Macdonald polynomials made easy

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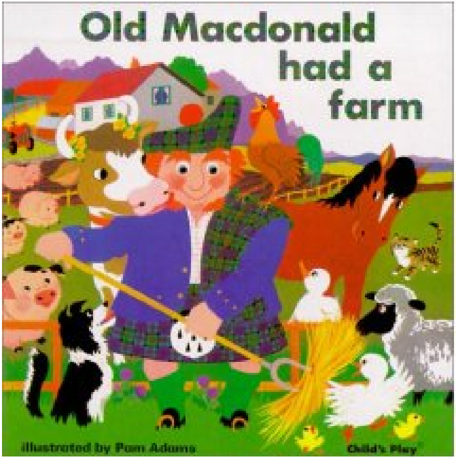
Or: Macdonald for preschool . . .



Outline

Motivation

Let $f(x) := f(x_1, \ldots, x_n)$ a Laurent polynomial in x and $\mathrm{CT}(f)$ its constant term.

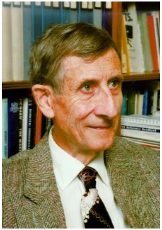
For example, if

$$f(x_1, x_2) = \left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_2}{x_1}\right)$$

then

$$\mathrm{CT}(f) = 2$$

In 1962 Freeman Dyson, while developing his *Statistical theory of energy levels of complex systems*, made a remarkable conjecture.



Dyson conjecture

$$\mathrm{CT}\left(\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)^k\left(1-\frac{x_j}{x_i}\right)^k\right)=\frac{(kn)!}{(k!)^n}$$

Dyson's conjecture was proved almost immediately by Gunson and (subsequent Nobel Laureate) Wilson.

George Andrews made the problem significantly harder by conjecturing a q-analogue.

Let

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

be a q-shifted factorial, and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q^{n-m+1})(1-q^{n-m+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^m)}$$

be a q-binomial coefficient.

For example

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4$$

Andrews' q-Dyson conjecture

$$\mathrm{CT}\left(\prod_{1\le i<j\le n}(x_i/x_j;q)_k(qx_j/x_i;q)_k\right)=\frac{(q;q)_{nk}}{(q;q)_k^n}$$

For example, if $k=1$ then

$$\mathrm{CT}\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{qx_2}{x_1}\right)=1+q=\frac{(q;q)_2}{(q;q)_1^2}=\begin{bmatrix}2\\1\end{bmatrix}$$

Andrews' conjecture was proved by Zeilberger and Bressoud in a famous (but difficult) paper.

Ian Macdonald made the problem even harder . . .

Macdonald observed that

$$\frac{(q;q)_{nk}}{(q;q)_k^n}$$

can also be written as

$$\begin{bmatrix} nk \\ k \end{bmatrix} \cdots \begin{bmatrix} 2k \\ k \end{bmatrix} \begin{bmatrix} k \\ k \end{bmatrix}$$

The numbers $1, 2, \dots, n$ are precisely the degrees of the fundamental invariants of the root system A_{n-1}. (Much more on this in my talk on Friday).

For other (finite and reduced) root systems these degrees are also known and Macdonald made the following deep conjecture.

Macdonald CT conjecture

Let Φ be a finite, reduced root system and Φ^+ the set of positive roots. Let D be the set of degrees of the fundamental invariants of Φ. Then

$$\mathrm{CT}\left(\prod_{\alpha\in\Phi^+}(e^{\alpha};q)_k(qe^{-\alpha};q)_k\right)=\prod_{d\in D}\begin{bmatrix}dk\\k\end{bmatrix}$$

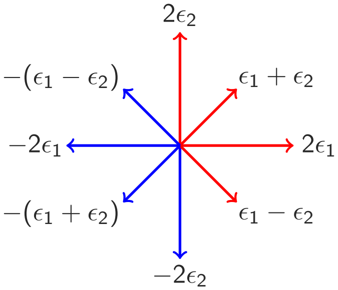
Nothing works better than an example ...

Let ϵ_i the ith standard unit vector in $\mathbb{R}^n$. Then the root system C_n is given by

$$\Phi = \{\pm 2\epsilon_i | 1 \leq i \leq n\} \cup \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\}$$

with positive roots

$$\Phi^+ = \{2\epsilon_i | 1 \leq i \leq n\} \cup \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\}$$



The root system C_2 with positive roots in red.

Identitfy $x_i = e^{\epsilon_i}$. Then the Macdonald conjecture for C_n is as follows.

If

$$f(x) = \prod_{i=1}^{n} (x_i^2; q)_k (q/x_i^2; q)_k \\ \times \prod_{1 \le i < j \le n} (x_i x_j; q)_k (x_i/x_j; q)_k (q x_j/x_i; q)_k (q/x_i x_j; q)_k$$

then

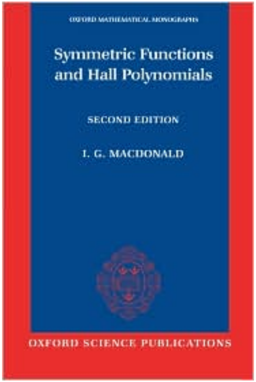
$$\mathrm{CT}(f) = \begin{bmatrix} 2kn \\ k \end{bmatrix} \cdots \begin{bmatrix} 4k \\ k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix}$$

To deal with the Macdonald conjectures, a theory of polynomials is needed that has constant terms naturally built in.

These are the G-Macdonald polynomials, where G is a reduced, finite root system.

In the remainder of this talk I will only consider the case A_{n-1}.

Symmetric functions



The Bible

A function $f(x) = f(x_1, \ldots, x_n)$ is called symmetric of it is invariant under permutations of the variables.

Some standard symmetric functions are the elementary symmetric functions

$$e_r(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

the complete symmetric functions

$$h_r(x) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

and the monomial symmetric functions

$$m_\lambda(x) = \sum x^\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$$

where the sum is over distinct permutations of the partition λ.

Examples:

$$e_0(x) = h_0(x) = m_0(x) = 1$$

$$e_1(x) = h_1(x) = m_{(1)}(x) = x_1 + \cdots + x_n$$

$$e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$$

$$h_2(x_1, x_2, x_3) = e_2(x_1, x_2, x_3) + x_1^2 + x_2^2 + x_3^2$$

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2)$$

If Λ_n is the ring $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ then $\{e_1, \dots, e_n\}$ and $\{h_1, \dots, h_n\}$ form algebraic bases of Λ_n.

It is no coincidence that the degrees of these polynomials are exactly $1, 2, \dots, n$, the degrees of the fundamental invariants of A_{n-1}.

The set of monomial symmetric functions $\{m_\lambda\}$, with λ ranging over all partitions of at most n parts, forms a linear bases of Λ_n.

The most important (linear) basis of Λ_n is given by the Schur functions.

A not so insightful definition of these is as the ratio of two alternants (which is in fact due to Jacobi)

$$s_\lambda(x) = \frac{\det_{1\leq i,j\leq n}(x_i^{\lambda_j+n-j})}{\det_{1\leq i,j\leq n}(x_i^{n-j})}$$

where the denominator is the famous Vandermonde determinant.

The more important description of the Schur functions is combinatorial in nature:

$$s_\lambda(x) = \sum_T x^T$$

where the sum is over all (semi-standard) Young tableaux T.
For example, there are eight tableaux of shape $(2,1)$ on three letters

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and therefore

$$\begin{aligned} s_{(2,1)}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 \\ &\qquad + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &= x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) + 2x_1 x_2 x_3 \\ &= m_{(2,1)}(x_1, x_2, x_3) + 2m_{(1,1,1)}(x_1, x_2, x_3) \end{aligned}$$

Macdonald polynomials

The Macdonald polynomials $P_\lambda(x; q, t)$ (of type A_{n-1}) are q, t-generalisations of the Schur functions and monomial symmetric functions, and form a linear basis of the ring

$$\Lambda_{\mathbb{F}} := \mathbb{F}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

where $\mathbb{F} = \mathbb{Q}(q, t)$.

When $t = q$ and $t = 1$ the Macdonald polynomials simplify to the Schur and monomial symmetric functions

$$P_\lambda(x; q, q) = s_\lambda(x)$$

$$P_\lambda(x; q, 1) = m_\lambda(x)$$

Other special cases include the Hall–Littlewood and the Jack polynomials.

The original definition of the Macdonald polynomials is neither easy nor very exlicit ...

For λ a partition let $m_i(\lambda)$ be the multiplicity of parts of size i.
For example, if $\lambda = (4, 2, 2, 1)$ then $m_2 = 2$ and $m_3 = 0$.

Let

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$$

For example

$$z_{(4,2,2,1)} = (1^1 1!) \times (2^2 2!) \times (4^1 1!) = 32$$

Let p_λ be a power-sum symmetric function

$$p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_n}(x)$$

with

$$p_r(x) = x_1^r + \cdots + x_n^r$$

Macdonald defined a q, t-analogue of Hall's scalar product by demanding that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \, z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

Macdonald's existence theorem

For each partition λ there exists a unique symmetric function $P_\lambda(x) \in \Lambda_{\mathbb{F}}$ such that

$$P_\lambda(x) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu(x)$$

and

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if } \lambda \neq \mu$$

where m_λ is the monomial symmetric function and $<$ refers to the dominance order on partitions.

Problem: The above existence theorem is inappropriate in a talk called *Macdonald polynomials made easy*.

Solution: Go nonsymmetric (Cherednik,Opdam) and nonhomogeneous (Okounkov,Knop,Sahi).

Before going nonsymmetric and inhomogeneous let me remark that the Macdonald polynomials are indeed related to constant term identities. In particular, assuming $t = q^k$, Macdonald defined a second scalar product on the ring $\Lambda_{\mathbb{F}}$ as

$$\langle f, g \rangle' := \frac{1}{n!} \operatorname{CT}\left(f(x) g(1/x) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (x_i/x_j; q)_k \right)$$

He then proved the orthogonality and quadratic norm evaluation.

Theorem

$$\langle P_\lambda, P_\mu \rangle' := \delta_{\lambda\mu} \prod_{1 \le i < j \le n} \prod_{r=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_j + r} t^{j-i}}{1 - q^{\lambda_i - \lambda_j - r} t^{j-i}}$$

Taking $\lambda = \mu = 0$ this in particular implies that

$$\frac{1}{n!}\,\mathrm{CT}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n} (x_i/x_j;q)_k\right) = \prod_{i=1}^{n} \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}$$

It requires only highschool maths to show that this is equivalent to Andrews' q-Dyson (ex-)conjecture

$$\mathrm{CT}\left(\prod_{1\leq i<j\leq n} (x_i/x_j;q)_k (qx_j/x_i;q)_k\right) = \prod_{i=1}^{n} \begin{bmatrix} ik \\ k \end{bmatrix}$$

Macdonald interpolation polynomials

For $u = 0, 1, 2 \dots$ define the Newton interpolation polynomial $M_u(x) = M_u(x; q)$ as

$$M_u(x) = q^{-\binom{u}{2}}(x-1)(x-q)\cdots(x-q^{u-1})$$

Clearly, up to normalisation, this polynomial is uniquely defined by its degree and the fact that

$$M_u(\langle v \rangle) = 0 \qquad \langle v \rangle := q^v$$

for $0 \leq v < u$. These are referred to as the vanishing conditions.

It is also obvious that we have the recursion

$$M_{u+1}(x) = (x-1)M_u(x/q)$$

The previous constructions have been generalised by Knop and Sahi, resulting in nonsymmetric and nonhomogeneous polynomials in n variables.

These are known as the interpolation Macdonald polynomials or vanishing Macdonald polynomials and are labelled by compositions $u = (u_1, \ldots, u_n)$.

A composition is called dominant if it is a partition.
More generally we set u^+ for the partition obtained by reordering the parts of the composition u.
If a composition u is dominant, define its spectral vector as

$$\langle u \rangle = (q^{u_1} t^{n-1}, q^{u_2} t^{n-2}, \ldots, q^{u_n} t^0)$$

For example, if $u = (8, 5, 5, 0)$ then

$$\langle (8, 5, 5, 0) \rangle = (q^8 t^3, q^5 t^2, q^5 t, 1)$$

If u is not dominant generalise this in the "obvious way"
For example, if $u = (5, 0, 8, 5)$ then

$$\langle (5, 0, 8, 5) \rangle = (q^5 t^2, 1, q^8 t^3, q^5 t)$$

(Left-most 5 gets the higher power of t.)

Definition (Knop–Sahi)

Let $x = (x_1, \dots, x_n)$. Up to normalisation, the interpolation Macdonald polynomial $M_u(x)$ is the unique polynomial of (maximal) degree $|u| := u_1 + \cdots + u_n$ such that

$$M_u(\langle v \rangle) = 0 \quad \text{for } |v| \le |u|,\ v \neq u$$

Note that an arbitrary polynomial of degree $|u|$ is of the form

$$\sum_{\substack{v \\ |v| \le |u|}} c_v x^v$$

where $x^v = x_1^{v_1} \cdots x_n^{v_n}$, so that we have exactly the right number of conditions.

It requires a little lemma to show that the conditions are consistent.

There is another way to describe the polynomials M_u, generalising the recurrence for the Newton interpolation polynomials.

Below *all* operators act on the left.

Let $s_i \in \mathfrak{S}_n$ be the elementary transposition interchanging the variables x_i and x_{i+1}. Then T_i is the operator (acting on polynomials in $x_1, \dots, x_n$) defined by

$$T_i := t + (s_i - 1)\,\frac{tx_{i+1} - x_i}{x_{i+1} - x_i}$$

The easiest way to remember T_i is that it is the unique operator that commutes with functions symmetric in x_i and x_{i+1}, such that

$$\begin{aligned} 1\,T_i &= t \\ x_{i+1}\,T_i &= x_i \end{aligned}$$

One may verify that the T_i for $i = 1, \ldots, n-1$ satisfy the defining relations of the Hecke algebra of the symmetric group:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i \quad \text{for } |i-j| \neq 1$$

$$(T_i + 1)(T_i - t) = 0$$

The T_i are degree preserving operators. To be able to generate the interpolation Macdonald polynomials we also need to be able to increase the degree (like in the recurrence for the Newton interpolation polynomials).

This requires the extension of the Hecke algebra to the affine Hecke algebra.

Let $x\tau = (x_n/q, x_1, \dots, x_{n-1})$.

Then the raising operator ϕ is defined as

$$f(x)\phi := f(x\tau)(x_n - 1)$$

Note that for $n = 1$ this exactly generates the recursion for the Newton interpolation polynomials:

$$M_u(x)\phi = M_u(x/q)(x-1) = M_{u+1}(x)$$

The algebraic construction of the interpolation Macdonald polynomials can now be described as follows.

- Initial condition
$$M_{(0,\ldots,0)}(x) = 1$$
- Affine operation=degree raising
$$M_{(u_2,\ldots,u_{n-1},u_1+1)}(x) = M_u(x)\phi$$
- Hecke operation=permuting the u
  If $u_i < u_{i+1}$
$$M_{us_i}(x) = M_u(x)\left(T_i + \frac{t-1}{\langle u\rangle_{i+1}/\langle u\rangle_i - 1}\right)$$

The above construction is analogous to that of the Schubert and Grotendieck polynomials.

$$
\begin{tikzcd}
 & & & & & 000 \arrow[d, "\phi"] \\
 & & & 100 \arrow[dl, "\phi"] & 010 \arrow[l, "2"'] \arrow[d, "\phi"] & 001 \arrow[l, "2"'] \arrow[d, "\phi"] \\
200 & 020 \arrow[l, "1"'] & 002 \arrow[l, "2"'] & 110 & 101 \arrow[l, "2"'] & 011 \arrow[l, "1"']
\end{tikzcd}
$$

For this to be consistent (for arbitrary n) we must have

$$T_{i+1}\phi = \phi T_i$$

$$
\begin{tikzcd}
100 & 010 \arrow[l, "1"'] \arrow[d, "\phi"] \\
002 \arrow[u, leftarrow, "\phi"] \arrow[d, "\phi"'] & 101 \arrow[d, "\phi"] \\
021 & 012 \arrow[l, "2"']
\end{tikzcd}
$$

For this to be consistent (for arbitrary n) we must have

$$T_1 \phi^2 = \phi^2 T_{n-1}$$

In summary, the generators $T_1, \ldots, T_{n-1}, \phi$ satisfy the affine Hecke algebra

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i \quad \text{for } |i-j| \neq 1$$

$$(T_i + 1)(T_i - t) = 0$$

$$T_{i+1} \phi = \phi T_i$$

$$T_1 \phi^2 = \phi^2 T_{n-1}$$

It requires another little lemma to show that the definition of the M_u using the affine Hecke algebra is consistent with the definition using the vanishing conditions.

Once the M_u are understood the rest of Macdonald polynomial theory is easy:

$$\begin{array}{ccc} M_u & \xrightarrow{\text{symmetrisation}} & MS_\lambda \\ {\scriptstyle\text{homogenisation}}\downarrow & & \downarrow{\scriptstyle\text{homogenisation}} \\ E_u & \xrightarrow{\text{symmetrisation}} & P_\lambda \end{array}$$

where “symmetrisation” is easy and “homogenisation” is even easier.

Specifically, homogenisation just means taking the top-degree term:

$$E_u(x) = \lim_{a\to 0} a^{|u|} M_u(x/a)$$

There is a further extension of the Hecke algebra that plays a central role in the theory. Let X_i denote the operator "multiplication by x_i":

$$f(x)X_i = f(x)x_i$$

One readily checks that

$$X_iX_j = X_jX_i$$

$$T_iX_{i+1}T_i = tX_i$$

$$T_i(X_i + X_{i+1}) = (X_i + X_{i+1})T_i$$

$$T_iX_j = X_jT_i \quad \text{for } j \neq i, i+1$$

Let Y_i be the Cherednik operator

$$Y_i = t^{i-1} T_i \cdots T_{n-1} \tau^{-1} T_1^{-1} \cdots T_{i-1}^{-1}$$

for $1 \leq i \leq n$.

A little-less-little lemma shows that

$$E_u(x) Y_i = \langle u \rangle_i E_u(x)$$

That is, the nonsymmetric Macdonald polynomials are the eigenfunctions of the Y_i.

With a bit of pain one checks the following amazing facts

$$Y_i Y_j = Y_j Y_i$$

$$T_i Y_{i+1} T_i = t Y_i$$

$$T_i(Y_i + Y_{i+1}) = (Y_i + Y_{i+1}) T_i$$

$$T_i Y_j = Y_j T_i \quad \text{for } j \neq i, i+1$$

In other words, at the level of the algebra the "difficult" operators Y_i are not at all harder than the "easy" operators X_i.

Finally one can check that

$$T_i^2 X_{i+1} Y_i = t Y_i X_{i+1}$$

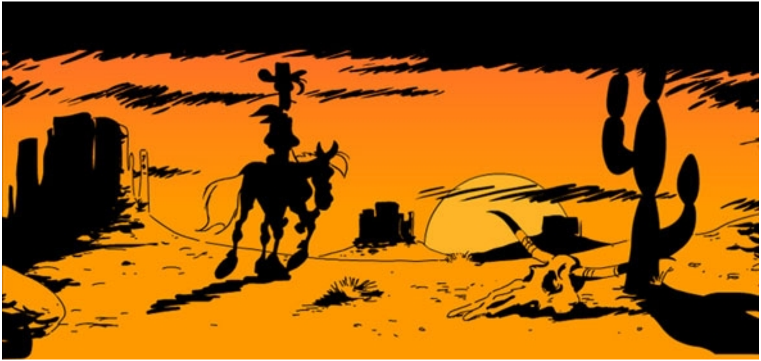
The algebra generated by the T_i, X_i, Y_i subject to all the is known as the double affine Hecke algebra (DAHA) (of type A_{n-1}), and was discovered by Ivan Cherednik.

The DAHA for arbitrary G can be used to prove the Macdonald CT conjecture.

Cherednik's CT theorem

Let Φ be a finite, reduced root system and Φ^+ the set of positive roots. Let D be the set of degrees of the fundamental invariants of Φ. Then

$$\mathrm{CT}\left(\prod_{\alpha\in\Phi^+}(e^{\alpha};q)_k(qe^{-\alpha};q)_k\right)=\prod_{d\in D}\begin{bmatrix}dk\\k\end{bmatrix}$$



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