THE $\mathfrak{sl}_3$ SELBERG INTEGRAL

S. OLE WARNAAR

ABSTRACT. Using an extension of the well-known evaluation symmetry, a new Cauchy-type identity for Macdonald polynomials is proved. After taking the classical limit this yields a new $\mathfrak{sl}_3$ generalisation of the famous Selberg integral. Closely related results obtained in this paper are an $\mathfrak{sl}_3$-analogue of the Askey–Habsieger–Kadell q-Selberg integral and an extension of the q-Selberg integral to a transformation between q-integrals of different dimensions.

1. INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra of rank n, with simple roots, fundamental weights and Chevalley generators given by α_i, Λ_i and e_i, f_i, h_i for $1 \le i \le n$. The roots of $\mathfrak{g}$ are normalised such that the maximal root θ has length $\sqrt{2}$, i.e., $(\theta,\theta)=2$, where $(\cdot\,,\cdot)$ is the standard bilinear symmetric form on the dual of the Cartan subalgebra.

Let V_λ and V_μ be highest weight modules of $\mathfrak{g}$ with highest weights λ and μ, and denote by $\mathrm{Sing}_{\lambda,\mu}[\nu]$ the space of singular vectors of weight ν in $V_\lambda \otimes V_\mu$:

$$\mathrm{Sing}_{\lambda,\mu}[\nu] = \big\{ v \in V_\lambda \otimes V_\mu : h_i v = \nu(h_i) v,\ e_i v = 0,\ 1 \le i \le n \big\}.$$

For fixed nonnegative integers $k_1,\dots,k_n$ assign $k := k_1 + \cdots + k_n$ integration variables $t_1,\dots,t_k$ to $\mathfrak{g}$ by attaching the k_i variables

$$t_{1+k_1+\cdots+k_{i-1}},\dots,t_{k_1+\cdots+k_i}$$

to the simple root α_i. In other words, the first k_1 integration variables are attached to α_1, the second k_2 to α_2 and so on. By a mild abuse of notation, also set

$$\alpha_{t_j} = \alpha_i \quad \text{if } k_1+\cdots+k_{i-1} < j \le k_1+\cdots+k_i.$$

Exploiting the connection between Knizhnik–Zamolodchikov equations and hypergeometric integrals, see e.g., [3, 22, 26], Mukhin and Varchenko [20] conjectured in 2000 that if the space

$$\mathrm{Sing}_{\lambda,\mu}\Big[\lambda+\mu-\sum_{i=1}^n k_i\alpha_i\Big]$$

is one-dimensional, then there exists a real integration domain Γ such that a closed-form evaluation exists (in terms of products of ratios of Gamma functions) for the $\mathfrak{g}$ Selberg integral

$$\int\limits_\Gamma \bigg[\prod_{i=1}^k t_i^{-(\lambda,\alpha_{t_i})}(1-t_i)^{-(\mu,\alpha_{t_i})} \prod_{1\le i<j\le k} |t_i-t_j|^{(\alpha_{t_i},\alpha_{t_j})}\bigg]^\gamma \mathrm{d}t_1\cdots\mathrm{d}t_k. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 33C70, 33D05, 33D52.

Work supported by the Australian Research Council.

For $\mathfrak{g}=\mathfrak{sl}_2$ the evaluation of (1.1) is well-known and corresponds to the celebrated Selberg integral [23, 18, 4]:

$$\int\limits_{0<t_1<\cdots<t_k<1} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} |t_i-t_j|^{2\gamma}\, \mathrm{d}t_1\cdots\mathrm{d}t_k = \prod_{i=0}^{k-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha+\beta+(i+k-1)\gamma)\Gamma(\gamma)}, \tag{1.2}$$

where

$$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>-\min\{1/k,\mathrm{Re}(\alpha)/(k-1),\mathrm{Re}(\beta)/(k-1)\}.$$

The prospect that generalisations of this extremely important integral exist for all simple Lie algebras has led to much recent progress in evaluating hypergeometric integrals, see e.g., [4, 8, 19, 25, 27, 29, 30, 31].

In [25] Tarasov and Varchenko obtained an evaluation of (1.1) for $\mathfrak{g}=\mathfrak{sl}_3$, $\lambda=\lambda_1\Lambda_1+\lambda_2\Lambda_2$, $\mu=\mu_2\Lambda_2$ and $k_1\le k_2$ as follows.

Theorem 1.1 (Tarasov–Varchenko)**.** *For $0\le k_1\le k_2$ let $t=(t_1,\dots,t_{k_1})$, $s=(s_1,\dots,s_{k_2})$, and let $\alpha_1,\alpha_2,\beta_2,\gamma\in\mathbb{C}$ such that $\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2),\mathrm{Re}(\beta_2)>0$ and $|\gamma|$ is sufficiently small. Then*

$$\begin{aligned}\int\limits_{C_\gamma^{k_1,k_2}[0,1]} &\prod_{i=1}^{k_1} t_i^{\alpha_1-1} \prod_{i=1}^{k_2} s_i^{\alpha_2-1}(1-s_i)^{\beta_2-1} \\ &\times \prod_{1\le i<j\le k_1}|t_i-t_j|^{2\gamma} \prod_{1\le i<j\le k_2}|s_i-s_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2}|t_i-s_j|^{-\gamma}\,\mathrm{d}t\,\mathrm{d}s \\ &= \prod_{i=0}^{k_1-1}\frac{\Gamma(\alpha_1+i\gamma)\Gamma(1+(i-k_2)\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_1+1+(i+k_1-k_2-1)\gamma)\Gamma(\gamma)} \\ &\quad\times \prod_{i=0}^{k_2-1}\frac{\Gamma(\alpha_2+i\gamma)\Gamma(\beta_2+i\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_2+\beta_2+(i+k_2-k_1-1)\gamma)\Gamma(\gamma)} \\ &\quad\times \prod_{i=0}^{k_1-1}\frac{\Gamma(\alpha_1+\alpha_2+(i-1)\gamma)}{\Gamma(\alpha_1+\alpha_2+\beta_2+(i+k_2-2)\gamma)} \\ &\quad\times \prod_{i=0}^{k_1-1}\frac{\Gamma(\alpha_2+\beta_2+(i+k_2-k_1-1)\gamma)}{\Gamma(\alpha_2+(i+k_2-k_1)\gamma)},\end{aligned}\tag{1.3}$$

where $\mathrm{d}t=\mathrm{d}t_1\cdots\mathrm{d}t_{k_1}$ and $\mathrm{d}s=\mathrm{d}s_1\cdots\mathrm{d}s_{k_2}$.

In the above, $C_\gamma^{k_1,k_2}[0,1]$ is a somewhat complicated integration chain defined in (3.8) on page 16. Since

$$C_\gamma^{0,k}[0,1]=\{(s_1,\dots,s_k)\in\mathbb{R}^k:\ 0<s_1<\cdots<s_k<1\}$$

the Tarasov–Varchenko integral simplifies to the Selberg integral when $(k_1,k_2)=(0,k)$.

In [29, 31] the present author developed a method for proving Selberg-type integrals using Macdonald polynomials. This resulted in an evaluation of (1.1) for

$\mathfrak{g} = \mathfrak{sl}_n$ where $\lambda = \sum_i \lambda_i \Lambda_i$, $\mu = \mu_n \Lambda_n$ and $k_1 \le k_2 \le \cdots \le k_n$, generalising the Selberg and Tarasov–Varchenko integrals. In this paper we again employ the theory of Macdonald polynomials to establish the following Cauchy-type identity. For λ and μ partitions (and not, as above, weights of $\mathfrak{g}$) let P_λ be a suitably normalised Macdonald polynomial. Furthermore, let $(a)_n$ be a q-shifted factorial and $(a)_\lambda$ a generalised q-shifted factorial. (For precise definitions of all of the above, see Section 2.1.)

Theorem 1.2. *Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. Then*

$$\sum_{\lambda,\mu} t^{|\lambda|-n|\mu|} \mathsf{P}_\lambda(X) \mathsf{P}_\mu(Y)\, (at^{m-1})_\lambda (qt^n/a)_\mu \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}} \\ = \prod_{i=1}^{n} \frac{(ax_i)_\infty}{(tx_i)_\infty} \prod_{j=1}^{m} \frac{(qy_j/a)_\infty}{(y_j)_\infty} \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}.$$

For $m = 0$ ($n = 0$) the above identity reduces to the q-binomial theorem for Macdonald polynomials in X (Y). Theorem 1.2 may thus be viewed as two coupled, multidimensional q-binomial theorems. In the special case $(X, Y, a, q, t) \mapsto (X/q, qY, -q^2, q^2, q^2)$ the theorem simplifies to Kawanaka's q-Cauchy identity for Schur functions [13] (with the proviso that Kawanaka's description of the summand is significantly more involved).

After a limiting procedure, which turns the sums over λ and μ into integrals, Theorem 1.2 becomes a new evaluation of the Selberg integral (1.1) for $\mathfrak{g} = \mathfrak{sl}_3$ as follows.

Theorem 1.3. *Let $t = (t_1, \dots, t_{k_1})$, $s = (s_1, \dots, s_{k_2})$ and let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2), \mathrm{Re}(\beta_1), \mathrm{Re}(\beta_2) > 0$, $|\gamma|$ is sufficiently small,*

$$\beta_1 + (i - k_2 - 1)\gamma \notin \mathbb{Z} \quad \textit{for } 1 \le i \le \min\{k_1, k_2\}$$

and

$$\beta_1 + \beta_2 = \gamma + 1.$$

Then

$$\int\limits_{C^{k_1,k_2}_{\beta_1,\gamma}[0,1]} \prod_{i=1}^{k_1} t_i^{\alpha_1-1}(1-t_i)^{\beta_1-1} \prod_{i=1}^{k_2} s_i^{\alpha_2-1}(1-s_i)^{\beta_2-1} \\ \times \prod_{1\le i<j\le k_1} |t_i - t_j|^{2\gamma} \prod_{1\le i<j\le k_2} |s_i - s_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |t_i - s_j|^{-\gamma} \, \mathrm{d}t\, \mathrm{d}s \\ = \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1 + i\gamma)\Gamma(\beta_1 + (i-k_2)\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_1 + \beta_1 + (i + k_1 - k_2 - 1)\gamma)\Gamma(\gamma)} \\ \times \prod_{i=0}^{k_2-1} \frac{\Gamma(\alpha_2 + i\gamma)\Gamma(\beta_2 + i\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_2 + \beta_2 + (i + k_2 - k_1 - 1)\gamma)\Gamma(\gamma)} \\ \times \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1 + \alpha_2 + (i-1)\gamma)}{\Gamma(\alpha_1 + \alpha_2 + (i + k_2 - 1)\gamma)}, \tag{1.4}$$

where $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_{k_1}$, $\mathrm{d}s = \mathrm{d}s_1 \cdots \mathrm{d}s_{k_2}$ *and* $C^{k_1,k_2}_{\beta,\gamma}[0,1]$ *the integration chain defined in* (3.5) *on page 15.*

Since

$$C^{k,0}_{\beta,\gamma}[0,1] = \{(t_1,\dots,t_k) \in \mathbb{R}^k : \ 0 < t_1 < \cdots < t_k < 1\},$$

the integral (1.4) again contains the Selberg integral (1.2) as special case. Unlike (1.3), however, (1.4) exhibits $\mathbb{Z}_2$ symmetry thanks to

$$C^{k_2,k_1}_{\beta_2,\gamma}[0,1] = C^{k_1,k_2}_{\beta_1,\gamma}[0,1] \prod_{i=0}^{k_1-1} \frac{\Gamma(\beta_1+i\gamma)}{\Gamma(\beta_1+(i-k_2)\gamma)} \prod_{i=0}^{k_2-1} \frac{\Gamma(\beta_2+(i-k_1)\gamma)}{\Gamma(\beta_2+i\gamma)} \tag{1.5}$$

for $\beta_1+\beta_2=\gamma+1$, and

$$\prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1+\alpha_2+(i-1)\gamma)}{\Gamma(\alpha_1+\alpha_2+(i+k_2-1)\gamma)} = \prod_{i=0}^{k_2-1} \frac{\Gamma(\alpha_1+\alpha_2+(i-1)\gamma)}{\Gamma(\alpha_1+\alpha_2+(i+k_1-1)\gamma)}.$$

If we specialise $\beta_2=\gamma$ in (1.3) and $(\beta_1,\beta_2)=(1,\gamma)$ in (1.4) then the respective products over gamma functions on the right coincide. Since also

$$C^{k_1,k_2}_{1,\gamma}[0,1] = C^{k_1,k_2}_{\gamma}[0,1] \tag{1.6}$$

(see Section 3.1 for more details) the two $\mathfrak{sl}_3$ integrals are indeed identical for this particular specialisation.

2. Macdonald Polynomials

2.1. **Definitions and notation.** Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \dots$ with finitely many λ_i unequal to zero. The length and weight of λ, denoted by $l(\lambda)$ and $|\lambda|$, are the number and sum of the nonzero λ_i, respectively. Two partitions that differ only in their string of zeros are identified, and the unique partition of length (and weight) 0 is itself denoted by 0. The multiplicity of the part i in the partition λ is denoted by $m_i=m_i(\lambda)$, and occasionally we will write $\lambda=(1^{m_1}2^{m_2}\dots)$.

We identify a partition with its diagram or Ferrers graph, defined by the set of points in $(i,j)\in\mathbb{Z}^2$ such that $1\leq j\leq\lambda_i$. The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal, so that, in particular, $m_i(\lambda)=\lambda'_i-\lambda'_{i+1}$. The statistic $n(\lambda)$ is given by

$$n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i=\sum_{i\geq 1}\binom{\lambda'_i}{2}.$$

The dominance partial order on the set of partitions of N is defined by $\lambda\geq\mu$ if $\lambda_1+\cdots+\lambda_i\geq\mu_1+\cdots+\mu_i$ for all $i\geq 1$. If $\lambda\geq\mu$ and $\lambda\neq\mu$ then $\lambda>\mu$.

If λ and μ are partitions then $\mu\subseteq\lambda$ if (the diagram of) μ is contained in (the diagram of) λ, i.e., $\mu_i\leq\lambda_i$ for all $i\geq 1$.

For $s=(i,j)\in\lambda$ the integers $a(s)$, $a'(s)$, $l(s)$ and $l'(s)$, known as the arm-length, arm-colength, leg-length and leg-colength of s, are defined as

$$a(s)=\lambda_i-j, \qquad a'(s)=j-1,$$
$$l(s)=\lambda'_j-i, \qquad l'(s)=i-1.$$

Note that $n(\lambda)=\sum_{s\in\lambda} l(s)$. Using the above we define the generalised hook-length polynomials c_λ and c'_λ as

$$c_\lambda = c_\lambda(q,t) := \prod_{s\in\lambda}\bigl(1-q^{a(s)}t^{l(s)+1}\bigr),$$
$$c'_\lambda = c'_\lambda(q,t) := \prod_{s\in\lambda}\bigl(1-q^{a(s)+1}t^{l(s)}\bigr).$$

The ordinary q-shifted factorial are given by

$$(a)_\infty = (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$$

and

$$(b)_z = (a;q)_z := \frac{(b)_\infty}{(bq^z)_\infty}.$$

Note in particular that for N a positive integer $(b)_N=(1-b)(1-bq)\cdots(1-bq^{N-1})$, and $1/(q)_{-N}=0$. Also note that $c'_{(k)}=(q)_k$. The q-shifted factorials can be generalised to allow for a partition as indexing set:

$$(b)_\lambda = (b;q,t)_\lambda := \prod_{s\in\lambda}\bigl(1-bq^{a'(s)}t^{-l'(s)}\bigr) = \prod_{i=1}^{l(\lambda)}(bt^{1-i})_{\lambda_i}.$$

With this notation,

$$c_\lambda = (t^n)_\lambda \prod_{1\le i<j\le n}\frac{(t^{j-i})_{\lambda_i-\lambda_j}}{(t^{j-i+1})_{\lambda_i-\lambda_j}}, \tag{2.1a}$$

$$c'_\lambda = (qt^{n-1})_\lambda \prod_{1\le i<j\le n}\frac{(qt^{j-i-1})_{\lambda_i-\lambda_j}}{(qt^{j-i})_{\lambda_i-\lambda_j}}, \tag{2.1b}$$

where n is an arbitrary integer such that $n\ge l(\lambda)$. We also introduce the usual condensed notation

$$(a_1,\dots,a_k)_N = (a_1)_N\cdots(a_k)_N$$

and likewise for q-shifted factorials indexed by partitions.

2.2. **Macdonald polynomials.** Let $\mathfrak{S}_n$ denote the symmetric group, and $\Lambda_n=\mathbb{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}$ the ring of symmetric polynomials in n independent variables.

For $X=\{x_1,\dots,x_n\}$ and $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition of length at most n the monomial symmetric function $m_\lambda(X)$ is defined as

$$m_\lambda(X)=\sum_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

where the sum is over all distinct permutations $\alpha=(\alpha_1,\dots,\alpha_n)$ of λ. If $l(\lambda)>n$ then $m_\lambda(X):=0$. The monomial symmetric functions $m_\lambda(X)$ for $l(\lambda)\le n$ form a $\mathbb{Z}$-basis of Λ_n.

A $\mathbb{Q}$-basis of Λ_n is given by the power-sum symmetric functions $p_\lambda(X)$, defined as

$$p_r(X)=\sum_{i=1}^n x_i^r$$

for $r\ge 0$ and $p_\lambda(X)=p_{\lambda_1}(X)\cdots p_{\lambda_n}(X)$. The power-sum symmetric functions may be used to define an extremely powerful notational tool in symmetric-function theory, known as plethystic or λ-ring notation, see [7, 14]. First we define the plethystic bracket by

$$f[x_1+\cdots+x_n]=f(x_1,\dots,x_n)$$

where f is a symmetric function. More simply we just write

$$f[X]=f(X)$$

where on the left we assume the additive notation for sets (or alphabets), i.e., $X=x_1+\cdots+x_n$ and on the right the more conventional $X=\{x_1,\dots,x_n\}$. With this notation $f[X+Y]$ takes on the obvious meaning of the symmetric function f acting on the disjoint union of the alphabets X and Y. Plethystic notation also allows for the definition of symmetric functions acting on differences $X-Y$ of alphabets, or for symmetric functions acting on such alphabets as $(X-Y)/(1-t)$, see e.g., [14]. In this paper we repeatedly need this last alphabet when both X and Y contain a single letter, say a and b, respectively. We may then take as definition

$$p_r\left[\frac{a-b}{1-t}\right]=\frac{a^r-b^r}{1-t^r},$$

and extend this by linearity to any symmetric function. Note in particular that

$$f\left[\frac{1-t^n}{1-t}\right]=f(t^{n-1},\dots,t,1)=:f(\langle 0\rangle)$$

corresponds to the so-called principal specialisation, where more generally,

$$\langle\lambda\rangle=\langle\lambda\rangle_n:=(q^{\lambda_1}t^{n-1},q^{\lambda_2}t^{n-2},\dots,q^{\lambda_n}t^0),$$

for $l(\lambda)\le n$.

After this digression we turn to the definition of the Macdonald polynomials and to some of its basic properties [15, 16]. First we define the scalar product $\langle\cdot,\cdot\rangle$ on symmetric functions by

$$\langle p_\lambda,p_\mu\rangle=\delta_{\lambda\mu}z_\lambda\prod_{i=1}^n\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

where $z_\lambda=\prod_{i\ge 1}m_i!\,i^{m_i}$ and $m_i=m_i(\lambda)$. If we denote the ring of symmetric functions in n variables over the field $\mathbb{F}=\mathbb{Q}(q,t)$ of rational functions in q and t by $\Lambda_{n,\mathbb{F}}$, then the Macdonald polynomial $P_\lambda(X)=P_\lambda(X;q,t)$ is the unique symmetric polynomial in $\Lambda_{n,\mathbb{F}}$ such that:

$$P_\lambda(X)=m_\lambda(X)+\sum_{\mu<\lambda}u_{\lambda\mu}m_\mu(X)$$

(where $u_{\lambda\mu}=u_{\lambda\mu}(q,t)$) and

$$\langle P_\lambda,P_\mu\rangle=0\quad\text{if}\quad\lambda\ne\mu.$$

The Macdonald polynomials $P_\lambda(X)$ with $l(\lambda)\le n$ form an $\mathbb{F}$-basis of $\Lambda_{n,\mathbb{F}}$. If $l(\lambda)>n$ then $P_\lambda(X):=0$. From the definition it follows that $P_\lambda(X)$ for $l(\lambda)\le n$ is homogeneous of (total) degree $|\lambda|$; $P_\lambda(zX)=z^{|\lambda|}P_\lambda(X)$. A second Macdonald polynomial $Q_\lambda(X)=Q_\lambda(X;q,t)$ is defined as

$$Q_\lambda(X)=b_\lambda P_\lambda(X),$$

where $b_\lambda=b_\lambda(q,t):=c_\lambda/c'_\lambda$. Then

$$\langle P_\lambda,Q_\mu\rangle=\delta_{\lambda\mu}.$$

This last result may equivalently be stated as the Cauchy identity

$$\sum_{\lambda} P_\lambda(X) Q_\lambda(Y) = \prod_{i,j=1}^{n} \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}.$$

We also need the skew Macdonald polynomials $P_{\lambda/\mu}(X)$ and $Q_{\lambda/\mu}(X)$ given by

$$P_\lambda[X+Y] = \sum_\lambda P_{\lambda/\mu}[X] P_\mu[Y]$$
$$Q_\lambda[X+Y] = \sum_\lambda Q_{\lambda/\mu}[X] Q_\mu[Y],$$

so that $P_{\lambda/0}(X) = P_\lambda(X)$ and $Q_{\lambda/\mu}(X) = b_\lambda b_\mu^{-1} P_{\lambda/\mu}(X)$. Equivalently,

$$Q_{\lambda/\mu}(X) = \sum_\nu f_{\mu\nu}^{\lambda} Q_\nu(X),$$

where $f_{\mu\nu}^{\lambda} = f_{\mu\nu}^{\lambda}$ are the q,t-Littlewood–Richardson coefficients:

$$P_\mu(X) P_\nu(X) = \sum_\lambda f_{\mu\nu}^{\lambda} P_\lambda(X).$$

From the homogeneity of the Macdonald polynomial it immediately follows that $f_{\mu\nu}^{\lambda}(q,t) = 0$ if $|\lambda| \neq |\mu| + |\nu|$. It may also be shown that $f_{\mu\nu}^{\lambda}(q,t) = 0$ if $\mu, \nu \not\subseteq \lambda$, so that $P_{\lambda/\mu}(X)$ vanishes if $\mu \not\subseteq \lambda$.

To conclude this section we introduce normalisations of the Macdonald polynomials convenient for dealing with basic hypergeometric series with Macdonald polynomial argument:

$$\mathsf{P}_{\lambda/\mu}(X) = t^{n(\lambda)-n(\mu)} \frac{c'_\mu}{c'_\lambda} P_{\lambda/\mu}(X) \tag{2.2a}$$

$$\mathsf{Q}_{\lambda/\mu}(X) = t^{n(\mu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu} Q_{\lambda/\mu}(X). \tag{2.2b}$$

Note that

$$\mathsf{Q}_{\lambda/\mu}(X) = t^{2n(\mu)-2n(\lambda)} \frac{c_\lambda c'_\lambda}{c_\mu c'_\mu} \mathsf{P}_{\lambda/\mu}(X). \tag{2.3}$$

If we also normalise the q,t-Littlewood–Richardson coefficients as

$$\mathsf{f}_{\mu\nu}^{\lambda} = t^{n(\mu)+n(\nu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu c'_\nu} f_{\mu\nu}^{\lambda},$$

then all of the preceding formulae have perfect analogues:

$$\mathsf{P}_\lambda[X+Y] = \sum_\mu \mathsf{P}_{\lambda/\mu}[Y] \mathsf{P}_\mu[X], \tag{2.4a}$$

$$\mathsf{Q}_\lambda[X+Y] = \sum_\mu \mathsf{Q}_{\lambda/\mu}[Y] \mathsf{Q}_\mu[X], \tag{2.4b}$$

$$\sum_\lambda \mathsf{P}_\lambda(X) \mathsf{Q}_\lambda(Y) = \prod_{i,j=1}^{n} \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}, \tag{2.5}$$

$$\mathsf{Q}_{\lambda/\mu}(X) = \sum_\nu \mathsf{f}_{\mu\nu}^{\lambda} \mathsf{Q}_\nu(X) \tag{2.6}$$

and

$$\mathsf{P}_\mu(X)\mathsf{P}_\nu(X)=\sum_\lambda \mathsf{f}^\lambda_{\mu\nu}\mathsf{P}_\lambda(X). \tag{2.7}$$

2.3. **Generalised evaluation symmetry.** One of the many striking results in Macdonald polynomial theory — first proved in unpublished work by Koornwinder — is the evaluation symmetry

$$\frac{\mathsf{P}_\lambda(\langle\mu\rangle)}{\mathsf{P}_\lambda(\langle 0\rangle)}=\frac{\mathsf{P}_\mu(\langle\lambda\rangle)}{\mathsf{P}_\mu(\langle 0\rangle)}, \tag{2.8}$$

where λ and μ are partitions of length at most n. As we shall see in Section 2.5, a simple generalisation of this result is the key to proving Theorem 1.2. Before stating this generalisation we put (2.8) in plethystic notation as

$$\mathsf{P}_\lambda\Big[\frac{1-t^n}{1-t}\Big]\mathsf{P}_\mu\big[\langle\lambda\rangle\big]=\mathsf{P}_\mu\Big[\frac{1-t^n}{1-t}\Big]\mathsf{P}_\lambda\big[\langle\mu\rangle\big],$$

where

$$f[\langle\lambda\rangle]=f\big[q^{\lambda_1}t^{n-1}+\cdots+q^{\lambda_n}t^0\big]=f\big(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n}t^0\big)=f(\langle\lambda\rangle).$$

Proposition 2.1 (Generalised evaluation symmetry. I)**.** *For λ and μ partitions of length at most n,*

$$\mathsf{P}_\lambda\Big[\frac{1-at^{nm}}{1-t}\Big]\mathsf{P}_\mu\Big[a\langle\lambda\rangle+\frac{1-a}{1-t}\Big]=\mathsf{P}_\mu\Big[\frac{1-at^{nm}}{1-t}\Big]\mathsf{P}_\lambda\Big[a\langle\mu\rangle+\frac{1-a}{1-t}\Big]. \tag{2.9}$$

Proof. Both sides are polynomials in a of degree $|\lambda|+|\mu|$ with coefficients in $\mathbb{Q}(q,t)$. It thus suffices to verify (2.9) for $a=t^p$, where p ranges over the nonnegative integers. We now write $\langle\lambda\rangle=\langle\lambda\rangle_n$ and use that

$$f\Big[a\langle\lambda\rangle_n+\frac{1-a}{1-t}\Big]\bigg|_{a=t^p}=f\big[\langle\lambda\rangle_{n+p}\big],$$

where, since $l(\lambda)\le n$,

$$f\big[\langle\lambda\rangle_{n+p}\big]=f\big(q^{\lambda_1}t^{n+p-1},\dots,q^{\lambda_n}t^p,t^{p-1},\dots,t^0\big).$$

As a result we obtain

$$\mathsf{P}_\lambda\big(\langle 0\rangle_{n+p}\big)\mathsf{P}_\mu\big(\langle\lambda\rangle_{n+p}\big)=\mathsf{P}_\mu\big(\langle 0\rangle_{n+p}\big)\mathsf{P}_\lambda\big(\langle\mu\rangle_{n+p}\big)$$

which follows from ordinary evaluation symmetry for Macdonald polynomials on $(n+p)$-letter alphabets. □

The generalised evaluation symmetry can also be stated without resorting to plethystic notation as a symmetry for skew Macdonald polynomials.

Proposition 2.2 (Generalised evaluation symmetry. II)**.**

$$(at^n)_\lambda\sum_\nu(a)_\nu\mathsf{Q}_{\mu/\nu}(a\langle\lambda\rangle)=(at^n)_\mu\sum_\nu(a)_\nu\mathsf{Q}_{\lambda/\nu}(a\langle\mu\rangle). \tag{2.10}$$

When $a=1$ both sums vanish unless $\nu=0$. Thanks to the principal specialisation formula [16, page 337]

$$\mathsf{Q}_\lambda(\langle 0\rangle)=(t^n)_\lambda \tag{2.11}$$

the $a=1$ case of (2.10) thus corresponds to (2.8) in the equivalent form

$$\frac{\mathsf{Q}_\lambda(\langle\mu\rangle)}{\mathsf{Q}_\lambda(\langle 0\rangle)}=\frac{\mathsf{Q}_\mu(\langle\lambda\rangle)}{\mathsf{Q}_\mu(\langle 0\rangle)}.$$

Proof of Proposition 2.2. By changing normalisation we may replace $(\mathsf{P}_\lambda, \mathsf{P}_\mu)$ in (2.9) by $(\mathsf{Q}_\lambda, \mathsf{Q}_\mu)$. Using [16, page 338]

$$(a)_\lambda = \mathsf{Q}_\lambda\left[\frac{1-a}{1-t}\right] \tag{2.12}$$

(for $a = t^n$ this is (2.11)) and (2.4b) this gives rise to

$$(at^n)_\lambda \sum_\nu \mathsf{Q}_{\mu/\nu}\big[a\langle\lambda\rangle\big]\mathsf{Q}_\nu\left[\frac{1-a}{1-t}\right] = (at^n)_\mu \sum_\nu \mathsf{Q}_{\lambda/\nu}\big[a\langle\mu\rangle\big]\mathsf{Q}_\nu\left[\frac{1-a}{1-t}\right].$$

Once again using (2.12) and dispensing with the remaining plethystic brackets yields (2.10). □

2.4. $\mathfrak{sl}_n$ **basic hypergeometric series.** Before we deal with the most important application of the generalised evaluation symmetry — the proof of Theorem 1.2 — we will show how it implies a multivariable generalisation of Heine's transformation formula.

Let

$$\tau_\lambda = \tau_\lambda(q,t) := (-1)^{|\lambda|} q^{n(\lambda')} t^{-n(\lambda)}$$

and $X = \{x_1, \dots, x_n\}$. Then the $\mathfrak{sl}_n$ basic hypergeometric series ${}_r\Phi_s$ is defined as

$${}_r\Phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; X\right] = \sum_\lambda \frac{(a_1, \dots, a_r)_\lambda}{(b_1, \dots, b_s)_\lambda}\, \tau_\lambda^{s-r+1}\, \mathsf{P}_\lambda(X). \tag{2.13}$$

For $n = 1$ this is in accordance with the standard definition of single-variable basic hypergeometric series ${}_r\phi_s$ as may be found in [1, 5]:

$$\begin{aligned} {}_r\Phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; \{z\}\right] &= \sum_{k=0}^\infty \frac{(a_1, \dots, a_r)_k}{(q, b_1, \dots, b_s)_k} \Big((-1)^k q^{\binom{k}{2}}\Big)^{s-r+1} z^k \\ &= {}_r\phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right], \end{aligned}$$

where in the second line the q-dependence of the ${}_r\phi_s$ series has been suppressed.

Theorem 2.3 ($\mathfrak{sl}_n$–$\mathfrak{sl}_m$ transformation formula)**.** *Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. Then*

$$\begin{aligned} &{}_{m+1}\Phi_m\left[\begin{matrix} a, ay_1/t, \dots, ay_m/t \\ ay_1, \dots, ay_m \end{matrix}; X\right] \\ &\qquad = \bigg(\prod_{i=1}^n \frac{(ax_i)_\infty}{(x_i)_\infty}\bigg)\bigg(\prod_{j=1}^m \frac{(y_j)_\infty}{(ay_j)_\infty}\bigg)\, {}_{n+1}\Phi_n\left[\begin{matrix} a, ax_1/t, \dots, ax_n/t \\ ax_1, \dots, ax_n \end{matrix}; Y\right]. \end{aligned}$$

For $m = 0$ this is the q-binomial theorem for Macdonald polynomials [11, 17]

$${}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; X\right] = \prod_{i=1}^n \frac{(ax_i)_\infty}{(x_i)_\infty} \tag{2.14}$$

and for $m = n = 1$ it is Heine's ${}_2\phi_1$ transformation formula [5, Equation (III.2)]

$${}_2\phi_1\left[\begin{matrix} a, ay/t \\ ay \end{matrix}; x\right] = \frac{(y, ax)_\infty}{(x, ay)_\infty}\, {}_2\phi_1\left[\begin{matrix} a, ax/t \\ ax \end{matrix}; y\right].$$

Proof of Theorem 2.3. First assume that $m=n$, multiply (2.10) by $\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y)$ and sum over λ and μ to get

$$\sum_{\nu,\mu,\lambda}(a)_\nu(at^n)_\lambda\mathsf{Q}_{\mu/\nu}(a\langle\lambda\rangle)\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y) \\ =\sum_{\nu,\mu,\lambda}(a)_\nu(at^n)_\mu\mathsf{Q}_{\lambda/\nu}(a\langle\mu\rangle)\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y). \tag{2.15}$$

If we multiply (2.7) by $\mathsf{P}_\nu(Y)$ and sum over ν then (2.5) and (2.6) permit this ν-sum to be carried out explicitly on both sides. As a result we obtain the skew Cauchy identity (see also [16, page 352])

$$\sum_\lambda \mathsf{P}_\lambda(X)\mathsf{Q}_{\lambda/\mu}(Y)=\mathsf{P}_\mu(X)\prod_{i,j=1}^n\frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}. \tag{2.16}$$

Applying this to (2.15) we can perform the sum over μ on the left and the sum over λ on the right, leading to

$$\sum_{\nu,\lambda}(a)_\nu(at^n)_\lambda\mathsf{P}_\lambda(X)\mathsf{P}_\nu(Y)\prod_{i,j=1}^n\frac{(aty_i\langle\lambda\rangle_j)_\infty}{(ay_i\langle\lambda\rangle_j)_\infty} \\ =\sum_{\nu,\mu}(a)_\nu(at^n)_\mu\mathsf{P}_\mu(Y)\mathsf{P}_\nu(X)\prod_{i,j=1}^n\frac{(atx_i\langle\mu\rangle_j)_\infty}{(ax_i\langle\mu\rangle_j)_\infty}.$$

Using the q-binomial theorem (2.14) to perform both sums over ν gives

$$\sum_\lambda(at^n)_\lambda\mathsf{P}_\lambda(X)\prod_{i=1}^n\frac{(ay_i)_\infty}{(y_i)_\infty}\prod_{i,j=1}^n\frac{(aty_i\langle\lambda\rangle_j)_\infty}{(ay_i\langle\lambda\rangle_j)_\infty} \\ =\sum_\mu(at^n)_\mu\mathsf{P}_\mu(Y)\prod_{i=1}^n\frac{(ax_i)_\infty}{(x_i)_\infty}\prod_{i,j=1}^n\frac{(atx_i\langle\mu\rangle_j)_\infty}{(ax_i\langle\mu\rangle_j)_\infty}.$$

Simplifying the products and replacing $a\mapsto at^{-n}$ completes the proof of the theorem for $m=n$.

The general m,n case trivially follows from $m=n$; assuming without loss of generality that $m\le n$ we set $y_{m+1},\dots,y_n=0$ and use that

$$\mathsf{P}_\lambda(y_1,\dots,y_m,\underbrace{0,\dots,0}_{n-m})=\begin{cases}\mathsf{P}_\lambda(y_1,\dots,y_m) & \text{if } l(\lambda)\le m,\\ 0 & \text{if } l(\lambda)>m.\end{cases}$$ □

2.5. **Proof of Theorem 1.2.** Using the generalised evaluation symmetry to prove Theorem 1.2 is much more difficult than the proof of Theorem 2.3, and we proceed by first proving an identity for skew Macdonald polynomials.

Theorem 2.4. *For λ and μ partitions of length at most n,*

$$\sum_\nu t^{-|\nu|}\mathsf{P}_{\mu/\nu}\Big[\frac{1-a}{1-t}\Big]\mathsf{Q}_{\lambda/\nu}\Big[\frac{1-q/at}{1-t}\Big] \\ =t^{-n|\mu|}\mathsf{P}_\mu\Big[\frac{1-at^n}{1-t}\Big]\mathsf{Q}_\lambda\Big[\frac{1-qt^{n-1}/a}{1-t}\Big]\prod_{i,j=1}^n\frac{(qt^{j-i-1}/a)_{\lambda_i-\mu_j}}{(qt^{j-i}/a)_{\lambda_i-\mu_j}}.$$

Recalling (2.12) and (2.3) it follows that the right-hand side is completely factorised. Moreover, for $a = 1$ the summand vanishes unless $\nu = \mu$ so that we recover the known factorisation of $\mathsf{Q}_{\lambda/\mu}[(1-q/t)/(1-t)]$, see [21, Equation (8.20)] or [28, Proposition 3.2]:

$$\mathsf{Q}_{\lambda/\mu}\left[\frac{1-q/t}{1-t}\right] = t^{(1-n)|\mu|}(qt^{n-1})_\lambda \mathsf{P}_\mu(\langle 0\rangle)\prod_{i,j=1}^{n}\frac{(qt^{j-i-1})_{\lambda_i-\mu_j}}{(qt^{j-i})_{\lambda_i-\mu_j}}.$$

Proof. In the first few steps we follow the proof of Theorem 2.3 but in an asymmetric manner. That is, we take (2.10), multiply both sides by $\mathsf{P}_\lambda(X)$ and sum over λ. By the Cauchy identity (2.16) followed by the q-binomial theorem (2.14) we can perform both sums on the right to find

$$\sum_{\lambda,\nu}(at^n)_\lambda(a)_\nu \mathsf{Q}_{\mu/\nu}(a\langle\lambda\rangle)\mathsf{P}_\lambda(X) = (at^n)_\mu\prod_{i=1}^{n}\frac{(ax_i)_\infty}{(x_i)_\infty}\prod_{i,j=1}^{n}\frac{(atx_i\langle\mu\rangle_j)_\infty}{(ax_i\langle\mu\rangle_j)_\infty}. \tag{2.17}$$

On the left we use (2.12) and (2.4b) (twice) to rewrite the sum over ν as

$$\sum_\nu (a)_\nu \mathsf{Q}_{\mu/\nu}(a\langle\lambda\rangle) = \mathsf{Q}_\mu\left[a\langle\lambda\rangle + \frac{1-a}{1-t}\right] = \sum_\nu \mathsf{Q}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]\mathsf{Q}_\nu(a\langle\lambda\rangle).$$

On the right we use (2.12) to trade $(at^n)_\mu$ for $Q_\mu[(1-at^n)/(1-t)]$. Also renaming the summation index λ as ω, (2.17) thus takes the form

$$\sum_{\nu,\omega}(at^n)_\omega \mathsf{Q}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]\mathsf{Q}_\nu(a\langle\omega\rangle)\mathsf{P}_\omega(X) = \mathsf{Q}_\mu\left[\frac{1-at^n}{1-t}\right]\prod_{i=1}^{n}\frac{(ax_i)_\infty}{(x_i)_\infty}\prod_{i,j=1}^{n}\frac{(atx_i\langle\mu\rangle_j)_\infty}{(ax_i\langle\mu\rangle_j)_\infty}.$$

By (2.3) it readily follows that we may replace all occurrences of Q in the above by P. Then specialising $X = b\langle\lambda\rangle$ we find

$$\begin{aligned}\sum_{\nu,\omega}(at^n)_\omega \mathsf{P}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]&\mathsf{P}_\nu(a\langle\omega\rangle)\mathsf{P}_\omega(b\langle\lambda\rangle)\\ &= \mathsf{P}_\mu\left[\frac{1-at^n}{1-t}\right]\prod_{i=1}^{n}\frac{(ab\langle\lambda\rangle_i)_\infty}{(b\langle\lambda\rangle_i)_\infty}\prod_{i,j=1}^{n}\frac{(abt\langle\lambda\rangle_i\langle\mu\rangle_j)_\infty}{(ab\langle\lambda\rangle_i\langle\mu\rangle_j)_\infty}.\end{aligned} \tag{2.18}$$

The next few steps focus on the left-hand side of this identity. First, by homogeneity followed by an application of the evaluation symmetry (2.8),

$$\mathrm{LHS}(2.18) = \sum_{\nu,\omega} a^{|\nu|}b^{|\omega|}(at^n)_\omega \mathsf{P}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]\frac{\mathsf{P}_\omega(\langle 0\rangle)}{\mathsf{P}_\lambda(\langle 0\rangle)}\,\mathsf{P}_\lambda(\langle\omega\rangle)\mathsf{P}_\nu(\langle\omega\rangle).$$

Using (2.7) this can be further rewritten as

$$\mathrm{LHS}(2.18) = \sum_{\eta,\nu,\omega} a^{|\nu|}b^{|\omega|}(at^n)_\omega\, \mathsf{f}^{\eta}_{\lambda\nu}\mathsf{P}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]\frac{\mathsf{P}_\omega(\langle 0\rangle)}{\mathsf{P}_\lambda(\langle 0\rangle)}\,\mathsf{P}_\eta(\langle\omega\rangle).$$

By another appeal to evaluation symmetry this yields

$$\mathrm{LHS}(2.18) = \sum_{\eta,\nu,\omega} a^{|\nu|}b^{|\omega|}(at^n)_\omega\, \mathsf{f}^{\eta}_{\lambda\nu}\mathsf{P}_{\mu/\nu}\left[\frac{1-a}{1-t}\right]\frac{\mathsf{P}_\eta(\langle 0\rangle)}{\mathsf{P}_\lambda(\langle 0\rangle)}\,\mathsf{P}_\omega(\langle\eta\rangle).$$

The sum over ω can now be performed by (2.14) so that

$$\mathrm{LHS}(2.18)=\sum_{\eta,\nu}a^{|\nu|}\,\mathsf{f}^{\eta}_{\lambda\nu}\mathsf{P}_{\mu/\nu}\biggl[\frac{1-a}{1-t}\biggr]\frac{\mathsf{P}_{\eta}(\langle 0\rangle)}{\mathsf{P}_{\lambda}(\langle 0\rangle)}\prod_{i=1}^{n}\frac{(abt^{n}\langle\eta\rangle_i)_{\infty}}{(b\langle\eta\rangle_i)_{\infty}}.$$

Equating this with the right-hand side of (2.18), manipulating the (infinite) q-shifted factorials and finally replacing $b\mapsto bt^{1-n}$ we find

$$\sum_{\eta,\nu}a^{|\nu|}\frac{(b)_{\eta}}{(abt^{n})_{\eta}}\,\mathsf{f}^{\eta}_{\lambda\nu}\mathsf{P}_{\mu/\nu}\biggl[\frac{1-a}{1-t}\biggr]\mathsf{P}_{\eta}(\langle 0\rangle)\\
=\frac{(b)_{\lambda}}{(ab)_{\lambda}}\,\mathsf{P}_{\lambda}(\langle 0\rangle)\mathsf{P}_{\mu}\biggl[\frac{1-at^{n}}{1-t}\biggr]\prod_{i,j=1}^{n}\frac{(abt^{n-i-j+1})_{\lambda_i+\mu_j}}{(abt^{n-i-j+2})_{\lambda_i+\mu_j}}.$$

For $a=1$ the summand vanishes unless $\nu=\mu$ and we recover [28, Proposition 3.1].

Next we specialise $b=q^{-N}$ and then replace λ and η by their complements with respect to the rectangular partition (N^n). Denoting these complementary partitions by $\hat{\lambda}$ and $\hat{\eta}$, we have $\hat{\lambda}_i=N-\lambda_{n-i+1}$ for $1\leq i\leq n$ (and a similar relation between η and $\hat{\eta}$). Using the relations [28, pp. 259 & 263]

$$\mathsf{f}^{\hat{\eta}}_{\hat{\lambda}\nu}=(-q^{N}t^{1-n})^{|\lambda-\eta|}q^{n(\eta')-n(\lambda')}t^{n(\lambda)-n(\eta)}\mathsf{f}^{\lambda}_{\eta\nu}\,\frac{(q^{-N})_{\lambda}}{(q^{-N})_{\eta}}\,\frac{\mathsf{P}_{\lambda}(\langle 0\rangle)}{\mathsf{P}_{\eta}(\langle 0\rangle)},$$

$$\frac{(a)_{\hat{\lambda}}}{(b)_{\hat{\lambda}}}=\Bigl(\frac{b}{a}\Bigr)^{|\lambda|}\frac{(a)_{(N^n)}}{(b)_{(N^n)}}\,\frac{(q^{1-N}t^{n-1}/b)_{\lambda}}{(q^{1-N}t^{n-1}/a)_{\lambda}},$$

and

$$\mathsf{P}_{\hat{\lambda}}(\langle 0\rangle)=(-1)^{|\lambda|}q^{N|\lambda|-n(\lambda')}t^{2N\binom{n}{2}+n(\lambda)-2(n-1)|\lambda|}\,\frac{(q^{-N},qt^{n-1})_{\lambda}}{(qt^{n-1})_{(N^n)}}\,\mathsf{P}_{\lambda}(\langle 0\rangle),$$

as well as the fact that the summand vanishes unless $|\nu|+|\eta|=|\lambda|$, we end up with

$$\sum_{\eta,\nu}t^{-|\nu|}(q/at)_{\eta}\mathsf{f}^{\lambda}_{\eta\nu}\,\mathsf{P}_{\mu/\nu}\biggl[\frac{1-a}{1-t}\biggr]\\
=t^{-n|\mu|}(qt^{n-1}/a)_{\lambda}\mathsf{P}_{\mu}\biggl[\frac{1-at^{n}}{1-t}\biggr]\prod_{i,j=1}^{n}\frac{(qt^{j-i-1}/a)_{\lambda_i-\mu_j}}{(qt^{j-i}/a)_{\lambda_i-\mu_j}}.$$

By (2.6) and (2.12)

$$\sum_{\nu}(b)_{\nu}\mathsf{f}^{\lambda}_{\mu\nu}=\sum_{\nu}\mathsf{f}^{\lambda}_{\mu\nu}\mathsf{Q}_{\nu}\biggl[\frac{1-b}{1-t}\biggr]=\mathsf{Q}_{\lambda/\mu}\biggl[\frac{1-b}{1-t}\biggr], \tag{2.19}$$

so that the sum over η can be performed. By a final appeal to (2.12) the proof is done. □

Equipped with Theorem 2.4 it is not difficult to prove Theorem 1.2. To streamline the proof given below we first prepare an easy lemma.

Lemma 2.5. *For $X=\{x_1,\dots,x_n\}$ and μ a partition of length at most n,*

$$\sum_{\lambda}\mathsf{Q}_{\lambda/\mu}\biggl[\frac{a-b}{1-t}\biggr]\mathsf{P}_{\lambda}(X)=\mathsf{P}_{\mu}(X)\prod_{i=1}^{n}\frac{(bx_i)_{\infty}}{(ax_i)_{\infty}}$$

and

$$\sum_{\lambda} \mathsf{P}_{\lambda/\mu}\Big[\frac{a-b}{1-t}\Big]\mathsf{Q}_\lambda(X) = \mathsf{Q}_\mu(X)\prod_{i=1}^n \frac{(bx_i)_\infty}{(ax_i)_\infty}.$$

For $\mu = 0$ this is just the q-binomial theorem (2.14) for Macdonald polynomials.

Proof. By (2.3) the two identities are in fact one and the same result and we only need to prove the first claim. To achieve this we multiply (2.19) by $\mathsf{P}_\lambda(aX)$ and sum over λ. By (2.7) and homogeneity this yields

$$\mathsf{P}_\mu(X)\sum_\nu (b)_\nu \mathsf{P}_\nu(aX) = \sum_\lambda \mathsf{Q}_{\lambda/\mu}\Big[\frac{a-ab}{1-t}\Big]\mathsf{P}_\lambda(X).$$

On the left we can sum over ν using the q-binomial theorem (2.14) leading to the desired result (with $b \mapsto ab$). □

Proof of Theorem 1.2. Elementary manipulations show that the theorem is invariant under the simultaneous changes $n \leftrightarrow m$, $tX \leftrightarrow Y$ and $a \mapsto qt/a$. Without loss of generality we may thus assume that $m \le n$. But

$$(at^{n-1})_\lambda \prod_{i,j=1}^n \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}}\bigg|_{\mu_{m+1}=\cdots=\mu_n=0} = (at^{m-1})_\lambda \prod_{i=1}^n\prod_{j=1}^m \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}}$$

so that the case $m < n$ follows from the case $m = n$ by setting $y_{m+1} = \cdots = y_n = 0$.

In the remainder we assume that $m = n$, in which case the theorem simplifies to

$$\begin{aligned}\sum_{\lambda,\mu} t^{|\lambda|-n|\mu|}\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y)\,(at^{n-1})_\lambda(qt^n/a)_\mu &\prod_{i,j=1}^n \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}} \\ &= \prod_{i=1}^n \frac{(ax_i)_\infty}{(tx_i)_\infty}\prod_{j=1}^n \frac{(qy_j/a)_\infty}{(y_j)_\infty}\prod_{i,j=1}^n \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}.\end{aligned} \tag{2.20}$$

To prove this we take Theorem 2.4, replace $a \mapsto q/a$, multiply both sides by

$$t^{|\lambda|}\mathsf{P}_\lambda(X)\mathsf{Q}_\mu(Y)$$

and sum over λ and μ. Hence

$$\begin{aligned}&\sum_{\lambda,\mu,\nu} \mathsf{P}_\lambda(X)\mathsf{Q}_\mu(Y)\mathsf{P}_{\mu/\nu}\Big[\frac{1-q/a}{1-t}\Big]\mathsf{Q}_{\lambda/\nu}\Big[\frac{t-a}{1-t}\Big] \\ &\quad = \sum_{\lambda,\mu} t^{|\lambda|-n|\mu|}\mathsf{P}_\lambda(X)\mathsf{Q}_\mu(Y)(at^{n-1})_\lambda\,\mathsf{P}_\mu\Big[\frac{1-qt^n/a}{1-t}\Big]\prod_{i,j=1}^n \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}}.\end{aligned}$$

On the right we apply (2.3) and (2.12) to rewrite

$$\mathsf{Q}_\mu(Y)\mathsf{P}_\mu\Big[\frac{1-qt^n/a}{1-t}\Big] = (qt^n/a)_\mu \mathsf{P}_\mu(Y),$$

and on the left we employ Lemma 2.5 to carry out the sums over λ and μ. Hence

$$\sum_{\nu} \mathsf{P}_\nu(X)\mathsf{Q}_\nu(Y)\prod_{i=1}^n \frac{(ax_i)_\infty}{(tx_i)_\infty}\prod_{j=1}^n \frac{(qy_j/a)_\infty}{(y_j)_\infty} \\ = \sum_{\lambda,\mu} t^{|\lambda|-n|\mu|}\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y)(at^{n-1})_\lambda(qt^n/a)_\mu \prod_{i,j=1}^n \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}}.$$

Performing the remaining sum on the left by (2.5) results in (2.20). □

We conclude this section with a remark about a generalisation of Theorem 1.2. Let $X=\{x_1,\dots,x_n\}$ and let λ be a partition of length n. Then

$$P_\lambda(X) = x_1\cdots x_n\, P_\mu(X),$$

where $\mu=(\lambda_1-1,\dots,\lambda_n-1)$. Now let $\mathcal{P}$ denote the set of weakly decreasing integer sequences of finite length. Then we may turn things around and use the above recursion to extend P_λ to all $\lambda\in\mathcal{P}$. It is then readily verified that

$$(qt^{n-1})_\lambda \mathsf{P}_\lambda(X) = t^{n(\lambda)}\frac{(qt^{n-1})_\lambda}{c'_\lambda}\, P_\lambda(X)$$

is well-defined for $\lambda\in\mathcal{P}$ (unlike $\mathsf{P}_\lambda(X)$).

We now state without proof the following generalisation of Theorem 1.2.

Theorem 2.6. *Let $X=\{x_1,\dots,x_n\}$ and $Y=\{y_1,\dots,y_m\}$. Then*

$$\sum_{\lambda\in\mathcal{P}}\sum_{\mu} t^{|\lambda|-n|\mu|}\mathsf{P}_\lambda(X)\mathsf{P}_\mu(Y)\frac{(at^{m-1},qt^{n-1})_\lambda(bt^n)_\mu}{(abt^{n-1})_\lambda}\prod_{i=1}^n\prod_{j=1}^m \frac{(at^{j-i-1})_{\lambda_i-\mu_j}}{(at^{j-i})_{\lambda_i-\mu_j}} \\ = \prod_{i=1}^n \frac{(qt^{i-1},bt^i,ax_i,q/ax_i)_\infty}{(abt^{i-1},qt^i/a,tx_i,b/x_i)_\infty}\prod_{j=1}^m\frac{(by_j)_\infty}{(y_j)_\infty}\prod_{i=1}^n\prod_{j=1}^m\frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}.$$

For $m=0$ this reduces to Kaneko's ${}_1\Psi_1$ sum for Macdonald polynomials [12] and for $n=0$ to the ${}_1\Phi_0$ sum (2.14). When $ab=q$ the summand on the left vanishes unless λ is an actual partition and we recover Theorem 1.2.

3. The $\mathfrak{sl}_3$ Selberg integral

3.1. **The integration chains $C^{k_1,k_2}_{\beta,\gamma}[0,1]$ and $C^{k_1,k_2}_{\gamma}[0,1]$.** Before proving Theorem 1.3 we give two descriptions of the chain $C^{k_1,k_2}_{\beta,\gamma}[0,1]$. We also identify the special case $\beta=1$ with the chain $C^{k_1,k_2}_{\gamma}[0,1]$ defined by Tarasov and Varchenko in [25].

Let

$$I^{k_1,k_2}[0,1]=\{(x_1,\dots,x_{k_1},y_1,\dots,y_{k_2})\in\mathbb{R}^{k_1+k_2}: \\ 0<x_1<\dots<x_{k_1}<1 \quad\text{and}\quad 0<y_1<\dots<y_{k_2}<1\}, \tag{3.1}$$

and fix a total ordering among the x_i and y_j as follows. Let $a=(a_1,\dots,a_{k_1})$ be a weakly increasing sequence of nonnegative integers not exceeding k_2:

$$0\le a_1\le\dots\le a_{k_1}\le k_2. \tag{3.2}$$

Then the domain $I_a^{k_1,k_2}[0,1] \subseteq I^{k_1,k_2}[0,1]$ is formed by imposing the additional inequalities

$$x_i < y_{a_i+1} < y_{a_i+2} < \cdots < y_{a_{i+1}} < x_{i+1} \quad \text{for } 0 \le i \le k_1,$$

where $x_0 := 0$, $x_{k_1+1} := 1$, $a_0 := 0$ and $a_{k_1+1} := k_2$. Equivalently,

$$\begin{cases} 0 < y_1 < y_2 < \cdots < y_{a_i} < x_i \\ x_i < y_{a_i+1} < \cdots < y_{k_2-1} < y_{k_2} < 1 \end{cases} \qquad \text{for } 1 \le i \le k_1. \tag{3.3}$$

Clearly, as a chain,

$$I^{k_1,k_2}[0,1] = \sum_a I_a^{k_1,k_2}[0,1], \tag{3.4}$$

where the sum is over all sequences $a = (a_1, \dots, a_{k_1})$ satisfying (3.2). To lift $I^{k_1,k_2}[0,1]$ to $C_{\beta,\gamma}^{k_1,k_2}[0,1]$ we replace the right-hand side of (3.4) by a weighted sum:

$$C_{\beta,\gamma}^{k_1,k_2}[0,1] = \sum_a \bigg(\prod_{i=1}^{k_1} \frac{\sin \pi(\beta - (i - a_i - k_1 + k_2)\gamma)}{\sin \pi(\beta - (i - k_1 + k_2)\gamma)} \bigg) I_a^{k_1,k_2}[0,1], \tag{3.5}$$

where it is assumed that $\beta, \gamma \in \mathbb{C}$ such that

$$\beta + (i - k_2 - 1)\gamma \notin \mathbb{Z} \quad \text{for } 1 \le i \le \min\{k_1, k_2\}.$$

This is a necessary and sufficient condition for

$$\prod_{i=1}^{k_1} \frac{\sin \pi(\beta - (i - a_i - k_1 + k_2)\gamma)}{\sin \pi(\beta - (i - k_1 + k_2)\gamma)}$$

to be free of poles for all admissible sequences a.

By viewing $(a_{k_1}, \dots, a_2, a_1)$ as a partition with largest part not exceeding k_2 and length not exceeding k_1, the operations of conjugation and/or complementation yield several alternative descriptions of the chain (3.5). Below we give one such description, reflecting the $\mathbb{Z}_2$ symmetry of Theorem 1.3 with respect to the interchange of the labels 1 and 2 in k_i, α_i and β_i.

Assume (3.1) and fix a total ordering among the x_i and y_j as follows. Let $b = (b_1, \dots, b_{k_2})$ be a weakly increasing sequence of nonnegative integers not exceeding k_1:

$$0 \le b_1 \le \cdots \le b_{k_2} \le k_1. \tag{3.6}$$

Then the domain $\bar{I}_b^{k_1,k_2}[0,1] \subseteq I^{k_1,k_2}[0,1]$ is formed by assuming the further inequalities

$$y_i < x_{b_i+1} < x_{b_i+2} < \cdots < x_{b_{i+1}} < y_{i+1} \quad \text{for } 0 \le i \le k_2,$$

where $y_0 := 0$, $y_{k_2+1} := 1$, $b_0 := 0$ and $b_{k_2+1} := k_1$. It is easily seen that if $\mu = (b_{k_2}, \dots, b_1)$ and $\lambda = (a_{k_1}, \dots, a_1)$, then μ' is the conjugate of λ with respect to $(k_2^{k_1})$, i.e., $\mu'_i = k_2 - \lambda_{k_1-i+1} = k_2 - a_i$ for $1 \le i \le k_1$. Hence, for a pair of admissible sequences (a, b) related by "conjugation–complementation",

$$\bar{I}_b^{k_1,k_2}[0,1] = I_a^{k_1,k_2}[0,1]$$

and

$$\prod_{i=1}^{k_2} \frac{\sin \pi(\beta + (i - b_i + k_1 - k_2 - 1)\gamma)}{\sin \pi(\beta + (i - k_2 - 1)\gamma)} = \prod_{i=1}^{k_1} \frac{\sin \pi(\beta - (i - a_i - k_1 + k_2)\gamma)}{\sin \pi(\beta - (i - k_1 + k_2)\gamma)}.$$

In other words,

$$C^{k_1,k_2}_{\beta,\gamma}[0,1] = \sum_b \biggl(\prod_{i=1}^{k_2} \frac{\sin\pi(\beta+(i-b_i+k_1-k_2-1)\gamma)}{\sin\pi(\beta+(i-k_2-1)\gamma)} \biggr) \bar{I}^{k_1,k_2}_b[0,1] \tag{3.7}$$

summed over all sequences $b=(b_1,\dots,b_{k_2})$ subject to (3.6). Comparing (3.5) and (3.7), and using that for $\beta_1+\beta_2=\gamma+1$,

$$\prod_{i=1}^{k_1} \frac{\sin\pi(\beta_1-(i-k_1+k_2))\gamma)}{\sin\pi(\beta_2+(i-k_1-1)\gamma)} = \prod_{i=0}^{k_1-1} \frac{\Gamma(\beta_1+i\gamma)}{\Gamma(\beta_1+(i-k_2)\gamma)} \prod_{i=0}^{k_2-1} \frac{\Gamma(\beta_2+(i-k_1)\gamma)}{\Gamma(\beta_2+i\gamma)}$$

it readily follows that the symmetry relation (1.5) holds.

To conclude this section we consider (3.5) for $\beta=1$:

$$C^{k_1,k_2}_{1,\gamma}[0,1] = \sum_a \biggl(\prod_{i=1}^{k_1} \frac{\sin\pi((i-a_i-k_1+k_2)\gamma)}{\sin\pi((i-k_1+k_2)\gamma)} \biggr) I^{k_1,k_2}_a[0,1].$$

The summand vanishes if $a_i=i-k_1+k_2$ for some $1\le i\le k_1$ so that we may add the additional restrictions

$$a_i \neq i-k_1+k_2 \qquad \text{for } 1\le i\le k_1$$

to the sum over a. Recalling (3.2) this in fact implies that the much stronger

$$a_i < i-k_1+k_2 \qquad \text{for } 1\le i\le k_1.$$

Therefore,

$$C^{k_1,k_2}_{1,\gamma}[0,1] = \sum_{\substack{a \\ a_i<i-k_1+k_2}} \biggl(\prod_{i=1}^{k_1} \frac{\sin\pi((i-a_i-k_1+k_2)\gamma)}{\sin\pi((i-k_1+k_2)\gamma)} \biggr) I^{k_1,k_2}_a[0,1].$$

Defining $M(i)=a_i+1$, so that

$$1\le M(1)\le M(2)\le\cdots\le M(k_1)\le k_2$$

and

$$M(i)\le i-k_i+k_2 \qquad \text{for } 1\le i\le k_1,$$

and writing $M=(M(1),\dots,M(k_1))$, we finally obtain

$$\begin{aligned} C^{k_1,k_2}_{1,\gamma}[0,1] &= \sum_M \biggl(\prod_{i=1}^{k_1} \frac{\sin\pi((i-M(i)-k_1+k_2+1)\gamma)}{\sin\pi((i-k_1+k_2)\gamma)} \biggr) I^{k_1,k_2}_M[0,1] \\ &=: C^{k_1,k_2}_{\gamma}[0,1]. \end{aligned} \tag{3.8}$$

In the above, by abuse of notation, $I^{k_1,k_2}_M[0,1]=I^{k_1,k_2}_a[0,1]$ if $M=(a_1+1,\dots,a_{k_1}+1)$. The chain $C^{k_1,k_2}_{\gamma}[0,1]$ is precisely that of Tarasov and Varchenko (up to an interchange of k_1 and k_2), see [25, page 177].

3.2. **Proof of Theorem 1.3.** We are now prepared to prove Theorem 1.3. In fact, we will prove a more general integral, generalising Kadell's extension of the Selberg integral [10] to $\mathfrak{sl}_3$. To shorten some of the subsequent equations we introduce another normalised Macdonald polynomial, and for $X = \{x_1, \dots, x_n\}$

$$\tilde{\mathsf{P}}_\lambda(X) = \frac{\mathsf{P}_\lambda(X)}{\mathsf{P}_\lambda(\langle 0\rangle)} = \frac{P_\lambda(X)}{P_\lambda(\langle 0\rangle)}.$$

Similarly we define a (normalised) Jack polynomial as

$$\tilde{\mathsf{P}}^{(\alpha)}_\lambda(X) = \lim_{q\to 1} \tilde{\mathsf{P}}_\lambda(X; q^\alpha, q).$$

Hence

$$\tilde{\mathsf{P}}^{(\alpha)}_\lambda(X) = \frac{P^{(\alpha)}_\lambda(X)}{P^{(\alpha)}_\lambda(1^n)},$$

where $P^{(\alpha)}_\lambda(X)$ is the Jack polynomial [16, 24].

Theorem 3.1. *Set* $X = \{x_1, \dots, x_{k_1}\}$*,* $Y = \{y_1, \dots, y_{k_2}\}$*,*

$$\mathrm{d}X = \mathrm{d}x_1 \cdots \mathrm{d}x_{k_1} \quad \textit{and} \quad \mathrm{d}Y = \mathrm{d}y_1 \cdots \mathrm{d}y_{k_1}.$$

For $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ *such that* $|\gamma|$ *is sufficiently small,*

$$\min\{\mathrm{Re}(\alpha_1) + \lambda_{k_1}, \mathrm{Re}(\alpha_2) + \mu_{k_2}, \mathrm{Re}(\beta_1), \mathrm{Re}(\beta_2)\} > 0,$$

$$\beta_1 + (i - k_2 - 1)\gamma \notin \mathbb{Z} \quad \textit{for } 1 \le i \le \min\{k_1, k_2\}$$

and

$$\beta_1 + \beta_2 = \gamma + 1$$

there holds

$$\begin{aligned}
\int\limits_{C^{k_1,k_2}_{\beta_1,\gamma}[0,1]} & \tilde{\mathsf{P}}^{(1/\gamma)}_\lambda(X) \tilde{\mathsf{P}}^{(1/\gamma)}_\mu(Y) \prod_{i=1}^{k_1} x_i^{\alpha_1 - 1}(1 - x_i)^{\beta_1 - 1} \prod_{i=1}^{k_2} y_i^{\alpha_2 - 1}(1 - y_i)^{\beta_2 - 1} \\
& \times \prod_{1\le i<j\le k_1} |x_i - x_j|^{2\gamma} \prod_{1\le i<j\le k_2} |y_i - y_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |x_i - y_j|^{-\gamma} \; \mathrm{d}X \, \mathrm{d}Y \\
&= \prod_{i=1}^{k_1} \frac{\Gamma(\alpha_1 + (k_1 - i)\gamma + \lambda_i)\Gamma(\beta_1 + (i - k_2 - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha_1 + \beta_1 + (2k_1 - k_2 - i - 1)\gamma + \lambda_i)\Gamma(\gamma)} \\
& \quad \times \prod_{i=1}^{k_2} \frac{\Gamma(\alpha_2 + (k_2 - i)\gamma + \mu_i)\Gamma(\beta_2 + (i - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha_2 + \beta_2 + (2k_2 - k_1 - i - 1)\gamma + \mu_i)\Gamma(\gamma)} \\
& \quad \times \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} \frac{\Gamma(\alpha_1 + \alpha_2 + (k_1 + k_2 - i - j - 1)\gamma + \lambda_i + \mu_j)}{\Gamma(\alpha_1 + \alpha_2 + (k_1 + k_2 - i - j)\gamma + \lambda_i + \mu_j)}.
\end{aligned}$$

Theorem 1.3 corresponds to the case special case $\lambda = \mu = 0$, and Kadell's integral arises by taking $k_1 = 0$ or $k_2 = 0$.

Proof. Throughout the proof $0<q<1$.

We take Theorem 1.2 with (λ,μ) replaced by (η,ν) and (n,m) replaced by (k_1,k_2). If we then specialise $X=z\langle\lambda\rangle_{k_1}$ and $Y=w\langle\mu\rangle_{k_2}$ and use the evaluation symmetry (2.8) on both Macdonald polynomials in the summand, we obtain

$$\begin{aligned}\sum_{\eta,\nu} t^{|\eta|-k_1|\nu|}\tilde{\mathsf{P}}_\lambda(\langle\eta\rangle_{k_1})\tilde{\mathsf{P}}_\mu(\langle\nu\rangle_{k_2})\mathsf{P}_\eta(z\langle 0\rangle_{k_1})\mathsf{P}_\nu(w\langle 0\rangle_{k_2})\\ \times (at^{k_2-1})_\lambda(qt^{k_1}/a)_\nu\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{(at^{j-i-1})_{\eta_i-\nu_j}}{(at^{j-i})_{\eta_i-\nu_j}}\\ =\prod_{i=1}^{k_1}\frac{(azq^{\lambda_i}t^{k_1-i})_\infty}{(zq^{\lambda_i}t^{k_1-i+1})_\infty}\prod_{j=1}^{k_2}\frac{(wq^{\mu_j+1}t^{k_2-j}/a)_\infty}{(wq^{\mu_j}t^{k_2-j})_\infty}\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{(wztq^{\lambda_i+\mu_j}t^{k_1+k_2-i-j})_\infty}{(wzq^{\lambda_i+\mu_j}t^{k_1+k_2-i-j})_\infty}.\end{aligned}$$

Next we set

$$(z,w,a,t)=(q^{\alpha_1-\gamma},q^{\alpha_2},q^{\beta_1+(k_1-k_2)\gamma},q^\gamma)$$

and introduce the auxiliary variable β_2 by $\beta_1+\beta_2=\gamma+1$. Equations (2.1), (2.3) and (2.12) imply the principal specialisation formula

$$\mathsf{P}_\lambda(\langle 0\rangle_n)=\frac{t^{2n(\lambda)}}{(qt^{n-1})_\lambda}\prod_{1\le i<j\le n}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i}}{1-t^{j-i}}\,\frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}}.$$

Using this as well as the definition of the q-Gamma function

$$\Gamma_q(x)=\frac{(q)_{x-1}}{(1-q)^{x-1}},\qquad x\in\mathbb{C},$$

we can rewrite the above identity as

$$\begin{aligned}(1-q)^{k_1+k_2}\sum_{\eta,\nu}\tilde{\mathsf{P}}_\lambda(x_1q^{(k_1-1)\gamma},x_2q^{(k_1-2)\gamma},\dots,x_{k_1})\\ \times\tilde{\mathsf{P}}_\mu(y_1q^{(k_2-1)\gamma},y_2q^{(k_2-2)\gamma},\dots,y_{k_1})\\ \times\prod_{i=1}^{k_1}x_i^{\alpha_1}(q^{1+(k_1-i)\gamma}x_i)_{\beta_1-1}\prod_{1\le i<j\le k_1}x_j^{2\gamma}\big(1-q^{(j-i)\gamma}x_i/x_j\big)(q^{1+(j-i-1)\gamma}x_i/x_j)_{2\gamma-1}\\ \times\prod_{i=1}^{k_2}y_i^{\alpha_2}(q^{1+(k_2-i)\gamma}y_i)_{\beta_2-1}\prod_{1\le i<j\le k_2}y_j^{2\gamma}\big(1-q^{(j-i)\gamma}y_i/y_j\big)(q^{1+(j-i-1)\gamma}y_i/y_j)_{2\gamma-1}\\ \times\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}y_j^{-\gamma}(q^{\beta_1+(k_1-k_2+j-i)\gamma}x_i/y_j)_{-\gamma}\\ =\prod_{i=1}^{k_1}\frac{\Gamma_q(\alpha_1+(k_1-i)\gamma+\lambda_i)\Gamma_q(\beta_1+(i-k_2-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\alpha_1+\beta_1+(2k_1-k_2-i-1)\gamma+\lambda_i)\Gamma_q(\gamma)}\\ \times\prod_{i=1}^{k_2}\frac{\Gamma_q(\alpha_2+(k_2-i)\gamma+\mu_i)\Gamma_q(\beta_2+(i-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\alpha_2+\beta_2+(2k_2-k_1-i-1)\gamma+\mu_i)\Gamma_q(\gamma)}\\ \times\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{\Gamma_q(\alpha_1+\alpha_2+(k_1+k_2-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma_q(\alpha_1+\alpha_2+(k_1+k_2-i-j)\gamma+\lambda_i+\mu_j)}.\end{aligned}$$

Here $x_i := q^{\eta_i}$ and $y_i := q^{\nu_i}$, so that

$$0 < x_1 < \cdots < x_{k_1} \le 1 \quad \text{and} \quad 0 < y_1 < \cdots < y_{k_1} \le 1. \tag{3.9}$$

The above is essentially a $(k_1 + k_2)$-dimensional q-integral (more on this in the next section) and all that remains is to let q tend to 1 from below. The resulting integrand, however, depends sensitively on the relative ordering between the x_i and y_j. Indeed [31],

$$\begin{aligned}\lim_{q\to 1^-} & y_j^{-\gamma}(q^{\beta_1+(k_1-k_2+j-i)\gamma}x_i/y_j)_{-\gamma} \\ &= |x_i - y_j|^{-\gamma} \times \begin{cases} 1 & \text{if } x_i < y_j \\ \dfrac{\sin\pi(\beta_1 - (i-j-k_1+k_2)\gamma)}{\sin\pi(\beta_1 - (i-j-k_1+k_2+1)\gamma)} & \text{if } x_i > y_j. \end{cases}\end{aligned}$$

Consequently, before we can take the required limit we must fix a complete ordering among the integration variables (compatible with (3.9)) and sum over all admissible orderings. This is exactly what is done at the beginning of this section and in the remainder we assume that

$$(x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}) \in I^{k_1,k_2}_{a_1,\dots,a_{k_1}}[0,1].$$

To find how to weigh this domain we recall that according to (3.3) $y_{a_i+1} > x_i > y_{a_i} > \cdots > y_1$. The correct weight is thus

$$\prod_{i=1}^{k_1}\prod_{j=1}^{a_i} \frac{\sin\pi(\beta_1 - (i-j-k_1+k_2)\gamma)}{\sin\pi(\beta_1 - (i-j-k_1+k_2+1)\gamma)} = \prod_{i=1}^{k_1} \frac{\sin\pi(\beta_1 - (i-a_i-k_1+k_2)\gamma)}{\sin\pi(\beta_1 - (i-k_1+k_2)\gamma)},$$

in accordance with $C^{k_1,k_2}_{\beta_1,\gamma}[0,1]$, see (3.5). □

4. The Askey–Habsieger–Kadell integral

For $0 < q < 1$ the q-integral on $[0,1]$ is defined as

$$\int_0^1 f(x)\mathrm{d}_q x = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k, \tag{4.1}$$

where it is assumed the series on the right converges. When $q \to 1^-$ the q-integral reduces, at least formally, to the Riemann integral of f on the unit interval. An obvious n-dimensional analogue of (4.1) is

$$\int\limits_{[0,1]^n} f(X)\mathrm{d}_q X = (1-q)^n \sum_{k_1,\dots,k_n=0}^{\infty} f(q^{k_1}, \dots, q^{k_n})q^{k_1+\cdots+k_n},$$

where the multiple sum on the right is assumed to be absolutely convergent and where $f(X) = f(x_1, \dots, x_n)$ and $\mathrm{d}_q X = \mathrm{d}_q x_1 \cdots \mathrm{d}_q x_n$.

In 1980 Askey [2] conjectured a q-analogue of the Selberg integral when the parameter γ is a nonnegative integer, say k:

$$\int\limits_{[0,1]^n} \prod_{i=1}^n x_i^{\alpha-1}(x_iq)_{\beta-1} \prod_{1\le i<j\le n} x_i^{2k}(q^{1-k}x_j/x_i)_{2k}\,\mathrm{d}_qX \\ = q^{\alpha k\binom{n}{2}+2k^2\binom{n}{3}} \prod_{i=0}^{n-1} \frac{\Gamma_q(\alpha+ik)\Gamma_q(\beta+ik)\Gamma_q(1+(i+1)k)}{\Gamma_q(\alpha+\beta+(n+i-1)k)\Gamma_q(1+k)},$$

for $\mathrm{Re}(\alpha)>0$ and $\beta\ne 0,-1,-2,\dots$. Askey's conjecture was proved independently by Habsieger [6] and Kadell [9].

Just as the ordinary Selberg integral, the Askey–Habsieger–Kadell integral can be generalised by the inclusion of symmetric functions in the integrand. Specifically, Kaneko [11] and Macdonald [16] proved that

$$\int\limits_{[0,1]^n} \tilde{\mathsf{P}}_\lambda(X;q,q^k) \prod_{i=1}^n x_i^{\alpha-1}(x_iq)_{\beta-1} \prod_{1\le i<j\le n} x_i^{2k}(q^{1-k}x_j/x_i)_{2k}\,\mathrm{d}_qX \\ = q^{\alpha k\binom{n}{2}+2k^2\binom{n}{3}} \prod_{i=1}^{n} \frac{\Gamma_q(\alpha+(n-i)k+\lambda_i)\Gamma_q(\beta+(i-1)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha+\beta+(2n-i-1)k+\lambda_i)\Gamma_q(k+1)}, \tag{4.2}$$

for $\mathrm{Re}(\alpha)>-\lambda_n$ and $\beta\ne 0,-1,-2,\dots$.

The $\mathfrak{sl}_n$–$\mathfrak{sl}_m$ transformation formula of Theorem 2.3 allows the for the Askey–Habsieger–Kadell integral as well as its generalisation (4.2) to be extended to a transformation between integrals of different dimensions. For $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_m)$ define

$$S_{\lambda\mu}^{(n,m)}(\alpha_1,\alpha_2,\beta;k) = \int\limits_{[0,1]^n} \tilde{\mathsf{P}}_\lambda(X;q,q^k) \prod_{i=1}^n x_i^{\alpha_1-1}(x_iq)_{\beta-(n-1)k-1} \\ \times \prod_{i=1}^n\prod_{j=1}^m \frac{(x_iq)_{\alpha_2+\beta+\mu_j+(m-n-j)k-1}}{(x_iq)_{\alpha_2+\beta+\mu_j+(m-n-j+1)k-1}} \prod_{1\le i<j\le n} x_i^{2k}(q^{1-k}x_j/x_i)_{2k}\,\mathrm{d}_qX.$$

Theorem 4.1. *Let $\lambda=(\lambda_1,\dots,\lambda_n)$, $\mu=(\mu_1,\dots,\mu_m)$ be partitions, k a nonnegative integer and $\alpha_1,\alpha_2,\beta\in\mathbb{C}$. Then*

$$S_{\lambda\mu}^{(n,m)}(\alpha_1,\alpha_2,\beta;k) = q^{\alpha_1k\binom{n}{2}-\alpha_2k\binom{m}{2}+2k^2\binom{n}{3}-2k^2\binom{m}{3}}\, S_{\mu\lambda}^{(m,n)}(\alpha_2,\alpha_1,\beta;k) \\ \times \prod_{i=1}^n \frac{\Gamma_q(\beta-(i-1)k)\Gamma_q(\alpha_1+\lambda_i+(n-i)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha_1+\beta+\lambda_i+(n-m-i)k)\Gamma_q(k+1)} \\ \times \prod_{i=1}^m \frac{\Gamma_q(\alpha_2+\beta+\mu_i+(m-n-i)k)\Gamma_q(k+1)}{\Gamma_q(\beta-(i-1)k)\Gamma_q(\alpha_2+\mu_i+(m-i)k)\Gamma_q(ik+1)}$$

for $\mathrm{Re}(\alpha_1)>-\lambda_n$, $\mathrm{Re}(\alpha_2)>-\mu_m$, and generic β.

By "generic β" it is meant that β should avoid a countable set of isolated singularities. More precisely, β should be such that none of $\beta-(n-1)k$, $\beta-(m-1)k$, $\alpha_1+\beta+\lambda_j+(n-m-j)k$ and $\alpha_2+\beta+\mu_j+(m-n-j)k$ take nonpositive integer values. Since $S_{0,\lambda}^{(0,n)}(\alpha_2,\alpha_1,\beta;k)=1$ the $m=0$ case of the theorem corresponds to (4.2) with $(\alpha,\beta)\mapsto(\alpha_1,\beta-(n-1)k)$.

Proof. The method of proof is identical to that employed in [28, Theorem 1.1] and we only sketch the details of what are essentially elementary manipulations.

We specialise $X \mapsto c\langle\lambda\rangle_n$ and $Y \mapsto b\langle\mu\rangle_m$ in Theorem 2.3 and apply the evaluation symmetry (2.8) to obtain

$$\begin{aligned}
&\sum_{\nu} \frac{(a, abq^{\mu_1}t^{m-2}, \dots, abq^{\mu_m}t^{-1})_\nu}{(abq^{\mu_1}t^{m-1}, \dots, abq^{\mu_m})_\nu} \mathsf{P}_\nu(c\langle 0\rangle_n)\tilde{\mathsf{P}}_\lambda(\langle\nu\rangle_n) \\
&\qquad = \bigg(\prod_{i=1}^n \frac{(acq^{\lambda_i}t^{n-i})_\infty}{(cq^{\lambda_i}t^{n-i})_\infty}\bigg)\bigg(\prod_{i=1}^m \frac{(bq^{\mu_i}t^{m-i})_\infty}{(abq^{\mu_i}t^{m-i})_\infty}\bigg) \\
&\qquad\qquad \times \sum_{\nu} \frac{(a, acq^{\lambda_1}t^{n-2}, \dots, acq^{\lambda_n}t^{-1})_\nu}{(acq^{\lambda_1}t^{n-1}, \dots, acq^{\lambda_n})_\nu} \mathsf{P}_\nu(b\langle 0\rangle_m)\tilde{\mathsf{P}}_\mu(\langle\nu\rangle_m).
\end{aligned}$$

Next we replace $t \mapsto q^k$ with k a positive integer and replace $a \mapsto q^\beta$, $b \mapsto q^{\alpha_2}$ and $c \mapsto q^{\alpha_1}$. Then we apply [28, Lemma 3.1] to write the ν-sums as n-fold unrestricted sums, and the claim follows. □

The above derivation can be repeated starting from Theorem 1.2. The result is an $\mathfrak{sl}_3$ variant of the q-integral (4.2). Problem with the theorem below is, however, that it does not converge in the $q \to 1^-$ limit unless m or n is 0. (This can be remedied by replacing $[0,1]^{m+n}$ by appropriate multiple Pochhammer double loops).

Theorem 4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_m)$ be partitions, k a nonnegative integer and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ such that $\beta_1 + \beta_2 = k + 1$. Then*

$$\begin{aligned}
&\int_{[0,1]^{n+m}} \tilde{\mathsf{P}}_\lambda(X; q, q^k) \prod_{i=1}^n x_i^{\alpha_1}(qx_i)_{\beta_1-1} \prod_{1\le i<j\le n} x_j^{2k}(q^{1-k}x_i/x_j)_{2k} \\
&\quad \times \tilde{\mathsf{P}}_\mu(Y; q, q^k) \prod_{i=1}^m y_i^{\alpha_2}(qy_i)_{\beta_2-1} \prod_{1\le i<j\le m} y_j^{2k}(q^{1-k}y_i/y_j)_{2k} \\
&\qquad\qquad \times \prod_{i=1}^n \prod_{j=1}^m y_j^{-k}(q^{\beta_1}x_i/y_j)_{-k} \, \mathrm{d}_q X \, \mathrm{d}_q Y \\
&= q^{\alpha_1 k\binom{n}{2} + \alpha_2 k\binom{m}{2} + 2k^2\binom{n}{3} + 2k^2\binom{n}{3} - k^2 n\binom{m}{2}} \\
&\quad \times \prod_{i=1}^n \frac{\Gamma_q(\alpha_1 + (n-i)k + \lambda_i)\Gamma_q(\beta_1 + (i-m-1)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha_1 + \beta_1 + (2n-m-i-1)k + \lambda_i)\Gamma_q(k)} \\
&\quad \times \prod_{i=1}^m \frac{\Gamma_q(\alpha_2 + (m-i)k + \mu_i)\Gamma_q(\beta_2 + (i-1)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha_2 + \beta_2 + (2m-n-i-1)k + \mu_i)\Gamma_q(k)} \\
&\quad \times \prod_{i=1}^n \prod_{j=1}^m \frac{\Gamma_q(\alpha_1 + \alpha_2 + (n+m-i-j-1)k + \lambda_i + \mu_j)}{\Gamma_q(\alpha_1 + \alpha_2 + (n+m-i-j)k + \lambda_i + \mu_j)},
\end{aligned}$$

for $\mathrm{Re}(\alpha_1) > -\lambda_n$, *for* $\mathrm{Re}(\alpha_2) > -\mu_m$, *and* $\beta_1, \beta_2 \neq 0, -1, -2, \dots$.

Acknowledgements. I thank Eric Rains for helpful discussions.

References

1. G.E. Andrews, R. Askey and R. Roy, Special functions, Encyclopedia Math. Appl., vol. 71, Cambridge University Press, Cambridge, 1999.

2. R. Askey, Some basic hypergeometric extensions of integrals of Selberg and Andrews, SIAM J. Math. Anal. 11 (1980), 938–951.
3. P.I. Etingof, I.B. Frenkel and A.A. Kirillov Jr., Lectures on Representation Theory and Knizhnik–Zamolodchikov Equations, Math. Surveys Monogr., vol. 58, Amer. Math. Soc., Providence, RI, 2003.
4. P. J. Forrester and S. O. Warnaar, The importance of the Selberg integral, Bull. Amer. Math. Soc. (N.S.) 45 (2008), 489–534.
5. G. Gasper and M. Rahman, Basic Hypergeometric Series, second edition, Encyclopedia Math. Appl., vol. 96, Cambridge University Press, Cambridge, 2004.
6. L. Habsieger, Une q-intégrale de Selberg et Askey, SIAM J. Math. Anal. 19 (1988), 1475–1489.
7. J. Haglund, The q,t-Catalan Numbers and the Space of Diagonal Harmonics, Univ. Lecture Ser., vol. 41, Amer. Math. Soc., Providence, RI, 2008.
8. S. M. Iguri, On a Selberg-Schur integral, Lett. Math. Phys. 89 (2009), 141–158.
9. K. W. J. Kadell, A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris, SIAM J. Math. Anal. 19 (1988), 969–986.
10. K. W. J. Kadell, The Selberg-Jack symmetric functions, Adv. Math. 130 (1997), 33–102.
11. J. Kaneko, q-Selberg integrals and Macdonald polynomials, Ann. Sci. École Norm. Sup. 29 (1996), 583–637.
12. J. Kaneko, A ${}_1\Psi_1$ summation theorem for Macdonald polynomials, Ramanujan J. 2 (1998), 379–386.
13. N. Kawanaka, A q-Cauchy identity for Schur functions and imprimitive complex reflection groups, Osaka J. Math. 38 (2001), 775–810.
14. A. Lascoux, Symmetric Functions and Combinatorial Operators on Polynomials, CBMS Reg. Conf. Ser. Math., vol. 99, Amer. Math. Soc., Providence, RI, 2003.
15. I. G. Macdonald, A new class of symmetric functions, Sém. Lothar. Combin. B20a (1988), 41pp.
16. I. G. Macdonald, Symmetric functions and Hall polynomials, second edition, Oxford University Press, New York, 1995.
17. I. G. Macdonald, Hypergeometric series II, arXiv:1309.5208.
18. M. L. Mehta, Random Matrices, third edition, Pure and Applied Mathematics, vol. 142, Elsevier/Academic Press, Amsterdam, 2004.
19. K. Mimachi and T. Takamuki, A generalization of the beta integral arising from the Knizhnik–Zamolodchikov equation for the vector representations of types B_n, C_n and D_n, Kyushu J. Math. 59 (2005), 117–126.
20. E. Mukhin and A. Varchenko, Remarks on critical points of phase functions and norms of Bethe vectors, Adv. Stud. Pure Math. 27 (2000), 239–246.
21. E. M. Rains, BC_n-symmetric abelian functions, Duke Math. J. 135 (2006), 99–180.
22. V. V. Schechtman and A. N. Varchenko, Arrangements of hyperplanes and Lie algebra homology, Invent. Math. 106 (1991), 139–194.
23. A. Selberg, Bemerkninger om et multipelt integral, Norsk. Mat. Tidsskr. 24 (1944), 71–78.
24. R. P. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math. 77 (1989), 76–115.
25. V. Tarasov and A. Varchenko, Selberg-type integrals associated with $\mathfrak{sl}_3$, Lett. Math. Phys. 65 (2003), 173–185.
26. A. Varchenko, Special Functions, KZ Type Equations, and Representation Theory, CBMS Reg. Conf. Series Math., vol. 98, Amer. Math. Soc., Providence, RI, 2003.
27. A. Varchenko, A Selberg integral type formula for an $\mathfrak{sl}_2$ one-dimensional space of conformal blocks, Mosc. Math. J. 10 (2010), 469–475.
28. S. O. Warnaar, q-Selberg integrals and Macdonald polynomials, Ramanujan J. 10 (2005), 237–268.
29. S. O. Warnaar, Bisymmetric functions, Macdonald polynomials and $\mathfrak{sl}_3$ basic hypergeometric series, Compositio Math. 114 (2008), 271–303.
30. S. O. Warnaar, The Mukhin–Varchenko conjecture for type A, in: 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), in: Discrete Math. Theor. Comput. Sci. Proc., vol. AJ, 2008, pp. 83–92.
31. S. O. Warnaar, A Selberg integral for the Lie algebra A_n, Acta Math. 203 (2009), 269–304.

School of Mathematics and Physics, The University of Queensland, QLD 4072, Australia