THE ANDREWS-GORDON IDENTITIES AND q-MULTINOMIAL COEFFICIENTS

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Abstract. We prove polynomial boson-fermion identities for the generating function of the number of partitions of n of the form $n = \sum_{j=1}^{L-1} jf_j$, with $f_1 \leq i-1$, $f_{L-1} \leq i'-1$ and $f_j + f_{j+1} \leq k$. The bosonic side of the identities involves q-deformations of the coefficients of x^a in the expansion of $(1+x+\dots+x^k)^L$. A combinatorial interpretation for these q-multinomial coefficients is given using Durfee dissection partitions. The fermionic side of the polynomial identities arises as the partition function of a one-dimensional lattice-gas of fermionic particles.

In the limit $L \to \infty$, our identities reproduce the analytic form of Gordon's generalization of the Rogers–Ramanujan identities, as found by Andrews. Using the $q \to 1/q$ duality, identities are obtained for branching functions corresponding to cosets of type $(\mathrm{A}_1^{(1)})_k \times (\mathrm{A}_1^{(1)})_\ell / (\mathrm{A}_1^{(1)})_{k+\ell}$ of fractional level ℓ.

1. Introduction

The Rogers–Ramanujan identities can be stated as the following q-series identities.

Theorem 1 (Rogers–Ramanujan). *For $a = 0, 1$ and $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{j=0}^{\infty} (1-q^{5j+1+a})^{-1}(1-q^{5j+4-a})^{-1}. \tag{1.1}$$

Since their independent discovery by Rogers [37–39], Ramanujan [35] and also Schur [43], many beautiful generalizations have been found, mostly arising from partition-theoretic or Lie-algebraic considerations, see [6, 33] and references therein.

Most surprising, in 1981 Baxter rediscovered the Rogers–Ramanujan identities (1.1) in his calculation of the order parameters of the hard-hexagon model [13], a lattice gas of hard-core particles of interest in statistical mechanics. It took however another ten years to fully realize the power of the (solvable) lattice model approach to finding q-series identities. In particular, based on a numerical study of the eigenspectrum of the critical three-state Potts model [20, 31] (yet another lattice model in statistical mechanics), the Stony Brook group found an amazing variety of new q-series identities of Rogers–Ramanujan type [29, 30]. Almost none of these identities had been encountered previously in the context of either partition theory or the theory of infinite dimensional Lie algebras.

More specific, in the work of [29, 30] expressions for Virasoro characters were given through systems of fermionic quasi-particles. Equating these *fermionic* character forms with the well-known Rocha-Caridi type *bosonic* expressions [36], led to many q-series identities for Virasoro characters, generalizing the Rogers–Ramanujan identities (which are associated to the $M(2, 5)$ minimal model).

The proof of the Rogers–Ramanujan identities by means of an extension to polynomial identities whose degree is determined by a fixed integer L, was initiated by Schur [43]. Before we elaborate on this approach, we need the combinatorial version of the Rogers–Ramanujan identities stating that

Theorem 2 (Rogers–Ramanujan)**.** *For $a = 0, 1$, the partitions of n into parts congruent to $1 + a$ or $4 - a \pmod 5$ are equinumerous with the partitions of n in which the difference between any two parts is at least 2 and 1 occurs at most $1 - a$ times.*

Denoting the number of occurences of the part j in a partition by f_j, the second type of partitions in the above theorem are those partitions of $n = \sum_{j\geq 1} jf_j$ which satisfy the following *frequency conditions*:

$$f_j + f_{j+1} \leq 1 \ \forall j \qquad \text{and} \qquad f_1 \leq 1 - a.$$

Schur notes that imposing the additional condition $f_j = 0$ for $j \geq L + 1$, the generating function of the "frequency partitions" satisfies the recurrence

$$g_L = g_{L-1} + q^L g_{L-2}. \tag{1.2}$$

Together with the appropriate initial conditions, Schur was able to solve these recurrences, to obtain an alternating-sign type solution, now called a bosonic expression. Taking $L \to \infty$ in these bosonic polynomials yields (after use of Jacobi's triple product identity) the right-hand side of (1.1). Since this indeed corresponds to the generating function of the "$\pmod 5$" partitions, this proves Theorem 2. Much later, Andrews [3] obtained a solution to the recurrence relation as a finite q-series with manifestly positive integer coefficients, now called a fermionic expression. Taking $L \to \infty$ in these fermionic polynomials yields the left-hand side of (1.1).

Recently much progress has been made in proving the boson-fermion identities of [29,30] (and generalizations thereof), by following the Andrews–Schur approach. That is, for many of the Virasoro-character identities, finitizations to polynomial boson-fermion identities have been found, which could then be proven either fully recursively (à la Andrews) or one side combinatorially and one side recursively (à la Schur), see [14–17,19,22–24,26,32,34,41,42,44–46].

In this paper we consider polynomial identities which imply the Andrews–Gordon generalization of the Rogers–Ramanujan identities. First, Gordon's theorem [27], which provides a combinatorial generalization of the Rogers–Ramanujan identities, reads

Theorem 3 (Gordon)**.** *For all $k \geq 1$, $1 \leq i \leq k+1$, let $A_{k,i}(n)$ be the number of partitions of n into parts not congruent to 0 or $\pm i \pmod{2k+3}$ and let $B_{k,i}(n)$ be the number of partitions of n of the form $n = \sum_{j\geq 1} jf_j$, with $f_1 \leq i - 1$ and $f_j + f_{j+1} \leq k$ (for all j). Then $A_{k,i}(n) = B_{k,i}(n)$.*

Subsequently the following analytic counterpart of this result was obtained by Andrews [5], generalizing the analytic form (1.1) of the Rogers–Ramanujan identities.

Theorem 4 (Andrews)**.** *For all $k \geq 1$, $1 \leq i \leq k+1$ and $|q| < 1$,*

$$\sum_{n_1,n_2,\dots,n_k\geq 0} \frac{q^{N_1^2+\cdots+N_k^2+N_i+\cdots+N_k}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}} = \prod_{\substack{j=1\\ j\not\equiv 0,\pm i \pmod{2k+3}}}^{\infty} (1-q^j)^{-1} \tag{1.3}$$

with

$$N_j = n_j + \cdots + n_k \tag{1.4}$$

and $(q)_a = \prod_{k=1}^{a}(1-q^k)$ for $a > 0$ and $(q)_0 = 1$.

Application of Jacobi's triple product identity admits for a rewriting of the right-hand side of (1.3) to

$$\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-)^j q^{j\big((2k+3)(j+1)-2i\big)/2}. \tag{1.5}$$

Equating (1.5) and the left-hand side of (1.3), gives an example of a boson-fermion identity. Here we consider, in the spirit of Schur, a "natural" finitization of Gordon's frequency condition such that this boson-fermion identity is a limiting case of polynomial identities. In particular, we are interested in the quantity $B_{k,i,i';L}(n)$, counting the number of partitions of n of the form

$$n=\sum_{j=1}^{L-1} j f_j$$

with frequency conditions

$$f_1 \le i-1, \qquad f_{L-1}\le i'-1 \qquad \text{and} \qquad f_j+f_{j+1}\le k \quad \text{for } j=1,\dots,L-2.$$

If we denote the generating function of partitions counted by $B_{k,i,i';L}(n)$ by $G_{k,i,i';L}(q)$, then clearly $\lim_{L\to\infty} G_{k,i,i';L}(q)=G_{k,i}(q)$, with $G_{k,i}$ the generating function associated with $B_{k,i}(n)$ of Theorem 3. Also note that $G_{k,i,1;L}=G_{k,i,k+1;L-1}$. Our main results can be formulated as the following two theorems for $G_{k,i,i';L}$.

Let $\begin{bmatrix} L\\ a\end{bmatrix}$ be the Gaussian polynomial or q-binomial coefficient defined by

$$\begin{bmatrix} L\\ a\end{bmatrix}=\begin{bmatrix} L\\ a\end{bmatrix}_q=\begin{cases} \dfrac{(q)_L}{(q)_a(q)_{L-a}} & 0\le a\le L\\ 0 & \text{otherwise.}\end{cases} \tag{1.6}$$

Further, let $\mathcal{I}_k$ be the incidence matrix of the Dynkin diagram of A_k with an additional tadpole at the k-th node:

$$(\mathcal{I}_k)_{j,\ell}=\delta_{j,\ell-1}+\delta_{j,\ell+1}+\delta_{j,\ell}\delta_{j,k} \qquad j,\ell=1,\dots,k, \tag{1.7}$$

and let C_k be the corresponding Cartan-type matrix, $(C_k)_{j,\ell}=2\delta_{j,\ell}-(\mathcal{I}_k)_{j,\ell}$. Finally let $\vec{n}$, $\vec{m}$ and $\vec{\mathrm{e}}_j$ be k-dimensional (column)-vectors with entries $\vec{n}_j=n_j$, $\vec{m}_j=m_j$ and $(\vec{\mathrm{e}}_j)_\ell=\delta_{j,\ell}$. Then

Theorem 5. *For all $k\ge 1$, $1\le i,i'\le k+1$ and $kL\ge 2k-i-i'+2$,*

$$G_{k,i,i';L}(q)=\sum_{n_1,n_2,\dots,n_k\ge 0} q^{\vec{n}^T C_k^{-1}(\vec{n}+\vec{\mathrm{e}}_k-\vec{\mathrm{e}}_{i-1})}\prod_{j=1}^{k}\begin{bmatrix} n_j+m_j\\ n_j\end{bmatrix}, \tag{1.8}$$

with (m,n)-system [14] given by

$$\vec{m}+\vec{n}=\frac{1}{2}\Big(\mathcal{I}_k\,\vec{m}+(L-2)\,\vec{\mathrm{e}}_k+\vec{\mathrm{e}}_{i-1}+\vec{\mathrm{e}}_{i'-1}\Big). \tag{1.9}$$

We note that $(C_k^{-1})_{j,\ell}=\min(j,\ell)$ and hence that, using the variables N_j of (1.4), we can rewrite the quadratic exponent of q in (1.8) as $N_1^2+\cdots+N_k^2+N_i+\cdots+N_k$. For $k\ge 2$, the "finitization" (1.8)–(1.9) of the left-hand side of (1.3) is new. For $k=1$ it is the already mentioned fermionic solution to the recurrence (1.2) as found by Andrews [3]. Another finitization, which does not seem to be related to a finitization of Gordon's frequency conditions, has recently been proposed in [22, 24, 32] (see also [15]). A more general expression, which includes (1.8)–(1.9) and that of [22, 24, 32] as special cases, will be discused in Section 6.

Our second result, which is maybe of more interest mathematically since it involves new generalizations of the Gaussian polynomials, can be stated as follows. Let $\begin{bmatrix} L\\ a\end{bmatrix}_k^{(p)}$ be the q-multinomial coefficient defined in equation (2.4) of the subsequent section. Also, define

$$r=k-i'+1 \tag{1.10}$$

and

$$s=\begin{cases} i & \text{for } i=1,3,\dots,2\lfloor\frac{k}{2}\rfloor+1 \\ 2k+3-i & \text{for } i=2,4,\dots,2\lfloor\frac{k+1}{2}\rfloor, \end{cases} \tag{1.11}$$

so that $r=0,1,\dots,k$ and $s=1,3,\dots,2k+1$. Then

Theorem 6. *For all $k>0$, $1\le i,i'\le k+1$ and $kL\ge 2k-i-i'+2$,*

$$G_{k,i,i';L}(q)=\sum_{j=-\infty}^{\infty}\Bigg\{q^{j\big((2j+1)(2k+3)-2s\big)}\begin{bmatrix} L \\ \frac{1}{2}(kL+k-s-r+1)+(2k+3)j\end{bmatrix}_k^{(r)} \\ -q^{\big(2j+1\big)\big((2k+3)j+s\big)}\begin{bmatrix} L \\ \frac{1}{2}(kL+k+s-r+1)+(2k+3)j\end{bmatrix}_k^{(r)}\Bigg\} \tag{1.12}$$

for $r\equiv k(L+1)\pmod 2$ and

$$G_{k,i,i';L}(q)=\sum_{j=-\infty}^{\infty}\Bigg\{q^{j\big((2j+1)(2k+3)-2s\big)}\begin{bmatrix} L \\ \frac{1}{2}(kL-k+s-r-2)-(2k+3)j\end{bmatrix}_k^{(r)} \\ -q^{\big(2j+1\big)\big((2k+3)j+s\big)}\begin{bmatrix} L \\ \frac{1}{2}(kL-k-s-r-2)-(2k+3)j\end{bmatrix}_k^{(r)}\Bigg\} \tag{1.13}$$

for $r\not\equiv k(L+1)\pmod 2$.

For $k\ge 3$, the finitizations (1.12) and (1.13) of the right-hand side of (1.3) are new. For $k=1$ (1.12) and (1.13) are Schur's bosonic polynomials. For $k=2$, $\begin{bmatrix} L \\ a\end{bmatrix}_2^{(p)}$ being a q-trinomial coefficient, (1.12) and (1.13) were (in a slightly different representation) first obtained in [10]. An altogether different alternating-sign expression for $G_{k,i,i';L}$ in terms of q-binomials has been found in [21]. A different finitization of the right-hand side of (1.3) involving q-binomials has been given in [3, 11]. A more general expression, which includes (1.12), (1.13) and that of [3, 11] as special cases, will be discused in Section 6.

Equating (1.8) and (1.12)–(1.13) leads to non-trivial polynomial identities, which in the limit $L\to\infty$ reduce to Andrews' analytic form of Gordon's identity. For $k=1$ these are the polynomial identities featuring in the Andrews–Schur proof of the Rogers–Ramanujan identities (1.1) [3].

The remainder of the paper is organized as follows. In the next section we introduce the q-multinomial coefficients and list some q-multinomial identities needed for the proof of Theorem 6. Then, in Section 3, a combinatorial interpretation of the q-multinomials is given using Andrews' Durfee dissection partitions. In Section 4 we give a recursive proof of Theorem 6 and in Section 5 we prove Theorem 5 combinatorially, interpreting the restricted frequency partitions as configurations of a one-dimensional lattice-gas of fermionic particles. We conclude this paper with a discussion of our results, a conjecture generalizing Theorems 5 and 6, and some new identities for the branching functions of cosets of type $(\mathrm{A}_1^{(1)})_k\times(\mathrm{A}_1^{(1)})_\ell/(\mathrm{A}_1^{(1)})_{k+\ell}$ with fractional level ℓ. Finally, proofs of some of the q-multinomial identities are given in the appendix.

2. q-MULTINOMIAL COEFFICIENTS

Before introducing the q-multinomial coefficients, we first recall some facts about ordinary multinomials. Following [10], we define $\binom{L}{a}_k$ for $a = 0, \dots, kL$ as

$$(1 + x + \cdots + x^k)^L = \sum_{a=0}^{kL} \binom{L}{a}_k x^a.$$

Multiple use of the binomial theorem yields

$$\binom{L}{a}_k = \sum_{j_1+\cdots+j_k=a} \binom{L}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{k-1}}{j_k}, \tag{2.1}$$

where $\binom{L}{a} = \binom{L}{a}_1$ is the usual binomial coefficient.

Some readily established properties of $\binom{L}{a}_k$ are the symmetry relation

$$\binom{L}{a}_k = \binom{L}{kL-a}_k \tag{2.2}$$

and the recurrence

$$\binom{L}{a}_k = \sum_{m=0}^{k} \binom{L-1}{a-m}_k. \tag{2.3}$$

For our subsequent working it will be convenient to define $k+1$ different q-deformations of the multinomial coefficient (2.1).

Definition 1. *For $p = 0, \dots, k$ we set*

$$\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} = \sum_{j_1+\cdots+j_k=a} q^{\sum_{\ell=1}^{k-1}(L-j_\ell)j_{\ell+1} - \sum_{\ell=k-p}^{k-1} j_{\ell+1}} \begin{bmatrix} L \\ j_1 \end{bmatrix}\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}\cdots\begin{bmatrix} j_{k-1} \\ j_k \end{bmatrix}, \tag{2.4}$$

with $\begin{bmatrix} L \\ a \end{bmatrix}$ the standard q-binomial coefficients of (1.6).

Note that $\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)}$ is unequal to zero for $a = 0, \dots, kL$ only. Also note the initial condition

$$\begin{bmatrix} 0 \\ a \end{bmatrix}_k^{(p)} = \delta_{a,0}. \tag{2.5}$$

In the following we state a number of q-deformations to (2.2) and (2.3). Although our list is certainly not exhaustive, we have restricted ourselves to those identities which in our view are simplest, and to those needed for proving Theorem 6. Most of these identities are generalizations of known q-binomial and q-trinomial identities which, for example, can be found in [9, 10, 16, 17].

First we put some simple symmetry properties generalizing (2.2), in a lemma.

Lemma 1. *For $p = 0, \dots, k$ the following symmetries hold:*

$$\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} = q^{(k-p)L-a} \begin{bmatrix} L \\ kL-a \end{bmatrix}_k^{(k-p)} \quad \textit{and} \quad \begin{bmatrix} L \\ a \end{bmatrix}_k^{(0)} = \begin{bmatrix} L \\ kL-a \end{bmatrix}_k^{(0)}. \tag{2.6}$$

The proof of this lemma is given in the appendix.

To our mind the simplest way of q-deforming (2.3) (which was communicated to us by A. Schilling) is

Proposition 1 (Fundamental recurrences; Schilling)**.** *For $p=0,\dots,k$, the q-multinomials satisfy*

$$\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} = \sum_{m=0}^{k-p} q^{m(L-1)} \begin{bmatrix} L-1 \\ a-m \end{bmatrix}_k^{(m)} + \sum_{m=k-p+1}^{k} q^{L(k-p)-m} \begin{bmatrix} L-1 \\ a-m \end{bmatrix}_k^{(m)}. \tag{2.7}$$

In the next section we give a combinatorial proof of this important result for the $p=0$ case. An analytic proof for general p has been given by Schilling in [42].

We now give some equations, proven in the appendix, which all reduce to the tautology $1=1$ in the $q\to 1$ limit.

Proposition 2. *For all $p=-1,\dots,k-1$, we have*

$$\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} + q^L \begin{bmatrix} L \\ kL-a-p-1 \end{bmatrix}_k^{(p+1)} = \begin{bmatrix} L \\ kL-a-p-1 \end{bmatrix}_k^{(p)} + q^L \begin{bmatrix} L \\ a \end{bmatrix}_k^{(p+1)}, \tag{2.8}$$

with $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_k^{(-1)} = 0$.

The power of these (q-deformed) tautologies is that they allow for an endless number of different rewritings of the fundamental recurrences. In particular, as shown in the appendix, they allow for the non-trivial transformation of (2.7) into

Proposition 3. *For all $p=0,\dots,k$, we have*

$$\begin{aligned} \begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} = &\sum_{\substack{m=0 \\ m\equiv p+k \ (\mathrm{mod}\ 2)}}^{k-p} q^{m(L-1)} \begin{bmatrix} L-1 \\ a-\frac{1}{2}(m-p+k) \end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=0 \\ m\not\equiv p+k \ (\mathrm{mod}\ 2)}}^{k-p-1} q^{m(L-1)} \begin{bmatrix} L-1 \\ kL-a-\frac{1}{2}(m+p+k+1) \end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=k-p+2 \\ m\equiv p+k \ (\mathrm{mod}\ 2)}}^{k} q^{\frac{1}{2}\left((2L-1)(k-p)-m\right)} \begin{bmatrix} L-1 \\ a-\frac{1}{2}(m-p+k) \end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=k-p+1 \\ m\not\equiv p+k \ (\mathrm{mod}\ 2)}}^{k} q^{kL+\frac{1}{2}\left((2L+1)(k-p)-m+1\right)-2a} \begin{bmatrix} L-1 \\ kL-a-\frac{1}{2}(m+p-k-1) \end{bmatrix}_k^{(m)}. \end{aligned} \tag{2.9}$$

It is thanks to these rather unappealing recurrences that we can prove Theorem 6.

Before concluding this section on the q-multinomial coefficients let us make some further remarks. First, for $k=1$ and $k=2$ we reproduce the well-known q-binomial and q-trinomial coefficients. In particular,

$$\begin{bmatrix} L \\ a \end{bmatrix}_1^{(0)} = \begin{bmatrix} L \\ a \end{bmatrix} \tag{2.10}$$

and

$$\begin{bmatrix} L \\ a \end{bmatrix}_2^{(p)} = \begin{pmatrix} L; L-a-p; q \\ L-a \end{pmatrix}_2 \qquad \text{for } p=0,1, \tag{2.11}$$

where on the right-hand side of (2.11) we have used the q-trinomial notation introduced by Andrews and Baxter [10].

Second, in [10], several recurrences involving q-trinomials with just a single superscript (p) are given. We note that such recurrences follow from (2.7) by taking the difference between various values of p. In particular we have for all $r = 0, \ldots, p$

$$\begin{aligned}\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} = \begin{bmatrix} L \\ a \end{bmatrix}_k^{(p-r)} &+ q^{L(k-p)-a} \sum_{m=0}^{p-r-1} \left(1 - q^{rL}\right) q^{m(L-1)} \begin{bmatrix} L-1 \\ kL-a-m \end{bmatrix}_k^{(m)} \\ &+ q^{L(k-p)-a} \sum_{m=p-r}^{p-1} \left(1 - q^{(p-m)L}\right) q^{m(L-1)} \begin{bmatrix} L-1 \\ kL-a-m \end{bmatrix}_k^{(m)}.\end{aligned} \tag{2.12}$$

This can be used to eliminate all multinomials $\left[\begin{smallmatrix} \cdot\cdot \\ \cdot\cdot \end{smallmatrix}\right]_k^{(m)}$ for $m = 0, \ldots, p-1, p+1, \ldots, k$ in favour of $\left[\begin{smallmatrix} \cdot\cdot \\ \cdot\cdot \end{smallmatrix}\right]_k^{(p)}$. The price to be paid for this is that the resulting expressions tend to get very complicated if k gets large.

A further remark we wish to make is that to our knowledge the general q-deformed multinomials as presented in (2.4) are new. The multinomial $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_k^{(0)}$ however was already suggested as a "good" q-multinomial by Andrews in [9], where the following generating function for q-multinomials was proposed for all $k > 1$:

$$p_{k,L}(x) = \sum_{a=0}^{L} x^a q^{\binom{a}{2}} \begin{bmatrix} L \\ a \end{bmatrix} p_{k-1,a}(xq^L), \tag{2.13}$$

with $p_{0,L}(x) = 1$. Clearly,

$$p_{k,L} = \sum_{a=0}^{kL} x^a q^{\binom{a}{2}} \begin{bmatrix} L \\ a \end{bmatrix}_k^{(0)}. \tag{2.14}$$

Also in the work of Date *et al.* the $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_k^{(0)}$ makes a brief appearance, see [21, Equation (3.29)].

The more general q-multinomials of equation (2.4) have been introduced independently by Schilling [42]. (The notation used in [42] and that of the present paper is almost identical apart from the fact that $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_k^{(p)}$ is replaced by $\left[\begin{smallmatrix} L \\ kL/2-a \end{smallmatrix}\right]_k^{(p)}$.)

3. Combinatorics of q-multinomial coefficients

In this section a combinatorial interpretation of the q-multinomials coefficients is given using Andrews' Durfee dissections [7]. We then show how the fundamental recurrences (2.7) with $p = 0$ follow as an immediate consequence of this interpretation.

As a first step it is convenient to change variables from q to $1/q$. Using the elementary transformation property of the Gaussian polynomials

$$\begin{bmatrix} L \\ a \end{bmatrix}_{1/q} = q^{-a(L-a)} \begin{bmatrix} L \\ a \end{bmatrix}_q, \tag{3.1}$$

we set

Definition 2. *For $p = 0, \dots, k$*

$$\left\{ \begin{matrix} L \\ a \end{matrix} \right\}_k^{(p)} := q^{-aL} \begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)} \Bigg|_{q\to 1/q} \tag{3.2}$$
$$= \sum_{N_1+\dots+N_k=a} q^{N_1^2+\dots+N_k^2+N_{k-p+1}+\dots+N_k} \begin{bmatrix} L \\ N_1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{k-1} \\ N_k \end{bmatrix}$$
$$= \sum_{\vec{n}^T C_k^{-1} \vec{e}_k = a} q^{\vec{n}^T C_k^{-1} (\vec{n} + \vec{e}_k - \vec{e}_{k-p})} \frac{(q)_L}{(q)_{L - \vec{n}^T C_k^{-1} \vec{e}_1} (q)_{n_1} (q)_{n_2} \dots (q)_{n_k}} .$$

3.1. **Successive Durfee squares and Durfee dissections.** As a short intermezzo, we review some of the ideas introduced by Andrews in [7], needed for our interpretation of (3.2). Those already familiar with such concepts as "(k,a)-Durfee dissection of a partition" and "(k,a)-admissible partitions" may wish to skip the following and resume in Section 3.2. Throughout the following a partition and its corresponding Ferrers graph are identified.

Definition 3. *The Durfee square of a partition is the maximal square of nodes (including the upper-leftmost node).*

The *size* of the Durfee square is the number of rows for which $r_\ell \geq \ell$, labelling the rows (=parts) of a partition by $r_1 \geq r_2 \geq \dots$. Copying the example from [7], the Ferrers graph and Durfee square of the partition $\pi_{\text{ex}} = 9+7+5+4+4+3+1+1$ is shown in Figure 1(a).

The portion of a partition of n below its Durfee square defines a partition of $m < n$. For this "smaller" partition one can again draw the Durfee square. Continuing this process of drawing squares, we end up with the successive Durfee squares of a partition. For the partition π_{ex} this is shown in Figure 1(b). If a partition π has k successive Durfee squares, with N_ℓ the size of the ℓ-th square, then π has exactly $N_1 + \dots + N_k$ parts with $N_1 + \dots + N_\ell$ parts $\geq N_\ell$ for all $\ell = 1, \dots, k$.

Following Andrews we now slightly generalize the previous notions.

Definition 4 (Durfee rectangle)**.** *The Durfee rectangle of a partition is the maximal rectangle of nodes whose height exceeds its width by precisely one row.*

The Durfee rectangle of the partition π_{ex} is shown in Figure 1(c). The *size* of the Durfee rectangle is its width.

One can now combine the Durfee squares and rectangles to define

Definition 5 (Durfee dissection)**.** *The (k,i)-Durfee dissection of a partition is obtained by drawing $i-1$ successive Durfee squares followed by $k-i+1$ successive Durfee rectangles.*

In the following it will be convenient to adopt a slightly unconventional labelling. In particular, we label the Durfee squares from 1 to $i-1$ and the rectangles from i to k. Correspondingly, N_ℓ is the size of the Durfee square or rectangle labelled by ℓ. We note that in the (k,i)-dissection of a partition corresponding to Durfee squares and rectangles of respective sizes $N_1 \geq N_2 \geq \dots \geq N_k$, all the N_ℓ beyond some fixed ℓ' may actually be zero.

Finally we come to the most important definition of this section.

Definition 6 ((k,i)-admissible)**.** *Let $N_1 \geq N_2 \geq \dots \geq N_k$ be the respective sizes of the Durfee squares and rectangles in the (k,i)-Durfee dissection of a partition π. Then π is (k,i)-admissible if*

- *π has no parts below its last successive Durfee rectangle (or square if $i = k+1$.)*
- *For $\ell = i, \dots, k$, the last row of the Durfee rectangle labelled by ℓ has N_ℓ nodes.*

The first condition is equivalent to stating that the number of parts of π equals $N_1 + \dots + N_k + \max(\ell' - i + 1, 0)$, where ℓ' labels the number of Durfee squares and rectangles of non-zero size; $N_\ell > 0$

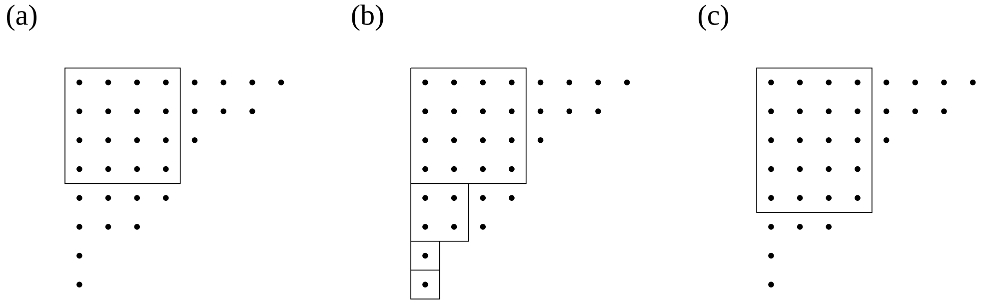


FIGURE 1. (a) Durfee square of the partition $\pi_{\text{ex}} = 9+7+5+4+4+3+1+1$. (b) The four successive Durfee squares of π_{ex}. (c) The Durfee rectangle of π_{ex}.

for $\ell \le \ell'$ and $N_\ell = 0$ for $\ell > \ell'$. The second condition is equivalent to stating that the last row of each Durfee rectangle is actually a part of π.

3.2. $(k,i;L,a)$**-admissible partitions and** q**-multinomial coefficients.** Using the previous definitions we are now prepared for the combinatorial interpretation of (3.2).

Definition 7 ($(k,i;L,a)$-admissible)**.** *Let $N_1 \ge N_2 \ge \ldots \ge N_k$ be the respective sizes of the Durfee squares and rectangles of a (k,i)-admissible partition π. Then π is said to be $(k,i;L,a)$-admissible if the largest part of π is less or equal to L and $N_1 + \cdots + N_k = a$.*

For a $(k,i;L,a)$-admissible partition π, the portion π_ℓ of π to the right of the Durfee square or rectangle labelled by ℓ (and below the Durfee square or rectangle labelled $N_{\ell-1}$), is a partition with largest part $\le N_{\ell-1} - N_\ell$ (where $N_0 = L$) and number of parts $\le N_\ell$. Recalling that the Gaussian polynomial (1.6) is the generating function of partitions with largest part $\le L-a$ and number of parts $\le a$ [6], we thus find that the generating function of $(k,i;L,a)$-admissible partitions is given by

$$\sum_{N_1+\cdots+N_k=a} q^{N_1^2} \begin{bmatrix} L \\ N_1 \end{bmatrix} \cdots q^{N_{i-1}^2} \begin{bmatrix} N_{i-2} \\ N_{i-1} \end{bmatrix} q^{N_i(N_i+1)} \begin{bmatrix} N_{i-1} \\ N_i \end{bmatrix} \cdots q^{N_k(N_k+1)} \begin{bmatrix} N_{k-1} \\ N_k \end{bmatrix} = \begin{Bmatrix} L \\ a \end{Bmatrix}_k^{(k-i+1)}. \tag{3.3}$$

Denoting an arbitrary partition of n with largest part $\le L$ and number of parts $\le a$ by a rectangle of width L and height a, the $(k,i;L,a)$-admissible partitions can be represented graphically as shown in Figure 2 for the case $k=2$.

Equipped with the above interpretation we return to the recurrence relation (2.7) for $p=0$. Using Definition 2 to rewrite this in terms of $\left\{ \begin{smallmatrix} L \\ a \end{smallmatrix} \right\}_k^{(p)}$, gives

$$\begin{Bmatrix} L \\ a \end{Bmatrix}_k^{(0)} = q^a \sum_{m=0}^{k} \begin{Bmatrix} L-1 \\ a-m \end{Bmatrix}_k^{(m)}. \tag{3.4}$$

This is obviously true if the following combinatorial statements hold.

Lemma 2.

- *Adding a column of a nodes to the left of a $(k,k-m+1;L-1,a-m)$-admissible partition with $m \in \{0,1,\ldots,k\}$, yields a $(k,k+1;L,a)$-admissible partition.*
- *Removing the first column (of a nodes) from a $(k,k+1;L,a)$-admissible partition yields a $(k,k-m+1;L-1,a-m)$-admissible partition for some $m \in \{0,1,\ldots,k\}$.*

To show the first statement, we note that a partition is $(k,k+1;L,a)$-admissible if it has exactly a parts, has largest part $\le L$ and has at most k successive Durfee squares. A $(k,k-m+1;L-1,a-m)$-admissible partition has at most a parts and has largest part $\le L-1$. Hence adding a column of

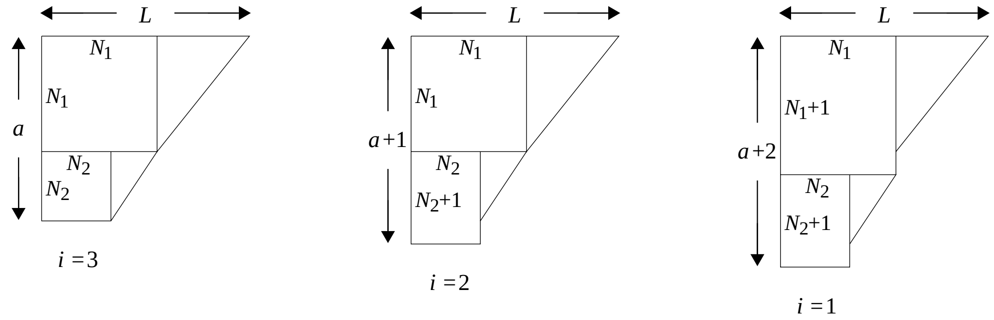


FIGURE 2. Graphical representation of the $(2,i;L,a)$-admissible partitions, generated by $\left\{\begin{smallmatrix} L \\ a \end{smallmatrix}\right\}_2^{(3-i)}$. The respective values of N_1 and N_2 are free to vary, only their sum taken the fixed value a. Note that the number of parts in the second and third figure are actually not fixed, but vary between a and $a-i+3$, depending on the number of Durfee rectangles of non-zero size.

a nodes to the left of such a partition, yields a partition π which has a parts and largest part $\leq L$. Remains to show that π has at most k successive Durfee squares. To see this first assume that the $(k,k-m+1;L-1,a-m)$-admissible partition only consists of Durfee squares and rectangles. That is, we have a partition of $N_1^2+\cdots+N_k^2+N_{k-m+1}+\cdots+N_k$, with $N_1+\cdots+N_k=a-m$. Adding a column of a dots trivially yields a partition π with k successive Durfee squares with respective sizes

$$N_1 \geq N_2 \geq \ldots \geq N_{k-m} \geq N_{k-m+1}+1 \geq \ldots \geq N_k+1 > 0, \tag{3.5}$$

with π having a column of N_ℓ nodes to the right of the ℓ-th successive Durfee square for each $\ell \leq k-m$. Now note that we in fact have treated the "worst" possible cases. All other $(k,k-m+1;L-1,a-m)$-admissible partitions can be obtained from the "bare" ones just treated by adding partitions with largest part $\leq N_{\ell-1}-N_\ell$ (where $N_0=L$) and number of parts $\leq N_\ell$ to the right of the Durfee square or rectangle labelled by ℓ for all ℓ. Let π be such a "dressed" partition, obtained from a bare $(k,k-m+1;L-1,a-m)$-admissible partition π_b, and let the images of π and π_b after adding a column of a dots be π' and π_b'. Further, let N_ℓ and M_ℓ be the size of the ℓ-th successive Durfee square of π_b' and π', respectively. Since π is obtained from π_b by adding additional nodes to its rows, we have $M_1+\cdots M_\ell \geq N_1+\cdots+N_\ell$ for all ℓ. From the fact that π_b' has at most k successive Durfee squares it thus follows that this is also true for π'.

To show the second statement of the lemma, note that from (2.3) we see that the map implied by the first statement is in fact a map onto the set of $(k,k+1,L,a)$-admissible partitions. Since for $m\neq m'$, the set of $(k,k-m+1;L-1,a-m)$-admissible partitions is distinct from the set of $(k,k-m'+1;L-1,a-m')$-admissible partitions, the second statement immediately follows.

To prove (2.7) is true for general p, we need to establish

$$\begin{Bmatrix} L \\ a \end{Bmatrix}_k^{(p)} = q^a \sum_{m=0}^{k-p} \begin{Bmatrix} L-1 \\ a-m \end{Bmatrix}_k^{(m)} + q^a \sum_{m=k-p+1}^{k} q^{L(p-k+m)} \begin{Bmatrix} L-1 \\ a-m \end{Bmatrix}_k^{(m)}. \tag{3.6}$$

Unfortunately, a generalization of Lemma 2 which would imply this more general result has so far eluded us.

Before concluding our discussion of q-multinomial coefficients we note that if the restriction on L is dropped in the $(k,i;L,a)$-admissible partitions, their generating function reduces to

$$\lim_{L\to\infty}\left\{\begin{matrix}L\\a\end{matrix}\right\}_k^{(k-i+1)} = \sum_{\substack{N_1+\cdots+N_k=a\\ n_1,\dots,n_k\geq 0}} \frac{q^{N_1^2+\cdots+N_k^2+N_i+\cdots+N_k}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_k}},$$

which, up to a factor $(q)_a$, is the representation of the Alder polynomials [2] as found in [5].

4. Proof of Theorem 6

With the results of the previous two sections, proving Theorem 6 is elementary. First we define $S_{k,i,i';L}$ as the set of partitions of n of the form $n=\sum_{j=1}^{L-1} jf_j$ satisfying the frequency conditions $f_1\leq i-1$, $f_{L-1}\leq i'-1$ and $f_j+f_{j+1}\leq k$ for $j=1,\dots,L-2$. Let π be a partition in $S_{k,i,i';L}$, with ℓ rows of length $L-1$. Using the frequency condition this implies $f_{L-2}\leq k-\ell$. Hence, by removing the first ℓ rows, π maps onto a partition in $S_{k,i,k-\ell+1;L-1}$. Conversely, by adding ℓ rows at the top to a partition in $S_{k,i,k-\ell+1;L-1}$, we obtain a partition in $S_{k,i,i';L}$. Since in the above ℓ can take the values $\ell=0,\dots,i'-1$, the following recurrences hold:

$$G_{k,i,i';L}(q)=\sum_{\ell=0}^{i'-1} q^{\ell(L-1)}G_{k,i,k-\ell+1;L-1}(q) \qquad \text{for} \quad i'=1,\dots,k+1. \tag{4.1}$$

In addition to this we have the initial condition

$$G_{k,i,i';2}(q)=\sum_{\ell=0}^{\min(i'-1,i-1)} q^{\ell}. \tag{4.2}$$

Using the recurrence relations, it is in fact an easy matter to verify that this is consistent with the condition

$$G_{k,i,i';0}(q)=\delta_{i,i'}. \tag{4.3}$$

Remains to verify that (1.12) and (1.13) satisfy the recurrence (4.1) and initial condition (4.3). Since in these two equations we have used the variables r and s instead of i' and i, let us first rewrite (4.1) and (4.3). Suppressing the k, s and q dependence, setting $G_{k,i,i';L}(q)=G_L(r)$, we get

$$G_L(r)=\sum_{\ell=0}^{k-r} q^{\ell(L-1)}G_{L-1}(\ell) \qquad \text{for} \quad r=0,\dots,k \tag{4.4}$$

and

$$G_0(r)=\begin{cases}\delta_{s+r,k+1} & \text{for } s=1,3,\dots,2\lfloor\frac{k}{2}\rfloor+1\\ \delta_{s-r,k+2} & \text{for } s=2\lfloor\frac{k}{2}\rfloor+3,\dots,2k+1.\end{cases} \tag{4.5}$$

To verify that (1.12) and (1.13) satisfy the initial condition (4.5), we set $L=0$ and use the fact that $r=0,1,\dots,k$ and $s=1,3,\dots,2k+1$. From this and equation (2.5) one immediately sees that the only non-vanishing term in (1.12) is given by $\left[\begin{smallmatrix}0\\(k-s-r+1)/2\end{smallmatrix}\right]_k^{(r)}=\delta_{s+r,k+1}$. Similarly the only non-vanishing term in (1.13) is $\left[\begin{smallmatrix}0\\(-k+s-r-2)/2\end{smallmatrix}\right]_k^{(r)}=\delta_{s-r,k+2}$. Now recall that (1.12) with $L=0$ is $G_0(r)$ for $r\equiv k$. From the allowed range of r this implies $s=1,3,\dots,2\lfloor\frac{k}{2}\rfloor+1$, in accordance with the top-line of (4.5). Also, since (1.13) with $L=0$ is $G_0(r)$ for $r\not\equiv k$, and because of the range of r, we get $s=2\lfloor\frac{k}{2}\rfloor+3,\dots,2k+1$, in accordance with the second line in (4.5).

Checking that (1.12) and (1.13) solve the recurrence relation (4.4) splits into several cases due to the parity dependence of $G_L(r)$ and of the q-multinomial recurrences (2.9). All of these cases are completely analogous and we restrict our attention to k and r being even, so that $G_L(r)$ is given by equation (1.12). Substituting recurrences (2.9), the first and second sum in (2.9) immediately give the right-hand side of (4.4). Consequently, the other two terms in (2.9) give rise to unwanted terms that have to cancel in order for (4.4) to be true. Dividing out the common factor $q^{(2L-1)(k-r)/2}$ and making the change of variables $m \to m-1$ in the last sum of (2.9), the unwanted terms read

$$\sum_{\substack{m=k-r+2\\ m \text{ even}}}^{k} q^{-\frac{1}{2}m} \sum_{j=-\infty}^{\infty} \Biggl\{ q^{j\left((2j+1)(2k+3)-2s\right)} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL-s-m+1)+(2k+3)j \end{bmatrix}_k^{(m)} - q^{\left(2j+1\right)\left((2k+3)j+s\right)} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL+s-m+1)+(2k+3)j \end{bmatrix}_k^{(m)} + q^{\left(2j-1\right)\left((2k+3)j-s\right)} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL+s-m+1)-(2k+3)j \end{bmatrix}_k^{(m-1)} - q^{j\left((2j-1)(2k+3)+2s\right)} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL-s-m+1)-(2k+3)j \end{bmatrix}_k^{(m-1)} \Biggr\}.$$

After changing the summation variable $j \to -j$ in the second and fourth term, this becomes

$$\sum_{\substack{m=k-r+2\\ m \text{ even}}}^{k} q^{-\frac{1}{2}m} \sum_{j=-\infty}^{\infty} q^{j\left((2j+1)(2k+3)-2s\right)} \times \Biggl\{ \begin{bmatrix} L-1 \\ \frac{1}{2}(kL-s-m+1)+(2k+3)j \end{bmatrix}_k^{(m)} - q^{s-2(2k+3)j} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL+s-m+1)-(2k+3)j \end{bmatrix}_k^{(m)} + q^{s-2(2k+3)j} \begin{bmatrix} L-1 \\ \frac{1}{2}(kL+s-m+1)-(2k+3)j \end{bmatrix}_k^{(m-1)} - \begin{bmatrix} L-1 \\ \frac{1}{2}(kL-s-m+1)+(2k+3)j \end{bmatrix}_k^{(m-1)} \Biggr\}.$$

We now show that the term within the curly braces vanishes for all m and j. To establish this, we apply the symmetry (2.6) to all four q-multinomials within the braces and divide by

$$q^{(k-m)(L-1)-\frac{1}{2}(kL-s-m+1)-(2k+3)j}.$$

After replacing L by $L+1$ and m by $k-p$, this gives

$$\begin{bmatrix} L \\ \frac{1}{2}(kL+s-p-1)-(2k+3)j \end{bmatrix}_k^{(p)} - \begin{bmatrix} L \\ \frac{1}{2}(kL-s-p-1)+(2k+3)j \end{bmatrix}_k^{(p)} + q^L \begin{bmatrix} L \\ \frac{1}{2}(kL-s-p-1)+(2k+3)j \end{bmatrix}_k^{(p+1)} - q^L \begin{bmatrix} L \\ \frac{1}{2}(kL+s-p-1)-(2k+3)j \end{bmatrix}_k^{(p+1)}.$$

Recalling the tautology (2.8) with $a = \frac{1}{2}(kL+s-p-1)-(2k+3)j$ this indeed gives zero.

5. Proof of Theorem 5

5.1. **From partitions to paths.** To prove expression (1.8) of Theorem 5, we reformulate the problem of calculating the generating function $G_{k,i,i';L}(q)$ into a lattice path problem. Hereto we represent each partition π in $S_{k,i,i';L}$ as a restricted lattice path $p(\pi)$, similar in spirit to the lattice path formulation of the left-hand side of (1.3) by Bressoud [18].[1]

[1]Finitizing Bressoud's lattice paths by fixing the length of his paths to L, results in the left-hand side of (6.1) of the next section. Hence the lattice paths introduced here are intrinsically different from those of [18] and in fact correspond to a finitization of the paths of [40].

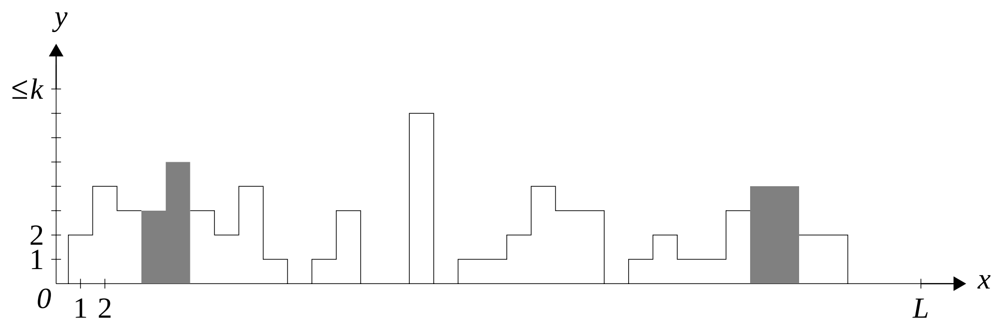


FIGURE 3. A lattice path of the partition $(f_1,\ldots,f_{L-1}) = (2,4,3,3,5,3,2,4,1,0,1,3,0,0,7,0,1,1,2,4,3,3,0,1,2,1,1,3,4,4,2,2,0,0)$. The shaded regions correspond to the two particles with largest charge (=8), as described below.

To map a partition π of $n = \sum_{j=1}^{L-1} jf_j$ onto a lattice path $p(\pi)$, draw a horizontal line-segment in the (x,y)-plane from $(j-\frac{1}{2}, f_j)$ to $(j+\frac{1}{2}, f_j)$ for each $j = 1,\ldots,L-1$. Also draw vertical line-segments from $(j+\frac{1}{2}, f_j)$ to $(j+\frac{1}{2}, f_{j+1})$ for all $j = 0,\ldots,L-1$, where $f_0 = f_{L-1} = 0$. As a result π is represented by a lattice path (or histogram) from $(\frac{1}{2}, 0)$ to $(L-\frac{1}{2}, 0)$. The frequency condition $f_j + f_{j+1} \le k$ translates into the condition that the sum of the *heights* of a path at x-positions j and $j+1$ does not exceed k. The restrictions $f_1 \le i-1$ and $f_{L-1} \le i'-1$ correspond to the restrictions that the heights at $x=1$ and $x=L-1$ are less than i and i', respectively. An example of a lattice path for $k \ge 8$, $i \ge 3$ and $i' \ge 1$, is shown in Figure 3.

The above map clearly is reversible, and any lattice path satisfying the above height conditions maps onto a partition in $S_{k,i,i';L}$. From now on we let $P_{k,i,i';L}$ denote the set of restricted lattice paths corresponding to the set of partions $S_{k,i,i';L}$.

From the map of partitions onto paths, the problem of calculating the generating function $G_{k,i,i';L}(q)$ can be reformulated as

$$G_{k,i,i';L}(q) = \sum_{p \in P_{k,i,i';L}} W(p) \tag{5.1}$$

with Boltzmann weight $W(p) = \prod_{j=1}^{L-1} q^{jf_j}$.

Before we actually compute the above sum, we remark that in the following k, i and i' will always be fixed. Hence, to simplify notation, we use G_L and P_L to denote $G_{k,i,i';L}$ and $P_{k,i,i';L}$, respectively.

5.2. **Fermi-gas partition function;** $i = i' = k+1$. To perform the sum (5.1) over the restricted lattice path, we follow a procedure similar to the one employed in our proof of Virasoro-character identities for the unitary minimal models [44, 45]. That is, the sum (5.1) is interpreted as the grand-canonical partition function of a one-dimensional lattice-gas of fermionic particles.

The idea of this approach is to view each lattice path as a configuration of particles on a one-dimensional lattice. Since not all lattice paths correspond to the same *particle content* $\vec{n}$, this gives rise to a natural decomposition of (5.1) into

$$G_L(q) = \sum_{\vec{n}} Z_L(\vec{n};q), \tag{5.2}$$

with Z_L the canonical partition function,

$$Z_L(\vec{n};q) = \sum_{p \in P_L(\vec{n})} W(p).$$

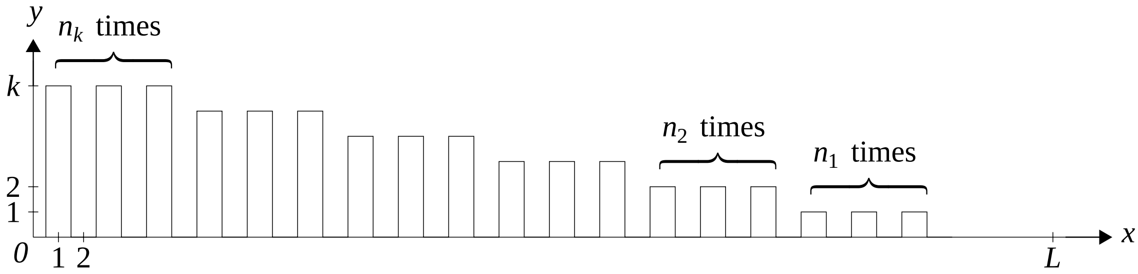


FIGURE 4. The minimal path of content $\vec{n}=(n_1,\dots,n_k)^T$.

Here $P_L(\vec{n})\subset P_L$ is the set of paths corresponding to a particle configuration with content $\vec{n}$. To avoid making the following description of the lattice gas unnecessarily complicated, we assume $i=i'=k+1$ in the remainder of this section. Subsequently we will briefly indicate how to modify the calculations to give results for general i and i'.

To describe how to interpret each path in $P_L=P_{k,k+1,k+1;L}$ as a particle configuration, we first introduce a special kind of paths from which all other paths can be constructed.

Definition 8 (minimal paths)**.** *The path shown in Figure 4 is called the minimal path of content $\vec{n}$.*

Definition 9 (charged particle)**.** *In a minimal path, each column with non-zero height t corresponds to a particle of charge t.*

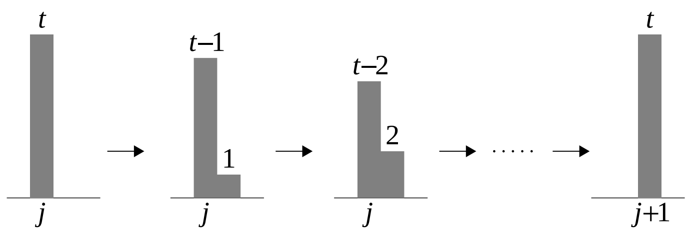
Note that in the minimal path the particles are ordered according to their charge and that adjacent particles are separated by a single empty column. The number of particles of charge t is denoted n_t and $\vec{n}=(n_1,\dots,n_k)^T$. For later use it will be convenient to give each particle a label, $p_{t,\ell}$ denoting the ℓ-th particle of charge t, counted from the right.

Since the length of a path is fixed by L, there are only a finite number of minimal paths. In particular, we have $2(n_1+\cdots+n_k)\le L$, so that there are $\binom{\lfloor L/2\rfloor+k}{k}$ different minimal paths.

In the following we show that all non-minimal paths in P_L can be constructed out of one (and only one) minimal path using a set of elementary moves. Hereto we first describe how various local configurations may be changed by moving a particle.[2] To suit the eye, the particle being moved in each example has been shaded.

To describe the moves we first consider the simplest type of motion, when the two columns immediately to the right of a particle are empty.

Definition 10 (free motion)**.** *The following sequence of moves is called free motion:*



Clearly, a particle of charge t in free motion takes t moves to fully shift position by one unit.

Now assume that in moving a particle of charge t, we at some stage encounter the local configuration shown in Figure 5(a). We then allow the particle to make $t-s$ more moves following the rules of

[2]In moving a particle we always mean motion from left to right.

free motion, to end up with the local configuration shown in (b). If instead of (a) we encounter the configurations (c) or (d), the particle can make no further moves.

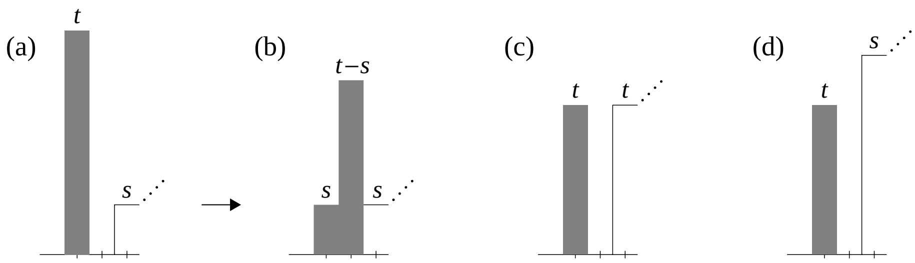


FIGURE 5.

In case of the configuration of Figure 5(b), there are three possibilities. Either we have one of the configurations shown in Figure 6(a) and (b), in which case the particle cannot move any further, or we have the configuration shown in Figure 6(c) (with $0 \leq u < t-s$), in which case we can make $t-u-s$ moves, going from (c) to (e). Ignoring the for our rules irrelevant column immediately to the left of

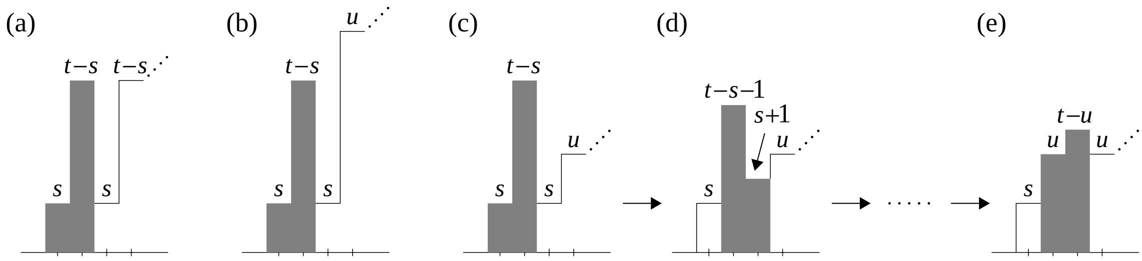


FIGURE 6.

the particle, the configuration of Figure 6(e) is essentially the same as that of Figure 5(b). To further move the particle we can thus refer to the rules given in Figure 6. That is, if the column immediately to the right of the white column of height u has height $t-u$ (corresponding to the configuration of Figure 6(a) with $s \to u$) the particle cannot move any further. Similarly, if the column immediately to the right of the white column of height u has height $v > t-u$ (corresponding to the configuration of Figure 6(b) with $u \to v$ and $s \to u$) the particle cannot move any further. However, if the height of the column immediately to the right of the white column of height u is $0 \leq v < t-u$ (corresponding to the configuration of Figure 6(c) with $u \to v$ and $s \to u$) we can make another $t-u-v$ moves.

Having introduced all necessary moves we come to the main propositions of this section

Proposition 4 (rules of motion)**.** *Each non-minimal configuration can be obtained from one and only one minimal configuration by letting the particles carry out elementary moves in the following order:*

- *Particle $p_{t,\ell}$ moves prior to $p_{s,\ell'}$ if $t < s$.*
- *Particle $p_{t,\ell'}$ moves prior to $p_{t,\ell}$ if $\ell' < \ell$.*

To prove this let us assume that we have completed the motion of all particles of charge less than t and all particles $p_{t,\ell'}$ with $\ell' < \ell$, and that we are currently moving the particle $p_{t,\ell}$. Therefore, for $x \leq 2n_k + \cdots + 2n_{t+1} + 2(n_t - \ell) := x_{\min}$, the lattice path still corresponds to the minimal path.

Now note that in moving the particle $p_{t,\ell}$, we never create a local configuration in which the sum of the height of two consecutive columns is greater than t, see the free motion and the Figures 5(a)(b)

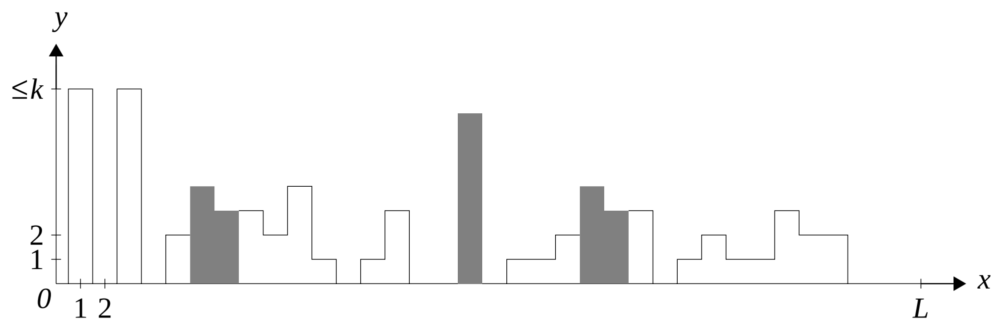


FIGURE 7. The lattice path obtained from Figure 1 after moving the largest particles to their minimal position. The shaded regions mark the three next-largest particles.

and 6(c)–(e). Since we have not moved any of the particles of charge greater than t, this means that to the right of x_{min} no two consecutive columns have summed heights greater than t.[3] Also note that as soon as $p_{t,\ell}$ meets two columns immediately to its right whose summed heights equal t, $p_{t,\ell}$ cannot move any further, see Figures 5(c) and 6(a). Consequently, $p_{t,\ell}$ always corresponds to the leftmost two consecutive columns right of $x_{\min}$ whose summed height equals t. (In fact, it corresponds to the leftmost two consecutive columns right of x_{min} with maximal summed heights.)

Now we define reversed moves by reading all the previous figures in a mirror. Using this motion we can move $p_{t,\ell}$ all the way back to its minimal position but not any further. To see this we note that the only situations in which $p_{t,\ell}$ cannot be moved further back is if it meets two consecutive columns to its left whose summed heights are greater or equal to t. Since we have just argued that such a configuration cannot occur between $x_{\min}$ and $p_{t,\ell}$ we can indeed move $p_{t,\ell}$ back to its minimal position using the reversed moves. Once it is back in its minimal position we either have the mirror image of Figure 5(c) (in case $\ell < n_t$) or (d) (in case $\ell = n_t$). Neither of these configurations allows for further reversed moves.

The above, however, gives a general procedure for reducing each non-minimal path to a minimal one by simply reversing the rules of motion in the proposition. That is, we first scan the path for all particles of charge k, by locating all occurrence of two consecutive columns of summed heights k. From left to right these label the particles p_{k,n_k} to $p_{k,1}$. Applying the previous reasoning with $t = k$, we can first move p_{k,n_k} back using reversed moves, than p_{k,n_k-1}, et cetera, until all particles of charge k have taken their "minimal position". Repeating this for the particles of charge $k-1$, then the particles of charge $k-2$, et cetera, each non-minimal path reduces to a unique minimal path.

As an example to the above, for the path of Figure 3 the shaded regions mark the (two) particles with largest charge (=8). Moving them back using the reversed motion, the leftmost particle being moved first, we end up with the path shown in Figure 7, in which now the (three) particles with next-largest charge have been marked. We leave it to the reader to further reduce the path to obtain the minimal path of content $(n_1, n_2, \ldots, n_8) = (2, 1, 2, 1, 3, 1, 3, 2)$.

The elementary moves and the reversed moves are clearly reversible. If a particle of charge t has made an elementary move changing a path from p to p', we can always carry a reversed move going from p' back to p. Since each path can be reduced to a unique minimal path using the reversed moves by carrying out the rules of motion of proposition 4 in reversed order, we have thus established that using the rules of motion we can generate each non-minimal path uniquely from a minimal path. Hence the proposition is proven.

[3] This also means that in moving a particle of charge t, the configurations of Figure 5(d) and Figure 6(b) in fact never arise.

We now have established the decomposition of the sum (5.1) into (5.2), where $Z_L(\vec{n};q)$ is the generating function of the paths generated by the minimal path labelled by $\vec{n}$, or, in other words, $Z_L(\vec{n};q)$ is the partition function of a lattice gas of fermions with particle content $\vec{n}$. The fermionic nature being that, unlike particles of different charge, particles of equal charge cannot exchange position.

Our next result concerns the actual computation of the partition function.

Proposition 5. *The partition function Z_L is given by*

$$Z_L(\vec{n};q) = q^{\vec{n}^T C_k^{-1}\vec{n}} \prod_{t=1}^{k} \begin{bmatrix} n_t+m_t \\ n_t \end{bmatrix}, \tag{5.3}$$

with $\vec{m}+\vec{n}=\frac{1}{2}(\mathcal{I}_k\,\vec{m}+L\,\vec{e}_k)$.

To prove this we first determine the contribution to Z_L of the minimal path of content $\vec{n}$,

$$\begin{aligned} W(p_{\min})/\ln q &= \sum_{t=1}^{k} t \sum_{\ell=1}^{n_t}\Big(2\ell-1+2\sum_{s=t+1}^{k} n_s\Big) = \sum_{t=1}^{k} t n_t\Big(n_t+2\sum_{s=t+1}^{k} n_s\Big) \\ &= \sum_{r=1}^{k}\sum_{t=r}^{k} n_t\Big(n_t+2\sum_{s=t+1}^{k} n_s\Big) = \sum_{r=1}^{k} N_r^2 = \vec{n}^T C_k^{-1}\vec{n}. \end{aligned} \tag{5.4}$$

To obtain the contribution to Z_L of the non-minimal configurations, we apply the rules of motion of Proposition 4. If e_ℓ denotes the number of elementary moves carried out by $p_{t,\ell}$, the generating function of moving the particles of charge t reads

$$\sum_{e_1=0}^{m_t}\sum_{e_2=0}^{e_1}\cdots\sum_{e_{n_t}=0}^{e_{n_t-1}} q^{e_1+e_2+\cdots+e_{n_t}} = \begin{bmatrix} m_t+n_t \\ n_t \end{bmatrix}. \tag{5.5}$$

Here we have used the fact that each elementary move generates a factor q and that $p_{t,\ell}$ cannot carry out more elementary moves than $p_{t,\ell-1}$. (If $p_{t,\ell}$ has made as many moves as $p_{t,\ell-1}$ we obtain either the local configuration of Figure 5(c) or Figure 6a, prohibiting any further moves.) The number m_t in (5.5) is the maximal number of elementary moves $p_{t,1}$ can make and remains to be determined. If the content of the minimal path is $\vec{n}$, the x-position of $p_{t,1}$ in $p_{\min}$ is $2(n_k+\cdots+n_t)-1 := x_0$. To fix m_t, let us assume that after the motion of the particles of charge less than t has been completed, the nontrivial part of the lattice path is encoded by the sequence of heights $(f_{x_0},\ldots,f_{L-1})$, with $f_{x_0}=t$, $f_{x_0+1}=0$ and $f_j+f_{j+1}<t$ for $j>x_0$. The particle $p_{t,1}$ can now move all the way to $x=L-1$ making

$$\begin{aligned} m_t &= (t-f_{x_0+2})+(t-f_{x_0+2}-f_{x_0+3})+(t-f_{x_0+3}-f_{x_0+4}) \\ &\quad +\cdots+(t-f_{L-2}-f_{L-1})+(t-f_{L-1}) \\ &= t(L-x_0-1)-2\sum_{j=x_0+2}^{L-1} f_j \end{aligned} \tag{5.6}$$

elementary moves. To simplify this, note that the sum on the right-hand side is nothing but twice the sum of the heights of the columns right of $x=x_0$, which is $2\sum_{s=1}^{t-1} s n_s$. Substituting this in (5.6) and using the definition of x_0, results in

$$m_t = tL-2\sum_{s=1}^{t-1} s n_s - 2t\sum_{s=t}^{k} n_t = tL-2\sum_{s=1}^{k}\min(s,t)n_s = L(C_k^{-1})_{t,k}-2\sum_{s=1}^{k}(C_k^{-1})_{t,s}n_s \tag{5.7}$$

in accordance with Proposition 5.

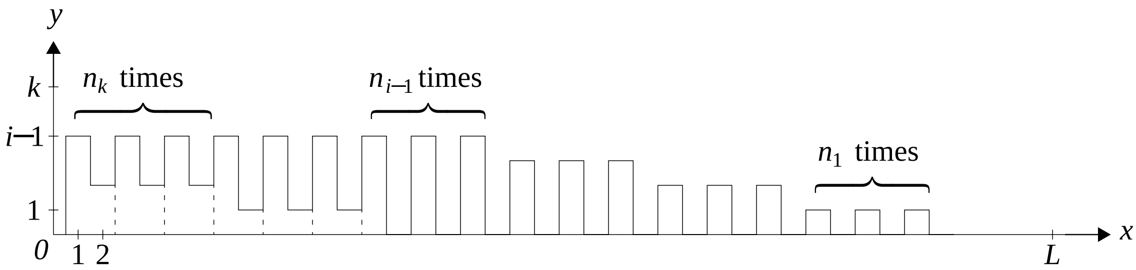


FIGURE 8. The minimal path of content $\vec{n} = (n_1, \dots, n_k)^T$ for general i. The dashed lines are drawn to mark the different particles.

Putting together the results (5.4), (5.5) and (5.7) completes the proof of Proposition 5. Substituting the form (5.3) of the partition function into (5.2) proves expression (1.8) of Theorem 5, for $i = i' = k+1$.

5.3. **Fermi-gas partition function; general i and i'.** Modifying the proof of Theorem 5 for $i = i' = k+1$ to all i and i' is straightforward and few details will be given. It is in fact interesting to note that unlike our proof for the character identities of the unitary minimal models [44, 45], the general case here does not require the introduction of additional "boundary particles".

First let us consider the general i' case, with $i = k+1$. This implies that the height f_{L-1} of the last column of the lattice paths is no longer free to take any of the values $1, \dots, k$, but is bound by $f_{L-1} \le i'-1$. For the particles of charge less or equal to $i'-1$ this does not impose any new restrictions on the maximal number of moves $p_{t,1}$ can make. For $t > i'-1$ however, m_t in (5.7) has to be decreased by $t - i' + 1$. Thus we find that m_t of (5.7) has to be replaced by $m_t - \max(0, t-i'+1) = m_t - t + \min(t, i'-1)$. Recalling $(C_k^{-1})_{s,t} = \min(s,t)$, this yields

$$m_t = (L-1)(C_k^{-1})_{t,k} + (C_k^{-1})_{t,i'-1} - 2\sum_{s=1}^{k} (C_k^{-1})_{t,s} n_s$$

and therefore, $m_t + n_t = \frac{1}{2}(\sum_{s=1}^{k} (\mathcal{I}_k)_{t,s} m_s + (L-1)\delta_{t,k} + \delta_{t,i'-1})$ which is in accordance with Proposition 5, for $i = k+1$.

Second, consider i general, but $i' = k+1$, so that $f_1 \le i-1$, $f_{L-1} \le k$. Now the modification is slightly more involved since the actual minimal paths change from those of Figure 4 to those of Figure 8. This leads to a change in the calculation of $W(p_{\min})$ to

$$\begin{aligned} W(p_{\min})/\ln q &= \sum_{t=1}^{k} \sum_{\ell=1}^{n_t} \Bigl(2\ell t - \min(t, i-1) + 2t \sum_{s=t+1}^{k} n_s\Bigr) \\ &= \sum_{t=1}^{k} t n_t \Bigl(n_t + 2\sum_{s=t+1}^{k} n_s\Bigr) + \sum_{t=i}^{k} (t-i+1) n_t \\ &= \vec{n}^T C_k^{-1} \vec{n} + \sum_{t=1}^{k} t n_t - \sum_{t=1}^{k} \min(t, i-1) n_t \;=\; \vec{n}^T C_k^{-1} (\vec{n} + \mathrm{e}_k - \mathrm{e}_{i-1}), \end{aligned}$$

which is indeed the general form of the quadratic exponent of q in (1.8). Also m_t again requires modification, which is in fact similar to the previous case: $m_t \to m_t - \max(0, t-i+1)$. To see this note that it takes $\max(0, t-i+1)$ elementary moves to move $p_{t,1}$ from its minimal position in Figure 4 to its minimal position in Figure 8.

Finally, combining the previous two cases, and using the fact that the modifications of m_t due to $f_1 \leq i-1$ and $f_{L-1} \leq i'-1$ are independent, we immediately arrive at the general form of (1.8) with (m,n)-system (1.9).

6. Discussion

In this paper we have presented polynomial identities which arise from finitizing Gordon's frequency partitions. The bosonic side of the identities involves q-deformations of the coefficient of x^a in the expansion of $(1+x+\cdots+x^k)^L$. The fermionic side follows from interpreting the generating function of the frequency partitions, as the grand-canonical partition function of a one-dimensional lattice gas.

Interestingly, in recent publications, Foda and Quano, and Kirillov, have given different polynomial identities which imply (1.3) [22, 24, 32]. In the notation of Section 1 these identities can be expressed as

Theorem 7 (Foda–Quano, Kirillov). *For all $k \geq 1$, $1 \leq i \leq k+1$ and $L \geq k-i+1$,*

$$\sum_{n_1,\dots,n_k\geq 0} q^{N_1^2+\cdots+N_k^2+N_i+\cdots+N_k} \prod_{j=1}^{k} \begin{bmatrix} L-N_j-N_{j+1}-2\sum_{\ell=1}^{j-1} N_\ell - \alpha_{i,j} \\ n_j \end{bmatrix} \tag{6.1}$$
$$= \sum_{j=-\infty}^{\infty} (-)^j q^{j\left((2k+3)(j+1)-2i\right)/2} \begin{bmatrix} L \\ \lfloor \frac{L-k+i-1-(2k+3)j}{2} \rfloor \end{bmatrix},$$

with $N_{k+1}=0$ and $\alpha_{i,j}=\max(0,j-i+1)$.

An explanation for this different finitization of (1.3) can be found in a theorem due to Andrews [4]:

Theorem 8 (Andrews). *Let $Q_{k,i}(n)$ be the number of partitions of n whose successive ranks lie in the interval $[2-i, 2k-i+1]$. Then $Q_{k,i}(n)=A_{k,i}(n)$.*

It turns out that it is the (natural) finitization of these successive rank partitions which gives rise to the above alternative polynomial finitization. That is, (6.1) is an identity for the generating function of partitions with largest part $\leq \lfloor (L+k-i+2)/2 \rfloor$, number of parts $\leq \lfloor (L-k+i-1)/2 \rfloor$, whose successive ranks lie in the interval $[2-i, 2k-i+1]$.

Let us now reexpress (6.1) into a form similar to equations (1.8), (1.12) and (1.13). Hereto we eliminate i in the right-hand side of (6.1) in favour of the variable s of equation (1.11) and split the result into two cases. This gives

$$\mathrm{RHS}(6.1) = \sum_{j=-\infty}^{\infty} \left\{ q^{j\left((2j+1)(2k+3)-2s\right)} \begin{bmatrix} L \\ \frac{1}{2}(L+k-s+1)+(2k+3)j \end{bmatrix}_1^{(0)} - q^{\left(2j+1\right)\left((2k+3)j+s\right)} \begin{bmatrix} L \\ \frac{1}{2}(L+k+s+1)+(2k+3)j \end{bmatrix}_1^{(0)} \right\}$$

for $L+k$ even, and

$$\mathrm{RHS}(6.1) = \sum_{j=-\infty}^{\infty} \left\{ q^{j\left((2j+1)(2k+3)-2s\right)} \begin{bmatrix} L \\ \frac{1}{2}(L-k+s-2)-(2k+3)j \end{bmatrix}_1^{(0)} - q^{\left(2j+1\right)\left((2k+3)j+s\right)} \begin{bmatrix} L \\ \frac{1}{2}(L-k-s-2)-(2k+3)j \end{bmatrix}_1^{(0)} \right\}$$

for $L+k$ odd. This we recognize to be exactly (1.12) and (1.13) with $r=0$, kL replaced by L and $\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}_k^{(0)}$ replaced by $\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}_1^{(0)}$.

Similarly, if we express the left-hand side of (6.1) through an (n,m)-system, we find precisely (1.8) but with

$$\vec{m}+\vec{n}=\frac{1}{2}\Big(\mathcal{I}_k\,\vec{m}+L\,\vec{\mathrm{e}}_1+\vec{\mathrm{e}}_{i-1}-\vec{\mathrm{e}}_k\Big). \tag{6.2}$$

This is just (1.9) with $r=0$ and $L\,\mathrm{e}_k$ replaced by $L\,\mathrm{e}_1$.

From the above observations it does not require much insight to propose more general polynomial identities which have (6.1) and those implied by the Theorems 5 and 6 as special cases. In particular, we have confirmed the following conjecture by extensive series expansions.

Conjecture 1. *For all $k\geq 1$, $1\leq \ell\leq k$, $1\leq i\leq k+1$, $1\leq i'\leq \ell+1$ and $\ell L\geq k+\ell-i-i'+2$*

$$\sum_{n_1,n_2,\dots,n_k\geq 0} q^{\vec{n}^T C_k^{-1}(\vec{n}+\vec{\mathrm{e}}_k-\vec{\mathrm{e}}_{i-1})}\prod_{j=1}^{k}\begin{bmatrix}n_j+m_j\\ n_j\end{bmatrix}$$

with (m,n)-system given by

$$\vec{m}+\vec{n}=\frac{1}{2}\Big(\mathcal{I}_k\,\vec{m}+(L-1)\,\vec{\mathrm{e}}_\ell+\vec{\mathrm{e}}_{i-1}+\vec{\mathrm{e}}_{i'-1}-\vec{\mathrm{e}}_k\Big) \tag{6.3}$$

equals

$$\sum_{j=-\infty}^{\infty}\Bigg\{q^{j\big((2j+1)(2k+3)-2s\big)}\begin{bmatrix}L\\ \frac{1}{2}(\ell L+k-s-r+1)+(2k+3)j\end{bmatrix}_\ell^{(r)} \\ -q^{\big(2j+1\big)\big((2k+3)j+s\big)}\begin{bmatrix}L\\ \frac{1}{2}(\ell L+k+s-r+1)+(2k+3)j\end{bmatrix}_\ell^{(r)}\Bigg\}$$

for $r\equiv \ell L+k \pmod 2$ and

$$\sum_{j=-\infty}^{\infty}\Bigg\{q^{j\big((2j+1)(2k+3)-2s\big)}\begin{bmatrix}L\\ \frac{1}{2}(\ell L-k+s-r-2)-(2k+3)j\end{bmatrix}_\ell^{(r)} \\ -q^{\big(2j+1\big)\big((2k+3)j+s\big)}\begin{bmatrix}L\\ \frac{1}{2}(\ell L-k-s-r-2)-(2k+3)j\end{bmatrix}_\ell^{(r)}\Bigg\}$$

for $r\not\equiv \ell L+k \pmod 2$. Here s is defined as in (1.11) and

$$r=\ell-i'+1$$

so that $r=0,\dots,\ell$.

For later reference, let us denote these more general polynomials as $G^{(\ell)}_{k,i,i';L}(q)$. Then $\ell=k$ corresponds to the polynomials considered in this paper and $\ell=1$ to those of Foda, Quano and Kirillov.

The above conjecture leads one to wonder whether there are in fact (at least) k different partition theoretical interpretations of (1.3), each of which has a natural finitization corresponding to the polynomials $G^{(\ell)}_{k,i,i';L}(q)$ with $\ell=1,\dots,k$.

Intimately related to the conjecture and perhaps even more surprising is the following observation, originating from the work of Andrews and Baxter [10]. For $k\geq 0$ and $1\leq i\leq k+1$, define a k-variable generating function

$$f(x_1,\dots,x_k)=\sum_{n_1,n_2,\dots,n_k\geq 0}\frac{q^{N_1^2+\dots+N_k^2+N_i+\dots+N_k}\,x_1^{2N_1}x_2^{2(N_1+N_2)}\cdots x_k^{2(N_1+\dots+N_k)}}{(x_1)_{n_1+1}(x_2)_{n_2+1}\cdots(x_k)_{n_k+1}},$$

where $(x)_n = \prod_{k=0}^{n-1}(1-xq^k)$. Obviously, $(1-x_1)\cdots(1-x_k)f(1,\dots,1)$ corresponds to the left-hand side of (1.3). Now define the polynomials $P(\ell_1,\dots,\ell_k) := P(\vec{\ell})$ as the coefficients in the series expansion of f,

$$f(x_1,\dots,x_k) = \sum_{\ell_1,\dots,\ell_k} P(\vec{\ell})\, x_1^{\ell_1}\cdots x_k^{\ell_k}.$$

From the readily derived functional equations for f and the recurrences (4.1) with (4.3) one can deduce that

$$P(\vec{m}+2C_k^{-1}\vec{n}) = G_{k,i,i';L}(q)$$

with $\vec{m}$ and $\vec{n}$ given by (1.9). Similarly the polynomials of Foda, Quano and Kirillov arise again as $P(\vec{m}+2C_k^{-1}\vec{n})$ where $\vec{m}$ and $\vec{n}$ now satisfy (6.2). Again we found numerically that also the polynomials featuring the conjecture appear naturally. That is,

$$P(\vec{m}+2C_k^{-1}\vec{n}) = G^{(\ell)}_{k,i,i';L}(q),$$

where now the generalized (m,n)-system (6.3) should hold (so that $\vec{m}+2C_k^{-1}\vec{n} = C_k^{-1}((L-1)\,\vec{\mathrm{e}}_\ell + \vec{\mathrm{e}}_{i-1}+\vec{\mathrm{e}}_{i'-1}-\vec{\mathrm{e}}_k))$.

Although all the polynomial identities implied by Conjecture 1 reduce to Andrews' identity (1.3) in the L to infinity limit, they still provide a powerful tool for generating new q-series results. That is, if we first replace $q\to 1/q$ and then take $L\to\infty$, new identities arise. To state these, we need some more notation. The inverse Cartan matrix of the Lie algebra $\mathrm{A}_{\ell-1}$ is denoted by $B_{\ell-1}$, and $\vec{\mu}$ and $\vec{\varepsilon}_j$ are $(\ell-1)$-dimensional (column) vectors with entries $\vec{\mu}_j=\mu_j$ and $(\vec{\epsilon}_j)_m=\delta_{j,m}$. Furthermore, we need the k-dimensional vector

$$\vec{Q}_{i,i',\ell} = \vec{\mathrm{e}}_i+\vec{\mathrm{e}}_{i+2}+\cdots+\vec{\mathrm{e}}_{i'}+\vec{\mathrm{e}}_{i'+2}+\cdots+\vec{\mathrm{e}}_{\ell+1}+\vec{\mathrm{e}}_{\ell+3}+\cdots,$$

with $\vec{\mathrm{e}}_j=\vec{0}$ for $j\ge k+1$. Using this notation, we are led to the following conjecture.

Conjecture 2. *For all $k\ge 1$, $1\le\ell\le k$, $1\le i\le k+1$, $1\le i'\le \ell+1$ and $|q|<1$, the q-series*

$$q^{(i'+i-2)/4}\sum_{\substack{m_1,m_2,\dots,m_k\ge 0\\ m_j\equiv(\vec{Q}_{i,i',\ell})_j \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^T C_k(\vec{m}+2\,\vec{\mathrm{e}}_k-2\,\vec{\mathrm{e}}_{i-1})}}{(q)_{m_\ell}} \prod_{\substack{j=1\\ j\ne\ell}}^{k} \begin{bmatrix} \frac{1}{2}\big(\mathcal{I}_k\,\vec{m}+\vec{\mathrm{e}}_{i-1}+\vec{\mathrm{e}}_{i'-1}-\vec{\mathrm{e}}_k\big)\\ m_j\end{bmatrix} \tag{6.4}$$

equals

$$\begin{aligned} &q^{(k+r-s+1)(k-r-s+1)/(4\ell)}\,\frac{1}{(q)_\infty}\sum_{n=0}^{\ell-1}\sum_{\substack{\mu_1,\dots,\mu_{\ell-1}\ge 0\\ n+\ell(B_{\ell-1}\vec{\mu})_1\equiv 0 \pmod{\ell}}} \frac{q^{\vec{\mu}^T B_{\ell-1}(\vec{\mu}-\vec{\epsilon}_r)}}{(q)_{\mu_1}\cdots(q)_{\mu_{\ell-1}}}\\ &\times\Bigg\{\sum_{\substack{j=-\infty\\ n+(k-s-r+1)/2+(2k+3)j\equiv 0 \pmod{\ell}}}^{\infty} q^{j\big((2k-2\ell+3)(2k+3)j+(2k+3)(k-\ell+1)-(2k-2\ell+3)s\big)/\ell}\\ &\qquad -\sum_{\substack{j=-\infty\\ n+(k+s-r+1)/2+(2k+3)j\equiv 0 \pmod{\ell}}}^{\infty} q^{\big((2k-2\ell+3)j+(k-\ell+1)\big)\big((2k+3)j+s\big)/\ell}\Bigg\} \end{aligned} \tag{6.5}$$

for $r \equiv k \pmod 2$, *and equals*

$$q^{(k+r-s+2)(k-r-s+2)/(4\ell)} \frac{1}{(q)_\infty} \sum_{n=0}^{\ell-1} \sum_{\substack{\mu_1,\dots,\mu_{\ell-1}\geq 0\\ n+\ell(B_{\ell-1}\vec{\mu})_1\equiv 0 \pmod \ell}} \frac{q^{\vec{\mu}^T B_{\ell-1}(\vec{\mu}-\vec{\epsilon}_r)}}{(q)_{\mu_1}\cdots(q)_{\mu_{\ell-1}}} \tag{6.6}$$
$$\times \Bigg\{ \sum_{\substack{j=-\infty\\ n-(k-s+r+2)/2-(2k+3)j\equiv 0 \pmod \ell}}^{\infty} q^{j\big((2k-2\ell+3)(2k+3)j+(2k+3)(k-\ell+2)-(2k-2\ell+3)s\big)/\ell}$$
$$- \sum_{\substack{j=-\infty\\ n-(k+s+r+2)/2-(2k+3)j\equiv 0 \pmod \ell}}^{\infty} q^{\big((2k-2\ell+3)j+(k-\ell+2)\big)\big((2k+3)j+s\big)/\ell}\Bigg\}$$

for $r \not\equiv k \pmod 2$.

Since Conjecture 1 is proven for $\ell = 1$ and k, we can for these particular values claim the above as theorem. In fact, for $\ell = 1$, the above was first conjectured in [30] and proven in [25]. In [12, 28] expressions for the branching functions of the $(\mathrm{A}_1^{(1)})_M \times (\mathrm{A}_1^{(1)})_N/(\mathrm{A}_1^{(1)})_{M+N}$ coset conformal field theories were given similar to (6.5) and (6.6). This similarity suggests that (6.4), (6.5) and (6.6) correspond to the branching functions of the coset $(\mathrm{A}_1^{(1)})_\ell \times (\mathrm{A}_1^{(1)})_{k-\ell-1/2}/(\mathrm{A}_1^{(1)})_{k-1/2}$ of fractional level.

A very last comment we wish to make is that there exist other polynomial identities than those discussed in this paper which imply the Andrews–Gordon identity 1.3 and which involve the q-multinomial coefficients.

Theorem 9. *For all* $k \leq 1$ *and* $1 \leq i \leq k+1$,

$$\sum_{a=0}^{L} \begin{Bmatrix} L \\ a \end{Bmatrix}_k^{(k-i+1)} = \sum_{j=-L}^{L} (-)^j q^{j\big((2k+3)(j+1)-2i\big)/2} \frac{(q)_L}{(q)_{L-j}(q)_{L+j}} . \tag{6.7}$$

Note that for $i = k+1$ the left-hand side is the generating function of partitions with at most k successive Durfee squares and with largest part $\leq L$.

The proof of Theorem 9 follows readily using the Bailey lattice of [1]. For $k = 1$ (6.7) was first obtained by Rogers [37]. For other k it is implicit in [1, 8].

Acknowledgements

I thank Anne Schilling for helpful and stimulating discussions on the q-multinomial coefficients. Especially her communication of equation (2.7) has been indispensable for proving Proposition 3. I thank Alexander Berkovich for very constructive discussions on the nature of the fermi-gas of Section 4. I wish to thank Professor G. E. Andrews for drawing my attention to the relevance of equation (6) and Barry McCoy for electronic lectures on the history of the Rogers–Ramanujan identities. Finally, helpful and interesting discussions with Omar Foda and Peter Forrester are greatfully acknowledged. This work is supported by the Australian Research Council.

Appendix A. Proof of q-multinomial relations

In this section we prove the various claims concerning the q-multinomial coefficients made in Section 2.

Let us start proving the symmetry properties (2.6) of Lemma 1. First we take the definition (2.4) and make the change variables $j_\ell \to L - j_{k-\ell+1}$ for all $\ell = 1, \dots, k$. This changes the restriction on the sum to $j_1 + \cdots + j_k = kL - a$, changes the exponent of q to

$$\sum_{\ell=1}^{k-1}(L - j_{k-\ell})j_{k-\ell+1} - \sum_{\ell=k-p}^{k-1}(L - j_{k-\ell}), \tag{A.1}$$

but leaves the product over the q-binomials invariant. We now perform a simple rewriting of (A.1) as follows

$$\begin{aligned}
(\text{A.1}) &= \sum_{\ell=1}^{k-1}(L - j_\ell)j_{\ell+1} + \sum_{\ell=1}^{p} j_\ell - pL && (\text{by } \ell \to k-\ell)\\
&= \sum_{\ell=1}^{k-1}(L - j_\ell)j_{\ell+1} - \sum_{\ell=p}^{k-1} j_{\ell+1} + (k-p)L - a && (\text{by } j_1 + \cdots + j_k = kL - a),
\end{aligned}$$

which proves the first claim of the lemma. The second statement in the lemma follows for example, by noting that $\left[{L \atop a}\right]_k^{(k)} = q^{-a}\left[{L \atop a}\right]_k^{(0)}$.

The proof of the tautologies (2.8) of Proposition 2 is somewhat more involved and we proceed inductively. For $L = 0$ (2.8) is obviously correct, thanks to

$$\begin{bmatrix} 0 \\ a \end{bmatrix}_k^{(p)} = \delta_{a,0}.$$

Now assume (2.8) holds true for all $L' = 0, \dots, L$. To show that this implies (2.8) for $L' = L + 1$, we substitute the fundamental recurrence (2.7) into (2.8) with L replaced by $L + 1$. After some cancellation of terms and division by $(1 - q^{L+1})$, this simplifies to

$$\sum_{m=0}^{M} q^{mL} \begin{bmatrix} L \\ a-m \end{bmatrix}_k^{(m)} = \sum_{m=0}^{M} q^{mL} \begin{bmatrix} L \\ kL-a-m+M \end{bmatrix}_k^{(m)}, \tag{A.2}$$

where we have replaced $k - p - 1$ by M. Since in (2.8) we have $p = -1, \dots, k-1$, (A.2) should hold for $M = 0, \dots, k$. A set of equations equivalent to this is obtained by taking $(\text{A.2})_{M=0}$ and $(\text{A.2})_M - (\text{A.2})_{M-1}$ for $M = 1, \dots, k$. In formula this new set of equations reads

$$\begin{aligned}
\sum_{m=0}^{M-1} q^{mL} \left\{ \begin{bmatrix} L \\ kL-a-m+M-1 \end{bmatrix}_k^{(m)} - q^L \begin{bmatrix} L \\ kL-a-m+M-1 \end{bmatrix}_k^{(m+1)} \right\}& \\
= \begin{bmatrix} L \\ kL-a+M \end{bmatrix}_k^{(0)} - q^{ML} \begin{bmatrix} L \\ a-M \end{bmatrix}_k^{(M)}&,
\end{aligned}$$

for $M = 0, \dots, k$. Now we use the induction assumption on the term within the curly braces, and the second symmetry relation of (2.6) on the first term of the right-hand side. This yields

$$\sum_{m=0}^{M-1} q^{mL} \left\{ \begin{bmatrix} L \\ a-M \end{bmatrix}_k^{(m)} - q^L \begin{bmatrix} L \\ a-M \end{bmatrix}_k^{(m+1)} \right\} = \begin{bmatrix} L \\ a-M \end{bmatrix}_k^{(0)} - q^{ML} \begin{bmatrix} L \\ a-M \end{bmatrix}_k^{(M)}.$$

Expanding the sum, all but two terms on the left-hand side cancel, yielding the right-hand side.

Finally we have to show Equation (2.9) of Proposition 3 to be true. We approach this problem indirectly and will in fact show that the right-hand side of (2.9) can be transformed into the right-hand side of (2.7) by multiple application of the tautologies (2.8) and the symmetries (2.6). For the sake of convenience, we restrict our attention to the case k and p even, and replace L in (2.7) and (2.9) by

$L+1$. The other choices for the parity of k and p follow in analogous manner, and the details will be omitted.

Rewriting the right-hand side of (2.9) by replacing L by $L+1$, using the even parity of k and p, and replacing p by $k-M$, gives

$$\begin{aligned}\text{(A.3)}\quad &\sum_{\substack{m=0\\ m \text{ even}}}^{M} q^{mL}\begin{bmatrix} L \\ a-\frac{1}{2}(m+M)\end{bmatrix}_k^{(m)} + \sum_{\substack{m=1\\ m \text{ odd}}}^{M-1} q^{mL}\begin{bmatrix} L \\ kL-a-\frac{1}{2}(m-M+1)\end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=M+2\\ m \text{ even}}}^{k} q^{\frac{1}{2}\big((2L+1)M-m\big)}\begin{bmatrix} L \\ a-\frac{1}{2}(m+M)\end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=M+1\\ m \text{ odd}}}^{k-1} q^{k(L+1)+\frac{1}{2}\big((2L+3)M-m+1\big)-2a}\begin{bmatrix} L \\ k(L+1)-a-\frac{1}{2}(m-M-1)\end{bmatrix}_k^{(m)}.\end{aligned}$$

The proof that this equals the right-hand side of (2.7) (with L replaced by $L+1$ and p by $k-M$) breaks up into two independent steps, both of which will be given as a lemma. First, we have

Lemma 3. *The top-line of equation* (A.3) *equals*

$$\text{(A.4)}\qquad \sum_{m=0}^{M} q^{mL}\begin{bmatrix} L \\ a-m\end{bmatrix}_k^{(m)}.$$

Second,

Lemma 4. *The bottom-two lines of equation* (A.3) *equal*

$$\text{(A.5)}\qquad \sum_{m=M+1}^{k} q^{(L+1)M-m}\begin{bmatrix} L \\ a-m\end{bmatrix}_k^{(m)}.$$

Clearly, application of these two lemmas immediately establishes the wanted result.

At the core of the proof of both lemmas is yet another result, which can be stated as

Lemma 5. *For M even and $\ell=0,\dots,\frac{1}{2}M$, the following function is independent of ℓ:*

$$\begin{aligned}F_\ell(M,a) = &\sum_{m=M-\ell}^{M} q^{mL}\begin{bmatrix} L \\ a-m\end{bmatrix}_k^{(m)} + \sum_{m=0}^{\ell-1} q^{mL}\begin{bmatrix} L \\ kL-a-m+\frac{1}{2}M-1\end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=\ell\\ m\equiv \ell \ (\mathrm{mod}\ 2)}}^{M-\ell-2} q^{mL}\left\{\begin{bmatrix} L \\ a-\frac{1}{2}(m+M-\ell)\end{bmatrix}_k^{(m)} + q^L\begin{bmatrix} L \\ kL-a-\frac{1}{2}(m-M+\ell)-1\end{bmatrix}_k^{(m+1)}\right\}.\end{aligned}$$

The proof of this is simple. First we apply the tautology (2.8) to the term within the curly braces, yielding

$$\begin{aligned}F_\ell(M,a) = &\sum_{m=M-\ell}^{M} q^{mL}\begin{bmatrix} L \\ a-m\end{bmatrix}_k^{(m)} + \sum_{m=0}^{\ell-1} q^{mL}\begin{bmatrix} L \\ kL-a-m+\frac{1}{2}M-1\end{bmatrix}_k^{(m)} \\ &+ \sum_{\substack{m=\ell\\ m\equiv \ell \ (\mathrm{mod}\ 2)}}^{M-\ell-2} q^{mL}\left\{\begin{bmatrix} L \\ kL-a-\frac{1}{2}(m-M+\ell)-1\end{bmatrix}_k^{(m)} + q^L\begin{bmatrix} L \\ a-\frac{1}{2}(m+M-\ell)\end{bmatrix}_k^{(m+1)}\right\}.\end{aligned}$$

After separating the $m=\ell$ term in the first and the $m=M-\ell-2$ term in the second term within the curly braces, we change the summation variable $m\to m-2$ in the sum over the second term within the braces. This results in

$$\begin{aligned}F_\ell(M,a) = &\sum_{m=M-\ell-1}^{M} q^{mL}\begin{bmatrix}L\\ a-m\end{bmatrix}_k^{(m)} + \sum_{m=0}^{\ell} q^{mL}\begin{bmatrix}L\\ kL-a-m+\frac12 M-1\end{bmatrix}_k^{(m)}\\ &+\sum_{\substack{m=\ell+1\\ m\equiv \ell+1 \ (\mathrm{mod}\ 2)}}^{M-\ell-3} q^{mL}\left\{\begin{bmatrix}L\\ a-\frac12(m+M-\ell-1)\end{bmatrix}_k^{(m)}\right.\\ &\qquad\left. + q^L\begin{bmatrix}L\\ kL-a-\frac12(m-M+\ell+1)-1\end{bmatrix}_k^{(m+1)}\right\} = F_{\ell+1}(M,a).\end{aligned}$$

The proof of the Lemmas 3 and 4 readily follows from Lemma 5. To prove Lemma 3, note that the top-line of (A.3) is nothing but $F_0(M)$. Since this is equal to $F_{\frac12 M}(M)$, we get

$$\text{top-line of (A.3)} = \sum_{m=\frac12 M}^{M} q^{mL}\begin{bmatrix}L\\ a-m\end{bmatrix}_k^{(m)} + \sum_{m=0}^{\frac12 M-1} q^{mL}\begin{bmatrix}L\\ kL-a-m+\frac12 M-1\end{bmatrix}_k^{(m)}. \tag{A.6}$$

Applying equation (A.2) with M replaced by $\frac12 M-1$, to the the second sum, we simplify to equation (A.4) thus proving our lemma.

To prove Lemma 4, we apply the first symmetry relation of (2.6) to all q-multinomials in the bottom-two lines of (A.3). After changing $m\to k-m$ in the second line and $m\to k-m-1$ in the third line, the last two lines of (A.3) combine to

$$f_a \sum_{\substack{m=0\\ m\ \text{even}}}^{k-M-2} q^{mL}\left\{\begin{bmatrix}L\\ kL-a-\frac12(m-M-k)\end{bmatrix}_k^{(m)} + q^L\begin{bmatrix}L\\ a-\frac12(m+M+k)-1\end{bmatrix}_k^{(m+1)}\right\}, \tag{A.7}$$

with $f_a=q^{(L+1)M-a}$. This we recognize as

$$f_a\left\{F_0(k-M,kL+k-a) - q^{(k-M)L}\begin{bmatrix}L\\ kL-a+M\end{bmatrix}_k^{(k-M)}\right\}.$$

Replacing the first term by $F_{\frac12(k-M)}(k-M,kL+k-a)$, gives

$$f_a\sum_{m=0}^{\frac12(k-M)-1} q^{mL}\begin{bmatrix}L\\ a-\frac12(M+k)-m-1\end{bmatrix}_k^{(m)} + f_a\sum_{m=\frac12(k-M)}^{k-M-1} q^{mL}\begin{bmatrix}L\\ kL-a-m+k\end{bmatrix}_k^{(m)}.$$

Applying equation (A.2) with M replaced by $\frac12(k-M)-1$ and a by $a-\frac12(M+k)-1$, to the first sum, this simplifies to

$$f_a\sum_{m=0}^{k-M-1} q^{mL}\begin{bmatrix}L\\ kL-a-m+k\end{bmatrix}_k^{(m)}.$$

Finally using the symmetry (2.6), recalling the definition of f_a and changing $m\to k-m$, we get equation (A.5).

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