Introduction to Mathematics with Maple

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Preface

> I attempted mathematics, and even went during the summer of 1828 with a private tutor to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense.
>
> Charles Darwin, *Autobiography* (1876)

Charles Darwin wasn't the last biologist to regret not knowing more about mathematics. Perhaps if he had started his university education at the beginning of the 21^{st} century, instead of in 1828, and had been able to profit from using a computer algebra package, he would have found the material less forbidding.

This book is about pure mathematics. Our aim is to equip the readers with understanding and sufficiently deep knowledge to enable them to use it in solving problems. We also hope that this book will help readers develop an appreciation of the intrinsic beauty of the subject!

We have said that this book is about mathematics. However, we make extensive use of the computer algebra package *Maple* in our discussion. Many books teach pure mathematics without any reference to computers, whereas other books concentrate too heavily on computing, without explaining substantial mathematical theory. We aim for a better balance: we present material which requires deep thinking and understanding, but we also fully encourage our readers to use *Maple* to remove some of the laborious computations, and to experiment. To this end we include a large number of *Maple* examples.

Most of the mathematical material in this book is explained in a fairly traditional manner (of course, apart from the use of *Maple*!). However, we depart from the traditional presentation of integral by presenting the Kurzweil—Henstock theory in Chapter 15.

Outline of the book

There are fifteen chapters. Each starts with a short abstract describing the content and aim of the chapter.

In Chapter 1, "Introduction", we explain the scope and guiding philosophy of the present book, and we make clear its logical structure and the role which *Maple* plays in the book. It is our aim to equip the readers with sufficiently deep knowledge of the material presented so they can use it in solving problems, and appreciate its inner beauty.

In Chapter 2, "Sets", we review set theoretic terminology and notation and provide the essential parts of set theory needed for use elsewhere in the book. The development here is not strictly axiomatic—that would require, by itself, a book nearly as large as this one—but gives only the most important parts of the theory. Later we discuss mathematical reasoning, and the importance of rigorous proofs in mathematics.

In Chapter 3, "Functions", we introduce relations, functions and various notations connected with functions, and study some basic concepts intimately related to functions.

In Chapter 4, "Real Numbers", we introduce real numbers on an axiomatic basis, solve inequalities, introduce the absolute value and discuss the least upper bound axiom. In the concluding section we outline an alternative development of the real number system, starting from Peano's axioms for natural numbers.

In Chapter 5, "Mathematical Induction", we study proof by induction and prove some important inequalities, particularly the arithmetic-geometric mean inequality. In order to employ induction for defining new objects we prove the so-called recursion theorems. Basic properties of powers with rational exponents are also established in this chapter.

In Chapter 6, "Polynomials", we introduce polynomials. Polynomial functions have always been important, if for nothing else than because, in the past, they were the only functions which could be readily evaluated. In this chapter we define polynomials as algebraic entities rather than functions and establish the long division algorithm in an abstract setting. We also look briefly at zeros of polynomials and prove the Taylor Theorem for polynomials in a generality which cannot be obtained by using methods of calculus.

In Chapter 7, "Complex Numbers", we introduce complex numbers, that is, numbers of the form $a + b\imath$ where the number $\imath$ satisfies $\imath^2 = -1$. Mathematicians were led to complex numbers in their efforts of solving algebraic equations, that is, of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$, with the a_k real numbers and n a positive integer (this problem is perhaps more widely known as finding the zeros of a polynomial). Our introduction follows the same idea although in a modern mathematical setting. Complex numbers now play important roles in physics, hydrodynamics, electromagnetic theory and electrical engineering, as well as pure mathematics.

In Chapter 8, "Solving Equations", we discuss the existence and uniqueness of solutions to various equations and show how to use *Maple* to find solutions. We deal mainly with polynomial equations in one unknown, but include some basic facts about systems of linear equations.

In Chapter 9, "Sets Revisited", we introduce the concept of equivalence for sets and study countable sets. We also briefly discuss the axiom of choice.

In Chapter 10, "Limits of Sequences", we introduce the idea of the limit of a sequence and prove basic theorems on limits. The concept of a limit is central to subsequent chapters of this book. The later sections are devoted to the general principle of convergence and more advanced concepts of limits superior and limits inferior of a sequence.

In Chapter 11, "Series", we introduce infinite series and prove some basic convergence theorems. We also introduce power series—a very powerful tool in analysis.

In Chapter 12, "Limits and Continuity of Functions", we define limits of functions in terms of limits of sequences. With a function f continuous on an interval we associate the intuitive idea of the graph f being drawn without lifting the pencil from the drawing paper. The mathematical treatment of continuity starts with the definition of a function continuous at a point; this definition is given here in terms of a limit of a function at a point. We develop the theory of limits of functions, study continuous functions, and particularly functions continuous on closed bounded intervals. At the end of the chapter we touch upon the concept of limit superior and inferior of a function.

In Chapter 13, "Derivatives", we start with the informal description of a derivative as a rate of change. This concept is extremely important in science and applications. In this chapter we introduce derivatives as limits, establish their properties and use them in studying deeper properties of functions and their graphs. We also extend the Taylor Theorem from polynomials to power series and explore it for applications.

In Chapter 14, "Elementary Functions", we lay the proper foundations for the exponential and logarithmic functions, and for trigonometric functions and their inverses. We calculate derivatives of these functions and use these for establishing important properties of these functions.

In Chapter 15, "Integrals", we present the theory of integration introduced by the contemporary Czech mathematician J. Kurzweil. Sometimes it is referred to as Kurzweil—Henstock theory. Our presentation generally follows Lee and Výborný (2000, Chapter 2).

The Appendix contains some examples of *Maple* programs. Finally, the book concludes with a list of References, an Index of *Maple* commands used in the book, and a general Index.

Notes on notation

Throughout the book there are a number of ways in which the reader's attention is drawn to particular points. Theorems, lemmas[1] and corollaries are placed inside rectangular boxes with double lines, as in

> **Theorem 0.1 (For illustrative purposes only!)** *This theorem is referred to only in the* Preface*, and can safely be ignored when reading the rest of the book.*

Note that these are set in slanting font, instead of the upright font used in the bulk of the book.

Definitions are set in the normal font, and are placed within rectangular boxes, outlined by a single line and with rounded corners, as in

> **Definition 0.1 (What is mathematics?)** There are almost as many definitions of what mathematics is as there are professional mathematicians living at the time.

[1]A lemma is sometimes known as an auxiliary theorem. It does not have the same level of significance as a theorem, and is usually proved separately to simplify the proof of the related theorem(s).

The number before the decimal point in all of the above is the number of the chapter: numbers following the decimal point label the different theorems, definitions, examples, etc., and are numbered consecutively (and separately) within each chapter. Corollaries are labeled by the number of the chapter, followed by the number of the theorem to which the corollary belongs then followed by the number of the individual corollary for that theorem, as in

Corollary 0.1.1 (Also for illustrative purposes only!) *Since* Theorem 0.1 *is referred to only in the* Preface, *any of its corollaries can also be ignored when reading the rest of the book.*

Corollary 0.1.2 (Second corollary for Theorem 0.1) *This is just as helpful as the first corollary for the theorem!*

Corollaries are set in the same font as theorems and lemmas.

Most of the theorems, lemmas and corollaries in this book are provided with proofs. All proofs commence with the word "***Proof.***" flush with the left margin. Since the words of a proof are set in the same font as the rest of the book, a special symbol is used to mark the end of a proof and the resumption of the main text. Instead of saying that a proof is complete, or words to that effect, we shall place the symbol □ at the end of the proof, and flush with the right margin, as follows:

Proof. This is not really a proof. Its main purpose is to illustrate the occurrence of a small hollow square, flush with the right margin, to indicate the end of a proof. □

Up to about the middle of the 20$^{\text{th}}$ century it was customary to use the letters 'q.e.d.' instead of □, q.e.d. being an abbreviation for *quod erat demonstrandum*, which, translated from Latin, means *which was to be proved*.

To assist the reader, there are a number of Remarks scattered throughout the book. These relate to the immediately preceding text. There are also Examples of various kinds, used to illustrate a concept by providing a (usually simple) case which can show the main distinguishing points of a concept. Remarks and Examples are set in sans serif font, like this, to help distinguish them.

Remark 0.1 The first book published on calculus was Sir Isaac Newton's *Philosophiae Naturalis Principia Mathematica* (Latin for *The Mathematical Principles of Natural Philosophy*), commonly referred to simply as *Principia*. As might be expected from the title, this was in Latin. We shall avoid the use of languages other than English in this book.

Since they use a different font, and have additional spacing above and below, it is obvious where the end of a Remark or an Example occurs, and no special symbol is needed to mark the return to the main text.

Scissors in the margin

Obviously, we must build on some previous knowledge of our readers. Chapter 1 and Chapter 2 summarise such prerequisites. Chapter 1 contains also a brief introduction to *Maple*. Starting with Chapter 3 we have tried to make sure that all proofs are in a strict logical order. On a few occasions we relax the logical requirements in order to illustrate some point or to help the reader place the material in a wider context. All such instances are clearly marked in the margin (see the outer margin of this page), the beginning by scissors pointing into the book, the end by scissors opening outwards. The idea here is to indicate readers can skip over these sections if they desire strict logical purity. For instance, we might use trigonometric functions before they are properly introduced,[2] but then this example will be scissored.

Exercises

There are exercises to help readers to master the material presented. We hope readers will attempt as many as possible. Mathematics is learned by doing, rather than just reading. Some of the exercises are challenging, and these are marked in the margin by the symbol (!). We do not expect that readers will make an effort to solve all these challenging problems, but should attempt at least some. Exercises containing fairly important additional information, not included in the main body of text, are marked by (i). We recommend that these should be read even if no attempt is made to solve them.

[2]Rather late in Chapter 14

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The authors wish to gratefully acknowledge the support of Waterloo Maple Inc, who provided us with copies of *Maple* version 7 software. We thank the Mathematics Department at The University of Queensland for its support and resourcing. We also thank those students we have taught over many years, who, by their questions, have helped us improve our teaching.

Our greatest debt is to our wives. As a small token of our love and appreciation we dedicate this book to them.

Peter Adams
Ken Smith
Rudolf Výborný

The University of Queensland, February 2004.

Contents

Chapter 1

Introduction

In this chapter we explain the scope and guiding philosophy of the present book, we try to make clear its logical structure and the role which *Maple* plays in the book. We also wish to orientate the readers on the logical structure which forms the basis of this book.

1.1 Our aims

Mathematics can be compared to a cathedral. We wish to visit a small part of this cathedral of human ideas of quantities and space. We wish to learn how mathematics can be built. Mathematics spans a very wide spectrum, from the simple arithmetic operations a pupil learns in primary school to the sophisticated and difficult research which only a specialist can understand after years of long and hard postgraduate study. We place ourselves somewhere higher up in the lower half of this spectrum. This can also be roughly described as where University mathematics starts. In natural sciences the criterion of validity of a theory is experiment and practice. Mathematics is very different. Experiment and practice are insufficient for establishing mathematical truth. Mathematics is deductive, the only means of ascertaining the validity of a statement is logic. However, the chain of logical arguments cannot be extended indefinitely: inevitably there comes a point where we have to accept some basic propositions without proofs. The ancient Greeks called these foundation stones *axioms*, accepted their validity without questioning and developed all their mathematics therefrom. In modern mathematics we also use axioms but we have a different viewpoint. The axiomatic method is discussed later in section 2.5. Ideally, teaching of mathematics would start with the axioms, however, this is hopelessly

Chapter 2

Sets

In this chapter we review set theoretic terminology and notation. Later we discuss mathematical reasoning.

2.1 Sets

The word set as used in mathematics means a collection of objects. It is customary to denote sets by capital letters like M, M_1, etc., below. At this stage the concept of a set is best illuminated by examples, so we list a few sets and name them for ease of reference.

Table 2.1 Some examples of sets

M_1,	the set of all rational numbers greater than 1;
M_2,	the set of all positive even integers;
M_3,	the set of all buildings on the St Lucia campus of the University of Queensland;
M_4,	the set of all readers of this book;
M_5,	the set of all persons who praise this book;
M_6,	the set of all persons who condemn this book;
M_7,	the set of all even numbers between 1 and 5.

In these examples we have used the word 'all' to make it doubly clear that, for instance, *every* rational number greater than 1 belongs to M_1. But usually, in mathematics, if someone mentions the set of rational numbers greater than 1 he or she means the set of all such numbers, and this is

Chapter 3

Functions

In this chapter we introduce relations, functions and various notations connected with functions, and study some basic concepts intimately related to functions.

3.1 Relations

In mathematics it is customary to define new concepts by using set theory. To say that two things are related is really the same as saying that the ordered pair (a, b) has some property. This, in turn, can be expressed by saying that the pair (a, b) belongs to some set. We define:

Definition 3.1 (Relation) A relation is a set of ordered pairs.

This means that if A and B are sets then a relation is a *part* of the cartesian product $A \times B$. If a set R is a relation and $(a, b) \in R$ then the elements a and b are related; we also denote this by writing aRb. For example, if related means that the first person is the father of the second person then this relation consists of all pairs of the form (father, daughter) or (father, son). Another example of a relation is the set $C = \{(1, 2), (2, 3), \ldots, (11, 12), (12, 1)\}$. This relation can be interpreted by saying that m and n are related (mCn) if n is the hour immediately following m on the face of a 12 hour clock.

We define the *domain of the relation* R to be the set of all first elements of pairs in R. Denoting by $\operatorname{dom} R$ the domain of R we have

$$\operatorname{dom} R = \{a\,;\ (a, b) \in R\}\,.$$

The *range of* R is denoted by $\operatorname{rg} R$ and is the set of all second elements of

Chapter 4

Real Numbers

In this chapter we introduce real numbers on an axiomatic basis, solve inequalities, introduce the absolute value and discuss the least upper bound axiom. In the concluding section we outline an alternative development of the real number system.

4.1 Fields

Real numbers satisfy three groups of axioms—field axioms, order axioms and the least upper bound axiom. We discuss each group separately.

A set F together with two functions $(x, y) \mapsto x + y$, $(x, y) \mapsto xy$ from $F \times F$ into F is called a *field* if the axioms in Table 4.1 are satisfied for all x, y, z in F.

Table 4.1 Field axioms

$\mathsf{A}_1 : x + y = y + x$	$\mathsf{M}_1 : xy = yx$
$\mathsf{A}_2 : x + (y + z) = (x + y) + z$	$\mathsf{M}_2 : x(yz) = (xy)z$
$\mathsf{A}_3 :$ There is an element $0 \in F$ such that $0 + x = x$ for all x in F	$\mathsf{M}_3 :$ There is an element $1 \in F$, $1 \neq 0$, such that $1x = x$ for all x in F;
$\mathsf{A}_4 :$ For every element $x \in F$ there exists an element $(-x) \in F$ such that $(-x) + x = 0$.	$\mathsf{M}_4 :$ For every element $x \in F$, $x \neq 0$, there exists an element $x^{-1} \in F$ such that $x^{-1}x = 1$;
$\mathsf{D} : x(y + z) = xy + xz.$	

Chapter 5

Mathematical Induction

In this chapter we study proof by induction and prove some important inequalities, particularly the arithmetic-geometric mean inequality. In order to employ induction for defining new objects we prove the so called recursion theorems. Basic properties of powers with rational exponents are also established in this chapter.

5.1 Inductive reasoning

The process of deriving general conclusions from particular facts is called induction. It is often used in the natural sciences. For example, an ornithologist watches birds of a certain species and then draws conclusions about the behaviour of all members of that species. General laws of motion were discovered from the motion of planets in the solar system. The following example shows that we encounter inductive reasoning also in mathematics.

Example 5.1 Let us consider the numbers $n^5 - n$ for the first few natural numbers:

n	$n^5 - n$
1	0
2	30
3	240
4	1020
5	3120
6	7770
7	16800

It seems likely that for every $n \in \mathbb{N}$ the number $n^5 - n$ is a multiple of 10.

Chapter 6

Polynomials

Polynomial functions have always been important, if for nothing else than because, in the past, they were the only functions which could be readily evaluated. In this chapter we define polynomials as algebraic entities rather than functions, establish the long division algorithm in an abstract setting, we also look briefly at zeros of polynomials and prove the Taylor Theorem for polynomials in a generality which cannot be obtained by using methods of calculus.

6.1 Polynomial functions

If $\boldsymbol{M}$ is a ring and a_0, a_1, a_2, ..., $a_n \in \boldsymbol{M}$ then a function of the form

$$A : x \mapsto a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{6.1}$$

is called a *polynomial*, or sometimes more explicitly, a polynomial with coefficients in $\boldsymbol{M}$. Obviously, one can add any number of zero coefficients, or rewrite Equation (6.1) in ascending order of powers of x without changing the polynomial. The domain of definition of the polynomial is naturally $\boldsymbol{M}$, but the definition of $A(x)$ makes sense for any x in a ring which contains $\boldsymbol{M}$. This natural extension of the domain of definition is often understood without explicitly saying so. If A and B are two polynomials then the polynomials $A + B$, $-A$ and AB are defined in the obvious way as

$$\begin{aligned} A + B &: x \mapsto A(x) + B(x) \\ -A &: x \mapsto -A(x) \\ AB &: x \mapsto A(x)B(x) \end{aligned}$$

The coefficients of $A+B$ are obvious; they are the sums of the corresponding coefficients of A and B. The zero polynomial function is the zero function,

Chapter 7

Complex Numbers

We introduce complex numbers; that is numbers of the form $a + b\imath$ where the number $\imath$ satisfies $\imath^2 = -1$. Mathematicians were led to complex numbers in their efforts to solve so-called algebraic equations; that is equations of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$, with $a_k \in \mathbb{R}$, $n \in \mathbb{N}$. Our introduction follows the same idea although in a modern mathematical setting. Complex numbers now play important roles in physics, hydrodynamics, electromagnetic theory, electrical engineering as well as pure mathematics.

7.1 Field extensions

During the history of civilisation the concept of a number was unceasingly extended, from integers to rationals, from positive numbers to negative numbers, from rationals to reals, etc. We now embark on an extension of reals to a field in which the equation

$$\xi^2 + 1 = 0 \tag{7.1}$$

has a solution. This will be the field of complex numbers.

Let us consider the following question. Is it possible to extend the field of rationals to a larger field in which the equation $\xi^2 - 2 = 0$ is solvable? The obvious answer is yes: the reals. Is there a smaller field? The answer is again yes: there is a smallest field which contains rationals and $\sqrt{2}$, namely the intersection of all fields which contain $\mathbb{Q}$ and the (real) number $\sqrt{2}$. Can this field be constructed directly without using the existence of $\mathbb{R}$? The answer is contained in Exercises 7.1.1–7.1.3.

The process of extension of a field can be most easily carried out in the case of a finite field. Let us consider the following problem: is it possible

Chapter 8

Solving Equations

In this chapter we discuss existence and uniqueness of solutions to various equations and show how to use *Maple* to find solutions. We deal mainly with polynomial equations in one unknown and add only some basic facts about systems of linear equations.

8.1 General remarks

Given a function f, solving the equation[1]

$$f(x) = 0 \tag{8.1}$$

means finding all x in $\operatorname{dom} f$ which satisfy Equation (8.1). Any x which satisfies Equation (8.1) is called a solution of Equation (8.1). In this chapter we assume that $\operatorname{dom} f$ is always part of $\mathbb{R}$ or $\mathbb{C}$ and it will be clear from the context which set is meant. Solving an equation like (8.1) usually consists of a chain of implications, starting with the equation itself and ending with an equation (or equations) of the form $x = a$. For instance:

$$\begin{aligned} \frac{1}{x+1} + \frac{1}{x+2} = 0 &\Rightarrow x + 2 + x + 1 = 0, \\ &\Rightarrow 2x + 3 = 0, \\ &\Rightarrow x = -\frac{3}{2}. \end{aligned} \tag{8.2}$$

For greater clarity we printed the implication signs, but implications are always understood automatically. It is a good habit always to check the solution by substituting the found value back into the original equation. This

[1] A more complicated equation of the form $F(x) = G(x)$ can be reduced to the form given by taking $f = F - G$.

Chapter 9

Sets Revisited

In this chapter we introduce the concept of equivalence for sets and study countable sets. We also discuss briefly the axiom of choice.

9.1 Equivalent sets

Two finite sets A and B have the same number of elements if there exists a bijection of A onto B. We can count the visitors in a sold-out theatre by counting the seats. The concept of equivalence for sets is an extension of the concept of two sets having the same number of elements, generalised to infinite sets.

Definition 9.1 A set A is said to be equivalent to a set B if there exists a bijection of A onto B; we then write $A \sim B$.

Example 9.1

(a) The set $\mathbb{N}$ and the set of even positive integers are equivalent. Indeed, $n \mapsto 2n$ is a bijection of $\mathbb{N}$ onto $\{2, 4, 6, \ldots\}$.

(b) $\mathbb{Z} \sim \mathbb{N}$. Let $f : n \mapsto 2n$ for $n > 0$, and $f : n \mapsto -2n + 1$ for $n \leq 0$. Then f is obviously onto and it is easy to check that it is one-to-one. Hence it is a bijection of $\mathbb{Z}$ onto $\mathbb{N}$.

(c) $]0, 1[\sim]0, \infty[$. The required bijection is $x \mapsto \dfrac{1}{x} - 1$.

We observe that the relation $A \sim B$ is reflexive, symmetric and transitive, and thus it is an equivalence relation. Reflexivity follows from the use of the function id_A, which provides a bijection of A onto itself. If the function f is a bijection of A onto B then it has an inverse and f_{-1} is a bijection of B onto A, so $A \sim B \Rightarrow B \sim A$. Finally, if f and g are bijections

Chapter 11

Series

In this chapter we introduce infinite series and prove some basic convergence theorems. We also introduce power series—a very powerful tool in analysis.

11.1 Definition of convergence

Study of the behaviour of the terms of a sequence when they are successively added leads to infinite series. For an arbitrary sequence $n \mapsto a_n \in \mathbb{C}$ we can form another sequence by successive additions as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i \end{aligned} \tag{11.1}$$

To indicate that we consider $n \mapsto s_n$ rather than $n \mapsto a_n$ we write

$$\sum a_i. \tag{11.2}$$

The symbol (11.2) is just an abbreviation for the sequence $n \mapsto s_n$, with s_n as in (11.1). We shall call (11.2) a "series" or an "infinite series"; a_n is the n^{th} term of (11.2); s_n is the n^{th} partial sum of (11.2).

It is usually clear from the context what the partial sums for a series like (11.2) are. On the other hand, if $a_i = \dfrac{1}{k^i}$ for some positive integer k

Chapter 12

Limits and Continuity of Functions

Limits of functions are defined in terms of limits of sequences. With a function f continuous on an interval we associate the intuitive idea of the graph f being drawn without lifting the pencil from the drawing paper. Mathematical treatment of continuity starts with the definition of a function continuous at a point; this definition is given here in terms of a limit of a function at a point. In this chapter we shall develop the theory of limits of functions, study continuous functions, and particularly functions continuous on closed bounded intervals. At the end of the chapter we touch upon the concept of limit superior and inferior of a function.

12.1 Limits

Looking at the graph of $f : x \mapsto \dfrac{x}{|x|}(1-x)$ (Figure 12.1), it is natural to say that the function value approaches 1 as x approaches 0 from the right. Formally we define:

Definition 12.1 A function f, with $\operatorname{dom} f \subset \mathbb{R}$, is said to have a limit l at $\hat{x}$ from the right if for every sequence $n \mapsto x_n$ for which $x_n \to \hat{x}$ and $x_n > \hat{x}$ it follows that $f(x_n) \to l$. If f has a limit l at $\hat{x}$ from the right, we write $\lim\limits_{x \downarrow \hat{x}} f(x) = l$.

Definition 12.2 If the condition $x_n > \hat{x}$ is replaced by $x_n < \hat{x}$ one obtains the definition of the limit of f at $\hat{x}$ from the left. The limit of f at $\hat{x}$ from the left is denoted by $\lim\limits_{x \uparrow \hat{x}} f(x)$.

Remark 12.1 The symbols $x \downarrow \hat{x}$ and $x \uparrow \hat{x}$ can be read as "x decreases

Exercise 13.1.2 *Show that* $D\dfrac{1}{x^n} = -n\dfrac{1}{x^{n+1}}$ *for* $x \neq 0$ *and* $n \in \mathbb{N}$. [Hint: Use equation (13.1) and the result from Example 13.1, namely that $Dx^n = nx^{n-1}$.]

Exercise 13.1.3 *Use* Maple *to find* $D(1+\sqrt{x})/(1-\sqrt{x})$.

13.2 Basic theorems on derivatives

If f is differentiable at x then the function

$$h \mapsto \mathcal{D}(h) = \frac{f(x+h)-f(x)}{h} \tag{13.3}$$

has a removable discontinuity at 0 and by defining $\mathcal{D}(0) = f'(x)$ becomes continuous. This leads to the next theorem, which is theoretical in character but needed at many practical occasions.

Theorem 13.1 *The function f is differentiable at x if and only if there exists a function $\mathcal{D}$ continuous at zero such that Equation (13.3) holds. If so then $\mathcal{D}(0) = f'(x)$.*

Remark 13.3 This theorem as well the next two are valid in the complex domain.

From this theorem it follows immediately:

Theorem 13.2 *If a function has a finite derivative at a point then the function is continuous at this point.*

Proof. As $h \to 0$, Equation (13.3) yields $f(x+h) - f(x) \to \mathcal{D}(0)0 = 0$ so f is continuous at x. □

Theorem 13.3 *If $f'(x)$ and $g'(x)$ exist and c is a constant then*

$$[cf(x)]' = cf'(x), \tag{13.4}$$

$$[f(x)+g(x)]' = f'(x) + g'(x), \tag{13.5}$$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x). \tag{13.6}$$

Chapter 14

Elementary Functions

In this chapter we lay the proper foundations for the exponential and logarithmic functions, for trigonometric functions and their inverses. We calculate derivatives of these functions and use this for establishing important properties of these functions.

14.1 Introduction

We have already mentioned that some theorems from the previous chapter are valid for differentiation in the complex domain. Specifically, this is so for Theorem 13.1 and 13.2, for basic rules of differentiation in Theorems 13.3, 13.5 and for the chain rule, Theorem 13.4. In contrast, Theorem 13.7 makes no sense in the complex domain since the concept of an increasing function applies only to functions which have real values. However, part (iii) of Theorem 13.7 can be extended as follows.

Theorem 14.1 *For $a \in \mathbb{C}$ and $R > 0$ denote by S the disc $\{z; |z-a| < R\}$ or the whole complex plane $\mathbb{C}$. If $f'(z) = 0$ for all $z \in S$ then f is constant in S.*

Proof. For $t \in [0,1]$ let $Z = tz_1 + (1-t)z_2$. If z_1 and z_2 are in S so is Z. Let $F : t \mapsto f(Z)$, $F_1(t) = \Re F(t)$ and $f_2(t) = \Im F(t)$. Then

$$F'(t) = f'(Z)(z_1 - z_2) = 0.$$

Consequently $F_1'(t) = F_2'(t) = 0$. By (iii) of Theorem 13.7 both F_1 and F_2 are constant in $[0,1]$, hence $f(z_1) = f(z_2)$ and f is constant in S. $\square$

An easy consequence of this Theorem is: $f : \mathbb{C} \mapsto \mathbb{C}$ is a polynomial of degree at most n if and only if $f^{(n+1)}(z) = 0$ for $z \in \mathbb{C}$.

Definition 15.1 (Riemann Sum) With a function $f : [a, b] \mapsto \mathbb{C}$ and a tagged division TX we associate the sum

$$\mathcal{R}(f, TX) = \sum_{i=1}^{n} f(X_i)(t_i - t_{i-1})$$

This sum is called a Riemann sum, or more explicitly the Riemann sum corresponding to the function f and the tagged division TX.

If no confusion can arise we shall abbreviate $\mathcal{R}(f, TX)$ to $\mathcal{R}(f)$ or $\mathcal{R}(TX)$ or even just $\mathcal{R}$. We shall only do this if the objects left out in the abbreviated notation (like TX in $\mathcal{R}(f)$) are fixed during the discussion.

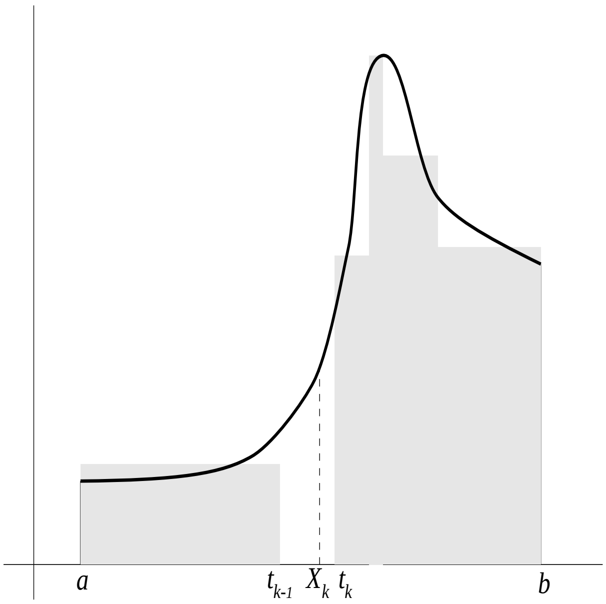

Fig. 15.1 A Riemann sum

The geometric meaning of $\mathcal{R}$ is indicated by Figure 15.1 for a non-negative function f. The sum $\mathcal{R}$ is simply the area of the rectangles which intersect the graph of f. Our intuition leads us to believe that, for a non-negative f, as a tagged division becomes finer and finer the sums $\mathcal{R}$ approach the area of the set $\{(x, y);\ a \le x \le b,\ 0 \le y \le f(x)\}$.

Index of *Maple* commands used in this book

The following table provides a brief description of the *Maple* commands used in this book, together with the pages on which they are described, and some of the pages on which they are used.

Most of the commands in the following table are set in the type used for *Maple* commands throughout this book, such as `collect`. Any which are set in normal type, such as Exit, are not, strictly, *Maple* commands, but are provided here to assist in running a *Maple* session.

Table A.1: List of *Maple* commands used in this book

Command	Description	Pages
`abs(x)`	The absolute value of the number x	75
`annuity`	Calculates quantities relating to annuities: requires the `finance` package to be loaded	31
`Apollonius`	Calculates and graphs the eight circles which touch three specified circles: requires the `geometry` package to be loaded	30
`assume`	Make an assumption about a variable: holds for the remainder of the worksheet (or until `restart`)	112
`assuming`	Make an assumption about a variable: holds only for the command it follows	111
`binomial`	Evaluates the binomial coefficient	151
`ceil(x)`	The ceiling of x (the smallest integer not less than x)	75, 118
`changevar`	Carry out a change of variables in an integral	474
`collect`	Collect similar terms together	24, 26
`combinat`	A collection of commands for solving problems in combinatorial theory: loaded by using the command `with(combinat)`	37
`combine`	Combine expressions into a single expression	25, 26
`conjugate`	The complex conjugate $\bar{z}$ of a complex number z	199

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List of *Maple* commands (continued)

Continued on next page

List of *Maple* commands (continued)

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List of *Maple* commands (continued)

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List of *Maple* commands (continued)