Beta Integrals

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Euler Beta Integral

- Wallis formula (1656)

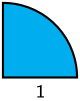
$$\frac{\pi}{2} = \frac{2^2}{1\cdot 3}\cdot\frac{4^2}{3\cdot 5}\cdot\frac{6^2}{5\cdot 7}\cdots$$

$$= \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$$

- Gamma function (Euler 1720s)

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^{x-1}}{x(x+1)\cdots(x+n-1)} \qquad x \neq 0, -1, -2, \ldots$$

$$= \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t \qquad \mathrm{Re}(x) > 0$$

Since 

$= \frac{\pi}{4}$ Wallis' formula is equivalent to

$$2\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \Gamma(1/2)\Gamma(3/2)$$

or, by $x^2 = t$, to

$$\int_0^1 t^{1/2-1}(1-t)^{3/2-1}\mathrm{d}t = \Gamma(1/2)\Gamma(3/2).$$

This led Euler to the discovery of a more general integral.

- Euler beta integral (1730s)

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Replacing $(\beta, t) \to (\zeta, t/\zeta)$ with $\zeta \in \mathbb{R}$ and letting $\zeta \to \infty$ using Stirling formula returns the integral representation of the gamma function.

Replacing $(\alpha, \beta, t) \to (\zeta^2+1, \zeta^2+1, 1/2 - x/(2\zeta))$ and letting $\zeta \to \infty$ yields the Gaussian integral

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = 1.$$

Much more on this later ...

For those with poor eyesight ...

Let $\beta = n + 1$ with $n = 0, 1, 2, \ldots$.

$$\int_0^1 t^{\alpha-1}(1-t)^n \mathrm{d}t = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{k+\alpha-1} \mathrm{d}t$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k+\alpha} \binom{n}{k}$$

$$= \frac{n!}{\alpha(\alpha+1)\ldots(\alpha+n)} \qquad \square$$

- Orthogonal polynomials

Set $t = (1-x)/2$ in the Euler beta integral and replace

$$(\alpha, \beta) \to (\alpha+1, \beta+1).$$

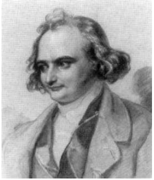
Then

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \mathrm{d}x = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

The distribution $\mathrm{d}w(x)$ on $[-1,1]$ given by

$$\mathrm{d}w(x) = (1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x$$

is that of the Jacobi (orthogonal) polynomials $P_n^{(\alpha,\beta)}(x)$.



$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \mathrm{d}w(x) = \delta_{mn} \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}.$$

Proof of the orthogonality and norm-evaluation follows immediately from the Rodrigues formula

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big]$$

(which may be taken as the definition of the Jacobi polynomials) and the Euler beta integral.

The one proof that all (good?) talks are supposed to have ...

To leading order the Rodrigues formula gives

$$x^{\alpha+\beta} P_n^{(\alpha,\beta)}(x) \text{ ``$=$'' } \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} x^{\alpha+\beta+2n}$$

so that

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} c_{nk}\, x^k$$

with

$$c_{nn} = \frac{(\alpha+\beta+n+1)\cdots(\alpha+\beta+2n)}{2^n n!}.$$

Without loss of generality assume that $m \leq n$.

Then

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \mathrm{d}w(x)$$

$$= \sum_{k=0}^{m} c_{mk} \frac{(-1)^n}{2^n n!} \int_{-1}^{1} x^k \frac{\mathrm{d}^n}{\mathrm{d}x^n} \Big[(1-x)^{\alpha+n} (1+x)^{\beta+n} \Big] \mathrm{d}x$$

(Rodrigues)

$$= \sum_{k=0}^{m} \frac{c_{mk}}{2^n} \delta_{kn} \int_{-1}^{1} (1-x)^{\alpha+n} (1+x)^{\beta+n} \mathrm{d}x$$

(k times integration by parts)

$$= \delta_{nm} \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (2n+\alpha+\beta+1) \Gamma(\alpha+\beta+n+1)}$$

(Euler beta integral & c_{nn}) □

Of course you should all care about the Jacobi polynomials since the Gegenbauer polynomials $C_n^{\lambda}(x)$ are nothing but

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)(2\lambda+1)\cdots(2\lambda+n-1)}{(\lambda+1/2)(\lambda+3/2)\cdots(\lambda+n-1/2)}\, P_n^{(\lambda-1/2,\lambda-1/2)}(x).$$

The fabulous Leopold Gegenbauer, Austria's favourite mathematician.

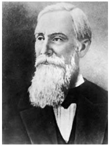
In fact, even non-Austrian's care (like the French and Russians) ...

$$T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!}\, P_n^{(-1/2,-1/2)}(x)$$ Chebyshev I

$$U_n(x) = \frac{2^{2n+1}((n+1)!)^2}{(2n+2)!}\, P_n^{(1/2,1/2)}(x)$$ Chebyshev II

$$P_n(x) = P_n^{(0,0)}(x)$$ Legendre

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x/\beta)$$ Laguerre



Selberg Integral

- Selberg integral (1944)

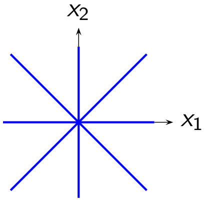
$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma} \mathrm{d}t$$

$$= n! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma(\alpha+\beta+(n+i-1)\gamma)\Gamma(\gamma)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \cdots$.

- Macdonald's conjectures (1982)

Let G be a finite reflection group or finite Coxeter group. That is, G is a finite group of isometries of $\mathbb{R}^n$ generated by reflections in hyperplanes through the origin.



The reflection group B_2 of order 8 (isomorphic to the signed permutations of $(1, 2)$), with 4 reflecting hyperplanes.

Normalise (up to sign) so that each hyperplane is of the form

$$a_1x_1 + \cdots + a_nx_n = 0$$

with

$$a_1^2 + \cdots + a_n^2 = 2.$$

Form the polynomial

$$P(x) = \prod_{\alpha=1}^{N} \Big(a_1^{(\alpha)} x_1 + \cdots + a_n^{(\alpha)} x_n \Big),$$

N being the number of hyperplanes.

Geometrically, $P(x)$ gives the product of the distances of the point $x = (x_1, \ldots, x_n)$ to the hyperplanes (up to a factor $2^{N/2}$).

By its action on $\mathbb{R}^n$ the reflection group G acts on polynomials in $x = (x_1, \dots, x_n)$.

The G-invariant polynomials form an $\mathbb{R}$-algebra $\mathbb{R}[f_1, \dots, f_n]$ generated by n algebraically independent polynomials $f_1, \dots, f_n$.

The $f_1, \dots, f_n$ are not unique but their degrees $d_1, \dots, d_n$ are.

Let φ be the Gaussian measure on $\mathbb{R}^n$:

$$d\varphi(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}\, dx.$$

Macdonald conjectured in 1982 that for every finite reflection group

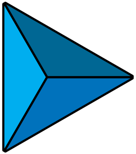
$$\int_{\mathbb{R}^n} |P(x)|^{2\gamma}\, d\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(d_i\gamma + 1)}{\Gamma(\gamma + 1)}.$$

For the trivial group A_0 of order 1 (mapping $\mathbb{R}$ to $\mathbb{R}$ by the identity map; i.e., no reflecting hyperplanes), $P(x) = 1$ and the conjecture corresponds to the Gaussian integral

$$\int_{\mathbb{R}} d\varphi(x) = 1.$$

- The reflection group A_{n-1}

A_{n-1} is the symmetry group of the $(n-1)$-simplex.



The 3-simplex or tetrahedron.

It is a group of order $n!$ (isomorphic to the symmetric group $\mathfrak{S}_n$) generated by the $\binom{n}{2}$ hyperplanes

$$x_i - x_j = 0 \qquad 1 \le i < j \le n.$$

The polynomial $P(x)$ is given by the Vandermonde product

$$P(x) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

The G-invariant polynomials are the symmetric polynomials in x, generated by the elementary symmetric functions $e_1, \dots, e_n$:

$$e_r(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Hence the degrees are given by $(d_1, d_2, \dots, d_n) = (1, 2, \dots, n)$.

Macdonald's conjecture for A_{n-1} is thus

$$\int_{\mathbb{R}^n} \prod_{1 \le i < j \le n} |x_i - x_j|^{2\gamma} \, \mathrm{d}\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(i\gamma + 1)}{\Gamma(\gamma + 1)}$$

better known as Mehta's integral.

This follows from the Selberg integral by taking

$$(\alpha, \beta) = (\zeta + 1, \zeta + 1) \qquad t_i = \frac{1}{2}\Big(1 - \frac{x_i}{\sqrt{2\zeta}}\Big) \qquad \zeta \to \infty.$$

- The reflection groups B_n and D_n

In these two cases the Macdonald conjecture is

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} |x_i|^{2\gamma} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma}\, \mathrm{d}\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

and

$$\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma}\, \mathrm{d}\varphi(x) = \frac{\Gamma(n\gamma+1)}{\Gamma(\gamma+1)} \prod_{i=1}^{n-1} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

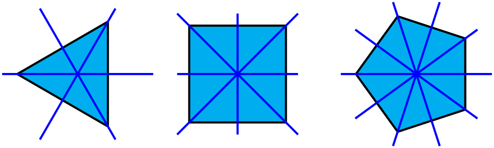
and follows again from the Selberg integral:

$$B_n: \quad (\alpha,\beta) = (\gamma+1/2, \zeta+1) \qquad t_i = \frac{x_i^2}{2\zeta} \qquad \zeta \to \infty$$

$$D_n: \quad (\alpha,\beta) = (1/2, \zeta+1) \qquad t_i = \frac{x_i^2}{2\zeta} \qquad \zeta \to \infty$$

- The dihedral group $I_2(m)$

$I_2(m)$ is the symmetry group of a regular m-gon,



The 3-gon, 4-gon and pentagon.

It is a group of order $2m$ generated by the m lines of reflection

$$\sqrt{2}x\sin\left(\frac{i\pi}{m}\right)-\sqrt{2}y\cos\left(\frac{i\pi}{m}\right)=0 \qquad 0\le i\le m-1.$$

The polynomial $P(x,y)$ is given by

$$\begin{aligned}P(x,y)&=\prod_{i=0}^{m-1}\left[\sqrt{2}y\cos\left(\frac{i\pi}{m}\right)-\sqrt{2}x\sin\left(\frac{i\pi}{m}\right)\right]\\&=-2^{1-m/2}(-r)^m\sin(m\phi).\end{aligned}$$

For $I_2(4)$ (symmetry group of the square) the invariant polynomials are of the form

$$\sum_{i,j} c_{ij}(xy)^{2i}(x^{2j}+y^{2j})$$

generated by x^2+y^2 and x^2y^2 of degree 2 and 4.

More generally, for $I_2(m)$ the invariant polynomials are generated by

$$x^2+y^2$$

and

$$x^m \sum_{i\geq 0} \left(-\frac{y^2}{x^2}\right)^i \binom{m}{2i}$$

so that the degrees are 2 and m.

Macdonald's conjecture for $I_2(m)$ (in polar coordinates) is thus

$$\frac{2^{2\gamma-m\gamma-1}}{\pi}\int_0^\infty r^{2m\gamma+1}e^{-r^2/2}dr\int_0^{2\pi}|\sin(m\phi)|d\phi = \frac{\Gamma(2\gamma+1)\Gamma(m\gamma+1)}{\Gamma^2(\gamma+1)}$$

which is (almost) trivially true.

- The exceptional reflection groups

For E_6, E_7, E_8, F_4 the proof is hard but follows from a uniform proof for all crystallographic reflection groups due to Opdam.

For the non-crystallographic groups H_3 and H_4 the proof is hard (Opdam, Garvan).

A_n Selberg Integral

- A_{n-1} versus A_1

We have seen that the Vandermonde product

$$\Delta(t) = \prod_{1 \le i < j \le n} (t_i - t_j)$$

and hence also the Selberg integral

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1 \le i < j \le n} |t_i - t_j|^{2\gamma} \mathrm{d}t$$

are connected to the reflection group A_{n-1}.

In the following we are going to depart from this point of view and will label the Selberg integral by the Lie algebra or root system A_1 as explained below.

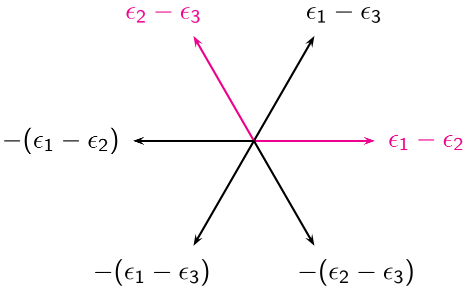
- The root system A_n

Recall that the reflection group A_n is generated by the $\binom{n+1}{2}$ hyperplanes

$$x_i - x_j = 0 \qquad 1 \leq i < j \leq n+1.$$

Let ϵ_i be the ith standard unit vector in $\mathbb{R}^{n+1}$.
The normals $\pm(\epsilon_i - \epsilon_j)$ for $1 \leq i < j \leq n+1$ are known as roots and form the root system A_n.

The roots $a_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$ form a basis in the root system and are known as simple roots.

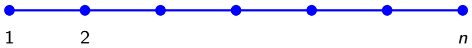


The root system A_2 with simple roots in pink.

The Cartan matrix C of A_n is given by

$$\big(a_i \cdot a_j\big)_{1\le i,j\le n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & & \ddots & \\ & & \ddots & & -1 \\ & & & -1 & 2 \end{pmatrix}$$

The A_n Dynkin diagram encodes the adjaceny matrix $2I - C$:



To each simple root a_s attach a set of variables

$$t^{(s)} = (t_1^{(s)}, \dots, t_{k_s}^{(s)})$$

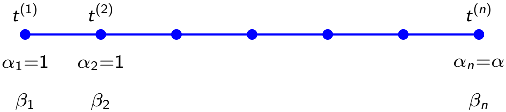
such that $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$.

Set $k_0 = k_{n+1} = 0$ and let $\alpha, \beta_1, \dots, \beta_n, \gamma \in \mathbb{C}$ subject to several mild restrictions, such as

$$\mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta_1) > 0, \dots, \mathrm{Re}(\beta_n) > 0.$$

Set

$$(\alpha_1, \dots, \alpha_{n-1}, \alpha_n) = (1, \dots, 1, \alpha).$$



Define the generalised Vandermonde product

$$\Delta(u, v) = \prod_{i,j \geq 1} (u_i - v_j)$$

and let

$$C^{k_1,\dots,k_n}[0,1] \subseteq [0,1]^{k_1+\dots+k_n}$$

be an integration domain, somewhat too technical for a talk.

- A_n Selberg integral

$$\int_{C^{k_1,\dots,k_n}[0,1]} \prod_{s=1}^{n}\prod_{i=1}^{k_s}\bigl(t_i^{(s)}\bigr)^{\alpha_s-1}\bigl(1-t_i^{(s)}\bigr)^{\beta_s-1}$$

$$\times \prod_{s=1}^{n-1}\bigl|\Delta\bigl(t^{(s)},t^{(s+1)}\bigr)\bigr|^{-\gamma}\prod_{s=1}^{n}\bigl|\Delta\bigl(t^{(s)}\bigr)\bigr|^{2\gamma}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}$$

$$=\prod_{1\le s\le r\le n}\prod_{i=1}^{k_s-k_{s-1}}\frac{\Gamma(\beta_s+\cdots+\beta_r+(i+s-r-1)\gamma)}{\Gamma(\alpha_r+\beta_s+\cdots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)}$$

$$\times\prod_{s=1}^{n}\prod_{i=1}^{k_s}\frac{\Gamma(\alpha_s+(i-k_{s+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}.$$

- The q-binomial theorem I

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$ the q-Pochhammer symbols are

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

and

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}.$$

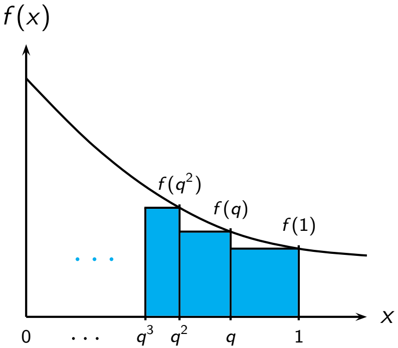
Then the q-binomial theorem is given by

$$\sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k} z^k = \frac{(bz;q)_\infty}{(z;q)_\infty}.$$

Let

$$\int_0^1 f(x)\mathrm{d}_q x = (1-q)\sum_{i=0}^{\infty} f(q^i)q^i$$

be the Jackson or q-integral.



Then the q-binomial theorem with $z = q^\alpha$ and $b = q^\beta$ may be written as the q-beta integral

$$\int_0^1 t^{\alpha-1}(tq;q)_{\beta-1}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

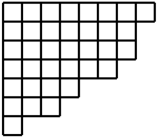
where Γ_q is the q-gamma function:

$$\Gamma_q(x) = (1-q)^{1-x}(q;q)_{x-1}.$$

In the $q \to 1^-$ limit the q-binomial theorem thus yields the Euler beta integral.

- Macdonald polynomials

Let x be an n-letter alphabet with letters $x_1, x_2, \ldots, x_n$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition

$\lambda =$ 

The q-shift operator T_{q,x_i} is defined as

$$T_{q,x_i}\big(f(x)\big) = f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n).$$

and Macdonald's commuting family $\{D_r\}_{r=0}^n$ of q-difference operators is given by

$$D_r = t^{\binom{r}{2}} \sum_{\substack{I \subseteq [n] \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i}.$$

Defining the generating series of the D_r as

$$D(u;q,t)=\sum_{r=0}^{n} D_r u^r$$

the Macdonald polynomials $P_\lambda(x;q,t)$ are the eigenfunctions of $D(u;q,t)$ with eigenvalue

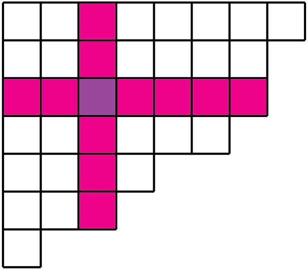
$$\prod_{i=1}^{n}(1+ut^{n-i}q^{\lambda_i}).$$

For $q=t$ the Macdonald polynomials simplify to the well-known Schur functions

$$P_\lambda(x;t,t)=s_\lambda(x)=\frac{\det_{1\le i,j\le n}(x_i^{\lambda_j+n-j})}{\det_{1\le i,j\le n}(x_i^{n-j})}.$$

- Cauchy identity

Given a partition λ, each of its squares s is assigned four integers, known as the arm-length $a(s)$, leg-length $l(s)$, arm-colength $a'(s)$ and leg-colength $l'(s)$.



The arm-length of ■ is 4. The leg-length of ■ is 3.
The arm- and leg-colengths of ■ are both 2.

The Cauchy identity for Macdonald polynomials is

$$\sum_{\lambda} P_\lambda(x;q,t) P_\lambda(y;q,t) \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = \prod_{i,j\geq 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

When $q=t$ this reduces to the well-known Cauchy determinant

$$\det_{1\leq i\leq j\leq n}\left(\frac{1}{1-x_iy_j}\right) = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1-x_iy_j)}.$$

- The q-binomial theorem II

The power sums p_r are given by $p_0 = 1$ and

$$p_r(x) = \sum_{i \geq 1} x_i^r.$$

The map $\epsilon_{b,t}$ — acting on symmetric functions of y — is defined by its action on the p_r:

$$\epsilon_{b,t}\big(p_r(y)\big) = \frac{1 - b^r}{1 - t^r}.$$

A theorem of Macdonald states that

$$\epsilon_{b,t}\big(P_\lambda(y; q, t)\big) = \prod_{s \in \lambda} \frac{t^{l'(s)} - b\, q^{a'(s)}}{1 - q^{a(s)} t^{l(s)+1}}.$$

It may also be shown that

$$\epsilon_{b,t}\Bigg(\prod_{i,j\geq 1}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\Bigg)=\prod_{i\geq 1}\frac{(b\,x_i;q)_\infty}{(x_i;q)_\infty}.$$

Applying the map $\epsilon_{b,t}$ to the Cauchy identity we thus obtain an n-dimensional analogue of the q-binomial theorem:

$$\sum_\lambda P_\lambda(x;q,t)\prod_{s\in\lambda}\frac{t^{l'(s)}-b\,q^{a'(s)}}{1-q^{a(s)+1}t^{l(s)}}=\prod_{i=1}^n\frac{(b\,x_i;q)_\infty}{(x_i;q)_\infty}$$

If $n=1$ then $x=(x_1)$, $\lambda=(k)$ and

$$P_{(k)}(x;q,t)=x_1^k,\qquad \prod_{s\in\lambda}\frac{t^{l'(s)}-b\,q^{a'(s)}}{1-q^{a(s)+1}t^{l(s)}}=\frac{(b;q)_k}{(q;q)_k}$$

so that we recover the classical q-binomial theorem (with $z\to x_1$).

Taking

$$\begin{aligned} x_i &= q^{\alpha+\gamma(n-i)} \quad \text{for} \quad 1 \le i \le n \\ t &= q^{\gamma} \\ b &= q^{\beta} \end{aligned}$$

in the n-dimensional q-binomial theorem yields an n-dimensional q-integral, generalising the q-beta integral.

In the $q \to 1^-$ limit this gives the Selberg integral.

To prove the A_n Selberg integral we need a further generalisation of the q-binomial theorem!

- The q-binomial theorem III

One may prove a q-binomial theorem of the form

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} P_{\lambda^{(1)}}(x^{(1)};q,t)\cdots P_{\lambda^{(n)}}(x^{(n)};q,t) \\ \times \bigl(\text{stuff with arms and legs}\bigr) = \text{infinite product}$$

with $x^{(s)} = (x_1^{(s)},\dots,x_{k_s}^{(s)})$ and $k_1 \le k_2 \le \cdots \le k_n$.

$\Rightarrow$ A $(k_1+\cdots+k_n)$-dimensional q-integral

$\Rightarrow$ The A_n Selberg integral.

- Open problems
  - Can we evaluate the integral

$$\int_{C^{k_1,\dots,k_n}[0,1]} \prod_{s=1}^{n}\prod_{i=1}^{k_s}\big(t_i^{(s)}\big)^{\alpha_s-1}\big(1-t_i^{(s)}\big)^{\beta_s-1} \times \prod_{s=1}^{n-1}\big|\Delta\big(t^{(s)},t^{(s+1)}\big)\big|^{-\gamma}\prod_{s=1}^{n}\big|\Delta\big(t^{(s)}\big)\big|^{2\gamma}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}$$

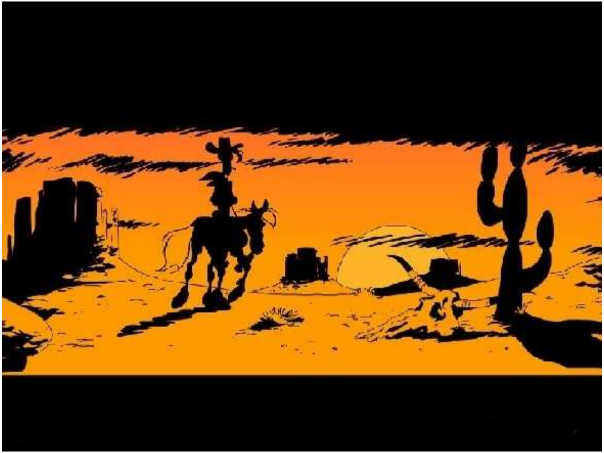
    when

$$(\alpha_1,\dots,\alpha_{n-1},\alpha_n)\neq(1,\dots,1,\alpha)?$$

  - Can we remove the ordering

$$0\le k_1\le k_2\le\cdots\le k_n\ ?$$

  - Can we generalise to other root systems and/or reflection groups?



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