

The Mukhin–Varchenko conjecture for type A

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FPSAC 2008, Valparaiso





Atle Selberg (June 14, 1917 – August 6, 2007)

The hypergeometric equation



$$x(1-x)\frac{d^2F}{dx^2} + (c - (a+b+1)x)\frac{dF}{dx} - abF = 0$$

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \quad (\text{Gauss})$$

where

$$(a)_k = a(a+1) \cdots (a+k-1)$$

is a **Pochhammer** symbol.

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a} dt \quad (\text{Euler})$$

When $x = 1$ Euler's solution reduces to the **beta integral**

$$\begin{aligned} F(a, b; c; 1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b, c-a-b) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \end{aligned}$$

Differential equation in x



Solution as integral



For a special choice of x , the integral may be evaluated
in terms of gamma functions

The idea is to generalise the previous three steps using Lie algebras.

For this to work we need

- Differential equations based on Lie algebras.
- Solutions as integrals.
- A whole lot more . . .

The Knizhnik–Zamolodchikov equation

Notation:

- Simple Lie algebra \mathfrak{g} of rank n .
- Chevalley generators $e_i, f_i, h_i, i \in [n] := \{1, \dots, n\}$.
- Simple roots $\alpha_i, i \in [n]$.
- Fundamental weights $\Lambda_i, i \in [n]$.
- The Casimir element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$.
- Highest weight modules V_λ and V_μ of highest weight λ and μ .

The **Knizhnik–Zamolodchikov** (KZ) equation for a function $u(z, w)$ taking values in $V_\lambda \otimes V_\mu$ is the system of partial differential equations

$$\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z - w} u$$

$$\kappa \frac{\partial u}{\partial w} = \frac{\Omega}{w - z} u$$

Let (\cdot, \cdot) the standard bilinear symmetric form on the dual of the Cartan subalgebra. Then

$$(\alpha_i, \Lambda_j) = \delta_{ij}$$

and

$$\left((\alpha_i, \alpha_j) \right)_{i,j=1}^n = \text{Cartan matrix of } \mathfrak{g}$$

- Example: $\mathfrak{g} = A_n$

$$\left((\alpha_i, \alpha_j) \right)_{i,j=1}^n = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & & \ddots & & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & & & -1 & 2 \end{pmatrix}$$



The Dynkin diagram of A_n .

To the simple root α_i attach a set of k_i integration variables

$$\{t_j\}_{j=k_1+\dots+k_{i-1}+1}^{k_1+\dots+k_i}$$

and write

$$\alpha_{t_j} = \alpha_i$$

if the variable t_j is attached to the root α_i .

Then the **master function** is defined as

$$\Phi(z, w; \mathbf{t}) = (z - w)^{(\lambda, \mu)} \prod_{i=1}^k (t_i - z)^{-(\lambda, \alpha_{t_i})} (t_i - w)^{-(\mu, \alpha_{t_i})} \\ \times \prod_{1 \leq i < j \leq k} (t_i - t_j)^{(\alpha_{t_i}, \alpha_{t_j})}$$

where $k = k_1 + \dots + k_n$.

Schechtman and Varchenko proved that in the subspace of singular vectors of weight $\lambda + \mu - \sum_i k_i \alpha_i$

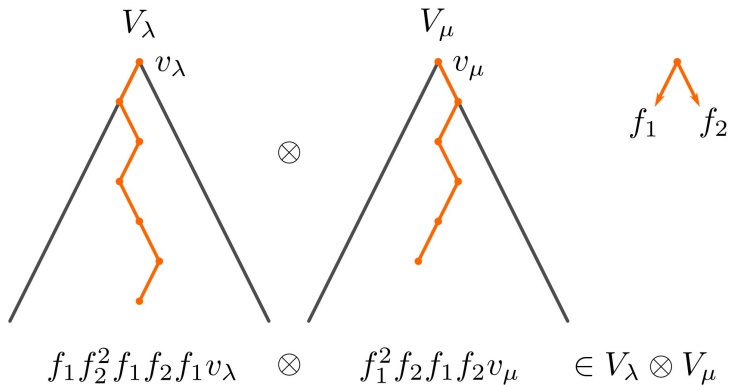
$$u(z, w) = \sum^* u_{IJ}(z, w) f^I v_\lambda \otimes f^J v_\mu$$

with

$$u_{IJ}(z, w) = \int_\gamma \Phi^{1/\kappa}(z, w; \mathbf{t}) A_{IJ}(z, w; \mathbf{t}) d\mathbf{t}$$

The sum is over multisets I and J with elements taken from $\{1, \dots, n\}$ such that their union contains the number i exactly k_i times, v_λ and v_μ are the highest weight vectors of V_λ and V_μ , and

$$f^I v = \left(\prod_{i \in I} f_i \right) v$$



- The case $\mathfrak{g} = \mathfrak{sl}_2 = A_1$, $k = 1$

Chevalley generators e, f, h ,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Casimir element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$$

Highest weights $\lambda = m_1 \Lambda_1$ and $\mu = m_2 \Lambda_1$.

$$u(z, w) = u_0(z, w) v_1 \otimes f v_2 + u_1(z, w) f v_1 \otimes v_2$$

with

$$u_0(z, w) = (z - w)^{\frac{m_1 m_2}{2\kappa}} \int_{\gamma} (z - t)^{-m_1/\kappa} (t - w)^{-m_2/\kappa - 1} dt$$

$$u_1(z, w) = (z - w)^{\frac{m_1 m_2}{2\kappa}} \int_{\gamma} (z - t)^{-m_1/\kappa - 1} (t - w)^{-m_2/\kappa} dt$$

with γ a **Pochhammer double loop** around w and z .

If $w = 0$ and $z = 1$ one can deform γ to

$$\gamma = \{t \in \mathbb{R}, 0 < t < 1\}$$

Both u_0 and u_1 become **Euler beta integrals** and can therefore be evaluated in terms of gamma functions.

The Mukhin–Varchenko conjecture



Define the **specialised master function** (i.e., $w = 0$, $z = 1$)

$$\Phi(\mathbf{t}) = \prod_{i=1}^k t_i^{-(\lambda, \alpha_{t_i})} (1 - t_i)^{-(\mu, \alpha_{t_i})} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{(\alpha_{t_i}, \alpha_{t_j})}$$

Then the example of \mathfrak{sl}_2 , $k = 1$ “justifies” the following conjecture:

Conjecture. If the space of singular vectors is one-dimensional then there exists a (real) domain of integration D such that

$$\int_D |\Phi(\mathbf{t})|^{1/\kappa} d\mathbf{t}$$

evaluates as a product of gamma functions.

- The case $\mathfrak{g} = \mathfrak{sl}_2 = A_1$, general k

Highest weights $\lambda = m_1\Lambda_1$ and $\mu = m_2\Lambda_1$,

$$(\alpha, \beta, \gamma) := (1 - m_1/\kappa, 1 - m_2/\kappa, 1/\kappa)$$

and

$$D = \{\mathbf{t} \in \mathbb{R}^k, 0 \leq t_k \leq \dots \leq t_1 \leq 1\}$$

$$\int_D \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{2\gamma} dt$$

$$= \prod_{i=0}^{k-1} \frac{\Gamma(\alpha + i\gamma)\Gamma(\beta + i\gamma)\Gamma(\gamma + i\gamma)}{\Gamma(\alpha + \beta + (k+i-1)\gamma)\Gamma(\gamma)}$$

This is the Selberg integral!

The Mukhin–Varchenko (ex-)conjecture for type A

- The case $\mathfrak{g} = \mathfrak{sl}_{n+1} = A_n$, general $k_1 \leq k_2 \leq \dots \leq k_n$

Highest weights $\lambda = \lambda_n \Lambda_n$ and $\mu = \mu_1 \Lambda_1 + \dots + \mu_n \Lambda_n$,

$$(\alpha, \beta_1, \dots, \beta_n, \gamma) := \left(1 - \frac{\lambda_n}{\kappa}, 1 - \frac{\mu_1}{\kappa}, \dots, 1 - \frac{\mu_n}{\kappa}, \frac{1}{\kappa}\right)$$

and

$$D = \{\mathbf{t} \in \mathbb{R}^k, \text{chain}\} \quad (\text{in the algebraic topology sense})$$

$$\begin{aligned} & \int_D |\Phi(\mathbf{t})|^\gamma d\mathbf{t} \\ &= \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma(\alpha\delta_{r,n} + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ & \quad \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha\delta_{s,n} + (i - k_{s+1} - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \end{aligned}$$

Proof is based on **Macdonald polynomial theory**.



To the initiated this should not come as a surprise. In **Macdonald** uses his polynomials to prove **Kadell's** extension of the Selberg integral:

$$\begin{aligned} & \int_D P_\lambda^{(1/\gamma)}(\mathbf{t}) \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{2\gamma} d\mathbf{t} \\ &= \prod_{1 \leq i < j \leq k} \frac{\Gamma((j-i+1)\gamma + \lambda_i - \lambda_j)}{\Gamma((j-i)\gamma + \lambda_i - \lambda_j)} \\ & \quad \times \prod_{i=1}^k \frac{\Gamma(\alpha + (k-i)\gamma + \lambda_i) \Gamma(\beta + (i-1)\gamma)}{\Gamma(\alpha + \beta + (2k-i-1)\gamma + \lambda_i)} \end{aligned}$$

where $P_\lambda^{(\alpha)}(\mathbf{t})$ is the Jack polynomial.



The End