j m_{i-j} - \sum_{p=1}^{n-r-1} (n-r-p) \delta_{n-i,p} \\ &+ i \left[ \left( L+n-i + \sum_{p=1}^{n-r-1} (n-r-p) \delta_{n-i,p} - 2 \sum_{j=1}^{n-1} j m_{n-j} \right) / n \right] \\ &= 2 \sum_{j=1}^{i-1} j m_{i-j} - \sum_{p=1}^{n-r-1} (n-r-p) \delta_{n-i,p} + \frac{i}{n} \left( L+n-r - 2 \sum_{j=1}^{n-1} j m_{n-j} \right) \\ &= \mathcal{L}_i(r=n) + \left( \frac{i(n-r)}{n} - \sum_{p=1}^{n-r-1} (n-r-p) \delta_{n-i,p} \right) = \mathcal{L}_i(r=n) + (C^{-1})_{r,i}, \end{aligned} \tag{42}$$

which proves the claim (31). Here we note that to obtain the first line of (42) one not only has to subtract the term  $\sum_p (n-r-p) \delta_{n-i,p}$  to account for the reduction of the principal region, but also to add this same term within the  $[\cdot]$ . This occurs since the effective length available for the tail of the  $i$ -strips has of course increased by the decrease of the principal region (see Fig. 9).

**C. Fermionic form for  $F_L(2\Lambda_0, \Lambda_k)$**

We now have computed the number of nodes of the parent associated to  $\vec{m}$  as well as the generating function for adding the  $g$ -components to this parent. Collecting these two results, we obtain the following expression for the generating function  $F_L(2\Lambda_0, \Lambda_k)$  of K-graphs in  $\mathcal{F}(2\Lambda_0, \Lambda_k)$ :

*Proposition 1:*

$$F_L(2\Lambda_0, \Lambda_k) = \sum q^{\vec{m}^t C^{-1} \vec{m}} \prod_{i=1}^{n-1} \begin{bmatrix} \mathcal{L}_i + m_i \\ m_i \end{bmatrix}, \tag{43}$$

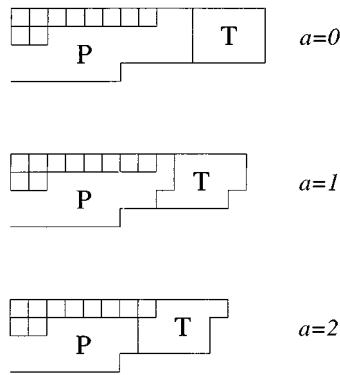


FIG. 9. Reduction of the 3-strips of Fig. 6. Recalling that  $i=r+a$ , we need for  $i=3$  and  $n=4$  to consider the cases  $a=0,1,2$ .

with  $\vec{\lambda}$  given by (31) and (32) and with the sum taken over all  $\vec{m} \in (\mathbb{Z}_{\geq 0})^{(n-1)}$  satisfying  $k + \sum_{i=1}^{n-1} im_i \equiv 0$ .

**VIII. THE GENERAL CHARACTER**

In this section we calculate  $F_L(\Lambda_0 + \Lambda_j, \Lambda_k)$  for arbitrary  $j(0 \leq j \leq n-1)$ . As we have already mentioned in Sec. II, we count the weights in the principal picture, so that any fermionic form can be reduced to one of the above form.

First consider the following injection:

$$\begin{aligned} \mathcal{P}_L(\Lambda_0 + \Lambda_j, \Lambda_k) &\rightarrow \mathcal{P}_{L+j}(2\Lambda_0, \Lambda_k), \\ p = (\lambda_0, \dots, \lambda_L) &\mapsto p', \end{aligned} \tag{44}$$

where  $p' = (2\Lambda_0, \Lambda_0 + \Lambda_1, \dots, \Lambda_0 + \Lambda_{j-1}, \lambda_0, \dots, \lambda_L)$ . In terms of K-graphs, we have

$$\begin{aligned} \mathcal{S}_L(\Lambda_0 + \Lambda_j, \Lambda_k) &\rightarrow \mathcal{S}_{L+j}(2\Lambda_0, \Lambda_k), \\ G &\mapsto G'. \end{aligned} \tag{45}$$

Here  $G'$  is obtained from  $G$  by placing the rectangle of width  $j$  and height  $H'$  in the left-hand side of  $G$  (see Fig. 10), where,  $H$  being the height of  $G$ ,  $H'$  is determined by  $H' - H = 0$  or  $n - j$ ,  $H' \in n\mathbb{Z} - k$ . It is clear that the image under the injection (45) is the set of K-graphs in  $\mathcal{S}_{L+j}(2\Lambda_0, \Lambda_k)$  having the lowest plain of width at least  $j$ .

Now let us recall that we have established the following bijection in the preceding sections:

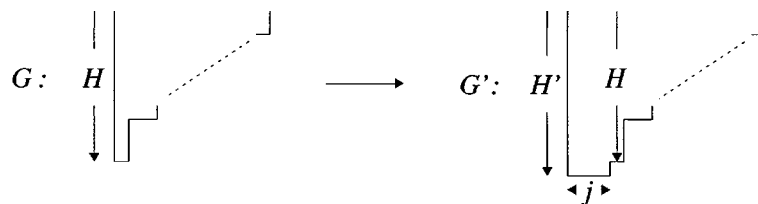


FIG. 10. Embedding of a K-graph in  $\mathcal{S}_L(\Lambda_0 + \Lambda_j, \Lambda_k)$  into  $\mathcal{S}_{L+j}(2\Lambda_0, \Lambda_k)$ .

$$\mathcal{P}_L(2\Lambda_0, \Lambda_k) \leftrightarrow \{(\vec{m}, (F_1, \dots, F_{n-1}))\}. \tag{46}$$

Here  $\vec{m}^t = (m_1, \dots, m_{n-1})$ , satisfying (19), characterizes the parent graph, and  $F_i$  is the Ferrers graph of a partition with largest part  $\leq \ell_i$  and number of parts  $\leq m_i$ . Regarding the image of the injection (45), the following question arises: How can we characterize K-graphs having the lowest plain of width at least  $j$  as elements in the right-hand side of the bijection (46)? The answer is given by the following.

*Proposition 2:* A K-graph in  $\mathcal{G}_L(2\Lambda_0, \Lambda_k)$  has a lowest plain of width  $w_1 \geq j$ , iff, for  $n-j+1 \leq i \leq n-1$ , the smallest part in the Ferrers graph  $F_i$  has at least  $i+j-n$  nodes.

To prove this, let  $G$  be such a K-graph,  $P$  its parent graph, and  $M(G)$  and  $M(P)$  the corresponding interpolating matrices. We write  $M(G)$  as in (14) and  $M(P)$  as

$$M(P) = \begin{pmatrix} c_0 + c_1 & \cdots & c_{i-1} + c_i & \cdots & c_{N-1} + c_N \\ n - c_1 & \cdots & n - c_i & \cdots & n - c_N \end{pmatrix} \quad (c_0 = 0).$$

After removing g-components such that the first  $i$  columns of  $M(G)$  equal  $M(P)$ , we must have that  $w_{i+1} \geq j + c_i$ .

We wish to prove the above assertion by induction on  $i$ . For  $i=0$ , the assertion is clear from the assumption of the proposition. Next, let us assume the assertion for  $i-1$ . Let us also assume that we have arrived at the minimal gap corresponding to  $c_{i-1} + c_i$ . In order to prove the assertion for  $i$ , we have to show that we can remove an  $(n - c_i)$ -component  $j - c_{i-1}$  times strictly horizontally. This can indeed be shown through straightforward, though tedious, consideration of the profile, and the conditions on its various segments. Proposition 2 follows from the above statement.

Applying Proposition 2, we immediately obtain

$$F_L(\Lambda_0 + \Lambda_j, \Lambda_k) = \sum q^{\vec{m}^t C^{-1} \vec{m} + Q} \prod_{i=1}^{n-1} \begin{bmatrix} \ell'_i + m_i \\ m_i \end{bmatrix}_q, \tag{47}$$

where

$$Q = \sum_{i=n-j+1}^{n-1} (i+j-n)m_i - \frac{j}{n} \sum_{i=1}^{n-1} im_i = -\vec{m}^t C^{-1} \vec{e}_{n-j},$$

$$\ell'_i = \ell_i |_{L \rightarrow L+j} - (i+j-n)\theta(i+j-n),$$

$$= [C^{-1}(L\vec{e}_{n-1} + \vec{e}_r + \vec{e}_{n-j} - 2\vec{m})]_i. \tag{48}$$

Here  $\ell_i$  is defined in (31), and  $\theta$  in (5).

In conclusion, we have the general form of the fermionic sum:

**Theorem 3:**

$$F_L(\Lambda_0 + \Lambda_j, \Lambda_k) = \sum q^{\vec{m}^t C^{-1} \vec{m} - \vec{m}^t C^{-1} \vec{e}_{n-j}} \prod_{i=1}^{n-1} \begin{bmatrix} \ell_i + m_i \\ m_i \end{bmatrix}_q,$$

$$\vec{\ell} = C^{-1}(L\vec{e}_{n-1} + \vec{e}_r + \vec{e}_{n-j} - 2\vec{m}), \tag{49}$$

where the sum is taken over all  $\vec{m} \in (\mathbb{Z}_{\geq 0})^{n-1}$  satisfying  $k + \sum_{i=1}^{n-1} im_i \equiv 0$ , and with  $r$  determined from  $L + j - 2k \equiv r, 0 < r \leq n$ .

**IX. SUMMARY AND DISCUSSION**

In this paper we have presented a method to compute finite analogs of the branching functions of the coset

$$\widehat{sl(n)}_1 \otimes \widehat{sl(n)}_1 / \widehat{sl(n)}_2. \tag{50}$$

Our approach, based on a direct counting of Ferrers graphs related to the crystal base formulation of the HWMs of  $\widehat{sl(n)}$ , leads to what are known as *fermionic polynomials*. This complements earlier results of Ref. 5 where the same finite analogs of branching functions were computed, and the result was expressed in terms of *bosonic polynomials*.

Equating these two results, as formulated in the Theorems 1 and 3, using Eq. (18), we obtain the main result of this paper:

**Theorem 4:** Let  $\vec{m}$  and  $\vec{e}_i$  be  $(n-1)$ -dimensional vectors with entries  $(\vec{m})_i = m_i$  and  $(\vec{e}_i)_j = \delta_{i,j}$ , respectively. Also, let  $C$  be the Cartan matrix of  $sl(n)$  and  $\mathscr{W}$  the Weyl group of  $\widehat{sl(n)}$ . Defining the function  $b_{L,i}$  as in (11), the following polynomial identity holds for all  $j, k = 0, \dots, n-1$ :

$$\begin{aligned} & \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^{n-1}} q^{\vec{m}^t C^{-1} \vec{m} - \vec{m}^t C^{-1} \vec{e}_{n-j}} \prod_{i=1}^{n-1} \left[ \begin{matrix} (\vec{m} + C^{-1}(L\vec{e}_{n-1} + \vec{e}_r + \vec{e}_{n-j} - 2\vec{m}))_i \\ m_i \end{matrix} \right]_q \\ &= q^{(|\Lambda_k|^2 - |\Lambda_j|^2)/2} \sum_{w \in \mathscr{W}} (\det w) b_{L,j-k}(\Lambda_k + \Lambda_{j-k+L} + \rho - w(\Lambda_0 + \Lambda_j + \rho)), \end{aligned} \tag{51}$$

with  $0 < r \leq n$  fixed by  $L + j - 2k \equiv r$  and with the sum over  $\vec{m}$  restricted by

$$\frac{k}{n} + (C^{-1}\vec{m})_{n-1} \in \mathbb{Z}. \tag{52}$$

Letting  $L \rightarrow \infty$  we obtain the following  $q$ -series identities for the branching functions of the coset (50).

*Corollary 1:* Let  $Q$  be the root lattice and  $\mathscr{W}'$  be the Weyl group of  $sl(n)$ . Then

$$\begin{aligned} & q^{(|\Lambda_j|^2 - |\Lambda_k|^2)/2} \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^{n-1}} \frac{q^{\vec{m}^t C^{-1} \vec{m} - \vec{m}^t C^{-1} \vec{e}_{n-j}}}{\prod_{i=1}^{n-1} (q)_{m_i}} \\ &= \frac{q^{|\Lambda_j + \rho|^2/2(n+2) - |\Lambda_k + \rho|^2/2(n+1)}}{(q)_{\infty}^{n-1}} \sum_{w \in \mathscr{W}'} (\det w) \Theta_{(n+2)(\bar{\Lambda}_k + \bar{\rho}) - (n+1)w(\bar{\Lambda}_j + \bar{\rho}), (n+1)(n+2)}(q), \end{aligned} \tag{53}$$

with the sum over  $\vec{m}$  again restricted by (52), and with  $\Theta_{\lambda, \mathscr{W}'}$  defined by

$$\Theta_{\lambda, \mathscr{W}'}(q) = \sum_{\alpha \in Q} q^{|\alpha - \lambda|/2}, \tag{54}$$

for  $\lambda \in \sum_{i=1}^{n-1} \mathbb{Z} \bar{\Lambda}_i$ .

We note that the left-hand side of (53) coincides with the character expressions of Lepowsky and Primc<sup>8</sup> for the  $\mathbb{Z}_n$ -parafermion conformal field theory.

The polynomial identities (51) proven in this work are, strictly speaking, not new, since under level-rank duality they map onto identities related to the coset

$$\widehat{sl(2)}_{n-1} \otimes \widehat{sl(2)}_1 / \widehat{sl(2)}_n. \quad (55)$$

The latter were conjectured in Refs. 9 and 17, and proven in Refs. 18–20. However, the proof presented here is *intrinsically* of  $\widehat{sl(n)}$  type, and we expect it admits generalization to the more general coset

$$\widehat{sl(n)}_\ell \otimes \widehat{sl(n)}_m / \widehat{sl(n)}_{\ell+m}. \quad (56)$$

Results related to general  $\widehat{sl(2)}$ -type cosets were discussed in Ref. 21. The fermionic character form for certain sectors of the higher-rank parafermions were proven in Ref. 22.

For the case of  $\widehat{sl(2)}$ , the paths considered in this paper admit yet another representation in terms of Ferrers graphs. These graphs, obeying entirely different conditions than our K-graphs, were introduced in Ref. 23. They are also more general, in the sense that they lead to character expressions for all  $\widehat{sl(2)}$  cosets of type  $\mathcal{E}_{2,\ell,1}$ , including rational values of  $\ell$ . Results for this type of coset have been discussed in Refs. 24 and 25.

#### NOTES ADDED:

After this work was completed, it was brought to our attention that the main concepts introduced in this work are analogous to, though intriguingly different from, concepts that are essential to the theory of modular representations of the symmetric group. (For the precise definition of some of the following terminology, see for example, the book by James and Kerber.<sup>26</sup>) In particular, our K-graphs of the coset  $\mathcal{E}_{n,1,1}$  are known as  $n$ -regular Young diagrams, our *parent graphs* are analogous to  $n$ -cores, our Gaussian polynomials generate the analogs of  $n$ -quotients, our  $g$ -components are analogous to *hook-ribbons*, and our counting procedure is very much related to the evaluation of Kostka–Green–Foulkes polynomials. However, there are differences, due to the fact that our K-graphs obey additional conditions.

Now, to make things even more intriguing, we also learned that the conditions obeyed by our K-graphs are almost identical to, though stronger than, those obeyed by Young diagrams that parametrize irreducible representations of  $sl(n)$  which remain irreducible under restriction to  $sl(n-1)$ . We hope to report on these interesting relationships in future publications.

In a recent preprint A. Schilling has given a recursive proof of fermionic forms for the cosets referred to in Eq. (56).<sup>27</sup>

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