Motivation
Symmetric functions
Macdonald polynomials
Macdonald interpolation polynomials

Macdonald polynomials made easy

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Or: Macdonald for preschool . . .
Outline

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Let $f(x) := f(x_1, \ldots, x_n)$ a Laurent polynomial in $x$ and $\text{CT}(f)$ its constant term.

For example, if

$$f(x_1, x_2) = \left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_2}{x_1}\right)$$

then

$$\text{CT}(f) = 2$$
In 1962 Freeman Dyson, while developing his *Statistical theory of energy levels of complex systems*, made a remarkable conjecture.

\[
\text{CT} \left( \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right)^k \left(1 - \frac{x_j}{x_i}\right)^k \right) = \frac{(kn)!}{(k!)^n}
\]

Dyson conjecture
Dyson’s conjecture was proved almost immediately by Gunson and (subsequent Nobel Laureate) Wilson.

George Andrews made the problem significantly harder by conjecturing a $q$-analogue.
Let

\[(a; q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1})\]

be a \emph{q-shifted factorial}, and

\[\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1 - q^{n-m+1})(1 - q^{n-m+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^m)}\]

be a \emph{q-binomial coefficient}.

For example

\[\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4\]
Andrews’ $q$-Dyson conjecture

$$\text{CT} \left( \prod_{1 \leq i < j \leq n} \left( \frac{x_i}{x_j}; q \right)_k \left( \frac{qx_j}{x_i}; q \right)_k \right) = \frac{(q; q)^{nk}}{(q; q)^n_k}$$

For example, if $k = 1$ then

$$\text{CT} \left( 1 - \frac{x_1}{x_2} \right) \left( 1 - \frac{qx_2}{x_1} \right) = 1 + q = \frac{(q; q)_2}{(q; q)^2_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
Andrews’ conjecture was proved by Zeilberger and Bressoud in a famous (but difficult) paper.

Ian Macdonald made the problem even harder . . .
Macdonald observed that
\[
\frac{(q; q)_{nk}}{(q; q)^n_k}
\]
can also be written as
\[
\begin{bmatrix}
  nk \\
  k 
\end{bmatrix} \cdots \begin{bmatrix}
  2k \\
  k 
\end{bmatrix} \begin{bmatrix}
  k \\
  k 
\end{bmatrix}
\]

The numbers 1, 2, \ldots, \( n \) are precisely the degrees of the fundamental invariants of the root system \( A_{n-1} \). (Much more on this in my talk on Friday).
For other (finite and reduced) root systems these degrees are also known and Macdonald made the following deep conjecture.

**Macdonald CT conjecture**

Let $\Phi$ be a finite, reduced root system and $\Phi^+$ the set of positive roots. Let $D$ be the set of degrees of the fundamental invariants of $\Phi$. Then

$$\text{CT} \left( \prod_{\alpha \in \Phi^+} (e^\alpha; q)_k (qe^{-\alpha}; q)_k \right) = \prod_{d \in D} \left[dk \atop k\right]$$
Nothing works better than an example . . .

Let $\epsilon_i$ the $i$th standard unit vector in $\mathbb{R}^n$. Then the root system $C_n$ is given by

$$\Phi = \{\pm 2\epsilon_i | 1 \leq i \leq n\} \cup \{\pm \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\}$$

with positive roots

$$\Phi^+ = \{2\epsilon_i | 1 \leq i \leq n\} \cup \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\}$$

The root system $C_2$ with positive roots in red.
Identify $x_i = e^{\epsilon_i}$. Then the Macdonald conjecture for $C_n$ is as follows.

If

$$f(x) = \prod_{i=1}^{n} (x_i^2; q)_k (q/x_i^2; q)_k$$

$$\times \prod_{1 \leq i < j \leq n} (x_i x_j; q)_k (x_i/x_j; q)_k (q x_j/x_i; q)_k (q/x_i x_j; q)_k$$

then

$$\text{CT}(f) = \begin{bmatrix} 2kn \\ k \end{bmatrix} \cdots \begin{bmatrix} 4k \\ k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix}$$
To deal with the Macdonald conjectures, a theory of polynomials is needed that has constant terms naturally built in.

These are the $G$-Macdonald polynomials, where $G$ is a reduced, finite root system.

In the remainder of this talk I will only consider the case $A_{n-1}$. 
Symmetric functions

The Bible
A function \( f(x) = f(x_1, \ldots, x_n) \) is called symmetric if it is invariant under permutations of the variables.

Some standard symmetric functions are the elementary symmetric functions

\[
e_r(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}
\]

the complete symmetric functions

\[
h_r(x) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}
\]

and the monomial symmetric functions

\[
m_\lambda(x) = \sum x^\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}
\]

where the sum is over distinct permutations of the partition \( \lambda \).
Examples:

\[ e_0(x) = h_0(x) = m_0(x) = 1 \]

\[ e_1(x) = h_1(x) = m_{(1)}(x) = x_1 + \cdots + x_n \]

\[ e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 \]

\[ h_2(x_1, x_2, x_3) = e_2(x_1, x_2, x_3) + x_1^2 + x_2^2 + x_3^2 \]

\[ m_{(2,1)}(x_1, x_2, x_3) = x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) \]
If $\Lambda_n$ is the ring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ then $\{e_1, \ldots, e_n\}$ and $\{h_1, \ldots, h_n\}$ form algebraic bases of $\Lambda_n$.

It is no coincidence that the degrees of these polynomials are exactly $1, 2, \ldots, n$, the degrees of the fundamental invariants of $A_{n-1}$.

The set of monomial symmetric functions $\{m_\lambda\}$, with $\lambda$ ranging over all partitions of at most $n$ parts, forms a linear bases of $\Lambda_n$. 
The most important (linear) basis of $\Lambda_n$ is given by the **Schur functions**.

A not so insightful definition of these is as the ratio of two alternants (which is in fact due to **Jacobi**)

$$s_\lambda(x) = \frac{\det_{1 \leq i, j \leq n}(x_i^{\lambda_j + n-j})}{\det_{1 \leq i, j \leq n}(x_i^{n-j})}$$

where the denominator is the famous **Vandermonde determinant**.
The more important description of the Schur functions is combinatorial in nature:

\[ s_\lambda(x) = \sum_T x^T \]

where the sum is over all (semi-standard) Young tableaux \( T \).

For example, there are eight tableaux of shape \((2, 1)\) on three letters

\[
\begin{array}{ccc}
1 & 1 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
1 & 1 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
1 & 2 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
1 & 2 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
1 & 3 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
1 & 3 & \text{1} \\
2 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
2 & 2 & \text{1} \\
3 & 3 & \text{2}
\end{array},
\begin{array}{ccc}
2 & 2 & \text{1} \\
3 & 3 & \text{2}
\end{array},
\end{array}

and therefore

\[ s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2x_2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3 + x_2^2x_3 + x_2x_3^2 \]

\[ = x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) + 2x_1x_2x_3 \]

\[ = m_{(2,1)}(x_1, x_2, x_3) + 2m_{(1,1,1)}(x_1, x_2, x_3) \]
The Macdonald polynomials $P_\lambda(x; q, t)$ (of type $A_{n-1}$) are $q, t$-generalisations of the Schur functions and monomial symmetric functions, and form a linear basis of the ring

$$\Lambda_\mathbb{F} := \mathbb{F}[x_1, \ldots, x_n]_\mathbb{S}_n$$

where $\mathbb{F} = \mathbb{Q}(q, t)$.

When $t = q$ and $t = 1$ the Macdonald polynomials simplify to the Schur and monomial symmetric functions

$$P_\lambda(x; q, q) = s_\lambda(x)$$

$$P_\lambda(x; q, 1) = m_\lambda(x)$$

Other special cases include the Hall–Littlewood and the Jack polynomials.
The original definition of the Macdonald polynomials is neither easy nor very explicit . . .

For $\lambda$ a partition let $m_i(\lambda)$ be the multiplicity of parts of size $i$.
For example, if $\lambda = (4, 2, 2, 1)$ then $m_2 = 2$ and $m_3 = 0$.

Let

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$$

For example

$$z_{(4,2,2,1)} = (1^11!) \times (2^22!) \times (4^11!) = 32$$

Let $p_\lambda$ be a **power-sum symmetric function**

$$p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_n}(x)$$

with

$$p_r(x) = x_1^r + \cdots + x_n^r$$
Macdonald defined a $q, t$-analogue of Hall’s scalar product by demanding that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

**Macdonald’s existence theorem**

For each partition $\lambda$ there exists a unique symmetric function $P_\lambda(x) \in \Lambda_F$ such that

$$P_\lambda(x) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu(x)$$

and

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if} \; \lambda \neq \mu$$

where $m_\lambda$ is the monomial symmetric function and $<$ refers to the dominance order on partitions.
**Problem:** The above existence theorem is inappropriate in a talk called *Macdonald polynomials made easy.*

**Solution:** Go nonsymmetric (Cherednik, Opdam) and nonhomogeneous (Okounkov, Knop, Sahi).
Before going nonsymmetric and inhomogeneous let me remark that the Macdonald polynomials are indeed related to constant term identities. In particular, assuming $t = q^k$, Macdonald defined a second scalar product on the ring $\Lambda_{\mathbb{F}}$ as

$$\langle f, g \rangle' := \frac{1}{n!} \text{CT} \left( f(x)g(1/x) \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_k \right)$$

He then proved the orthogonality and quadratic norm evaluation.

**Theorem**

$$\langle P_\lambda, P_\mu \rangle' := \delta_{\lambda\mu} \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_j + r} t^{j-i}}{1 - q^{\lambda_i - \lambda_j - r} t^{j-i}}$$
Taking $\lambda = \mu = 0$ this in particular implies that

$$
\frac{1}{n!} \text{CT} \left( \prod_{\substack{i,j=1 \atop i \neq j}}^{n} \frac{x_i}{x_j} ; q \right)_k = \prod_{i=1}^{n} \left[ \frac{i k - 1}{k - 1} \right]
$$

It requires only high school maths to show that this is equivalent to 
Andrews’ $q$-Dyson (ex-)conjecture

$$
\text{CT} \left( \prod_{1 \leq i < j \leq n} \frac{x_i}{x_j} ; q \right)_k \left( q x_j / x_i ; q \right)_k = \prod_{i=1}^{n} \left[ \frac{i k}{k} \right]
$$
For $u = 0, 1, 2 \ldots$ define the Newton interpolation polynomial $M_u(x) = M_u(x; q)$ as

$$M_u(x) = q^{-\binom{u}{2}}(x - 1)(x - q) \cdots (x - q^{u-1})$$

Clearly, up to normalisation, this polynomial is uniquely defined by its degree and the fact that

$$M_u(\langle v \rangle) = 0 \quad \langle v \rangle := q^v$$

for $0 \leq v < u$. These are referred to as the vanishing conditions.

It is also obvious that we have the recursion

$$M_{u+1}(x) = (x - 1)M_u(x/q)$$
The previous constructions have been generalised by Knop and Sahi, resulting in nonsymmetric and nonhomogeneous polynomials in \( n \) variables.

These are known as the interpolation Macdonald polynomials or vanishing Macdonald polynomials and are labelled by compositions \( u = (u_1, \ldots, u_n) \).
A composition is called **dominant** if it is a partition. More generally we set \( u^+ \) for the partition obtained by reordering the parts of the composition \( u \).

If a composition \( u \) is dominant, define its **spectral vector** as

\[
\langle u \rangle = (q^{u_1} t^{n-1}, q^{u_2} t^{n-2}, \ldots, q^{u_n} t^0)
\]

For example, if \( u = (8, 5, 5, 0) \) then

\[
\langle (8, 5, 5, 0) \rangle = (q^8 t^3, q^5 t^2, q^5 t, 1)
\]

If \( u \) is not dominant generalise this in the “obvious way” and take

For example, if \( u = (5, 0, 8, 5) \) then

\[
\langle (5, 0, 8, 5) \rangle = (q^5 t^2, 1, q^8 t^3, q^5 t)
\]

(Left-most 5 gets the higher power of \( t \).)
Definition (Knop–Sahi)

Let \( x = (x_1, \ldots, x_n) \). Up to normalisation, the interpolation Macdonald polynomial \( M_u(x) \) is the unique polynomial of (maximal) degree \( |u| := u_1 + \cdots + u_n \) such that

\[
M_u(\langle v \rangle) = 0 \quad \text{for} \quad |v| \leq |u|, \ v \neq u
\]

Note that an arbitrary polynomial of degree \( |u| \) is of the form

\[
\sum_{|v| \leq |u|} c_v x^v
\]

where \( x^v = x_1^{v_1} \cdots x_n^{v_n} \), so that we have exactly the right number of conditions.

It requires a little lemma to show that the conditions are consistent.
There is another way to describe the polynomials $M_u$, generalising the recurrence for the Newton interpolation polynomials.

Below all operators act on the left.

Let $s_i \in \mathfrak{S}_n$ be the elementary transposition interchanging the variables $x_i$ and $x_{i+1}$. Then $T_i$ is the operator (acting on polynomials in $x_1, \ldots, x_n$) defined by

$$T_i := t + (s_i - 1) \frac{tx_{i+1} - x_i}{x_{i+1} - x_i}$$

The easiest way to remember $T_i$ is that it is the unique operator that commutes with functions symmetric in $x_i$ and $x_{i+1}$, such that

$$1T_i = t$$
$$x_{i+1}T_i = x_i$$
One may verify that the $T_i$ for $i = 1, \ldots, n - 1$ satisfy the defining relations of the Hecke algebra of the symmetric group:

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\
T_i T_j = T_j T_i \quad \text{for } |i - j| \neq 1 \\
(T_i + 1)(T_i - t) = 0
\]

The $T_i$ are degree preserving operators. To be able to generate the interpolation Macdonald polynomials we also need to be able to increase the degree (like in the recurrence for the Newton interpolation polynomials).
This requires the extension of the Hecke algebra to the affine Hecke algebra.

Let \( x_{\tau} = (x_n/q, x_1, \ldots, x_{n-1}) \).

Then the raising operator \( \phi \) is defined as

\[
f(x)\phi := f(x_{\tau})(x_n - 1)
\]

Note that for \( n = 1 \) this exactly generates the recursion for the Newton interpolation polynomials:

\[
M_u(x)\phi = M_u(x/q)(x - 1) = M_{u+1}(x)
\]
The algebraic construction of the interpolation Macdonald polynomials can now be described as follows.

- **Initial condition**
  \[ M_{(0,\ldots,0)}(x) = 1 \]

- **Affine operation=degree raising**
  \[ M_{(u_2,\ldots,u_{n-1},u_1+1)}(x) = M_u(x) \phi \]

- **Hecke operation=permuting the \( u \)**
  If \( u_i < u_{i+1} \)
  \[ M_{u_{s_i}}(x) = M_u(x) \left( T_i + \frac{t-1}{\langle u \rangle_{i+1}/\langle u \rangle_i - 1} \right) \]

The above construction is analogous to that of the Schubert and Grotendieck polynomials.
For this to be consistent (for arbitrary $n$) we must have

$$T_{i+1}\phi = \phi T_i$$
For this to be consistent (for arbitrary $n$) we must have

$$T_1 \phi^2 = \phi^2 T_{n-1}$$
In summary, the generators $T_1, \ldots, T_{n-1}, \phi$ satisfy the affine Hecke algebra

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\
T_i T_j = T_j T_i \quad \text{for } |i - j| \neq 1 \\
(T_i + 1)(T_i - t) = 0 \\
T_{i+1} \phi = \phi T_i \\
T_1 \phi^2 = \phi^2 T_{n-1}
\]
It requires another little lemma to show that the definition of the $M_u$ using the affine Hecke algebra is consistent with the definition using the vanishing conditions.

Once the $M_u$ are understood the rest of Macdonald polynomial theory is easy:

\[
\begin{align*}
M_u & \xrightarrow{\text{symmetrisation}} MS_\lambda \\
E_u & \xrightarrow{\text{symmetrisation}} P_\lambda
\end{align*}
\]

where “symmetrisation” is easy and “homogenisation” is even easier.

Specifically, homogenisation just means taking the top-degree term:

\[
E_u(x) = \lim_{a \to 0} a^{\|u\|} M_u(x/a)
\]
There is a further extension of the Hecke algebra that plays a central role in the theory. Let $X_i$ denote the operator “multiplication by $x_i$“:

$$f(x)X_i = f(x)x_i$$

One readily checks that

$$X_i X_j = X_j X_i$$

$$T_i X_{i+1} T_i = tX_i$$

$$T_i (X_i + X_{i+1}) = (X_i + X_{i+1}) T_i$$

$$T_i X_j = X_j T_i$$ for $j \neq i, i+1$
Let $Y_i$ be the Cherednik operator

$$Y_i = t^{i-1} T_i \cdots T_{n-1} T_{n-1} T_{n-2} \cdots T_{i+1}$$

for $1 \leq i \leq n$.

A little-less-little lemma shows that

$$E_u(x) Y_i = \langle u \rangle_i E_u(x)$$

That is, the nonsymmetric Macdonald polynomials are the eigenfunctions of the $Y_i$. 

Macdonald polynomials made easy
With a bit of pain one checks the following amazing facts

\[
\begin{align*}
Y_i Y_j &= Y_j Y_i \\
T_i Y_{i+1} T_i &= t Y_i \\
T_i(Y_i + Y_{i+1}) &= (Y_i + Y_{i+1}) T_i \\
T_i Y_j &= Y_j T_i \quad \text{for } j \neq i, i+1
\end{align*}
\]

In other words, at the level of the algebra the “difficult” operators \( Y_i \) are not at all harder than the “easy” operators \( X_i \).
Finally one can check that

\[ T_i^2 X_{i+1} Y_i = t Y_i X_{i+1} \]

The algebra generated by the \( T_i, X_i, Y_i \) subject to all the is known as the double affine Hecke algebra (DAHA) (of type \( A_{n-1} \)), and was discovered by Ivan Cherednik.
The DAHA for arbitrary $G$ can be used to prove the **Macdonald CT** conjecture.

**Cherednik’s CT theorem**

Let $\Phi$ be a finite, reduced root system and $\Phi^+$ the set of positive roots. Let $D$ be the set of degrees of the fundamental invariants of $\Phi$. Then

$$
\text{CT} \left( \prod_{\alpha \in \Phi^+} (e^\alpha; q)_k(qe^{-\alpha}; q)_k \right) = \prod_{d \in D} \left[ \frac{dk}{k} \right]
$$
The End