Join and Meet Digraphs of the Poset of Graphs of Order 5

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August 13, 2002

Abstract

The poset $\mathcal{G}(5)$ comprises the 34 unlabelled simple graphs of order 5, with partial ordering $G \leq H$ whenever G is a spanning subgraph of H. For every independent $\mathcal{S} \subseteq \mathcal{G}(5)$ we determine the *join* $\mathcal{S}^{\vee} \subseteq \mathcal{G}(5)$, an independent set comprising the spanning-universal graphs for the graphs in $\mathcal{S}$, and the *meet* $\mathcal{S}^{\wedge} \subseteq \mathcal{G}(5)$, an independent set comprising the greatest common subgraphs for the graphs in $\mathcal{S}$. The results are presented as digraphs of order 862. We establish numerous structural properties of these digraphs and their interaction. Electronic versions of the results, suitable for computation, are available at our website
`http://www.maths.uq.edu.au/~pa/research/joinmeetG5.html`

2000 *Mathematics subject classification:* primary 05C35, secondary 05C20, 06A07.

1 Introduction

There are 34 distinct unlabelled simple graphs of order 5, ranging from the empty graph K_5^c to the complete graph K_5 (Figure 1). For simplicity, henceforth in this paper we shall call these the "graphs of order 5". The reference numbers for these graphs used in Figure 1 are due to Peter Steinbach [3], and are well-suited to our main objective, which is the study of two related operations on sets of these graphs. We shall use the Steinbach numbers throughout this paper without further explanation.

If G and H are graphs of order 5 and G is a subgraph of H, then we write $G \leq H$, or $G < H$ if G is a proper subgraph of H. In this context, G is in fact a spanning subgraph of H, and H is a spanning supergraph of G, since both graphs can be regarded as sharing the same vertex set. Under the partial ordering $\leq$, the graphs of order 5 form a partially ordered set (poset) of order 34, which we denote by $\mathcal{G}(5)$. Table 1 (see [1]) gives the poset structure of $\mathcal{G}(5)$ by specifying all pairs G, H for which H is an *immediate successor* of G, that is, $G < H$ and if G' is any graph such that $G \leq G' \leq H$ then $G' = G$ or $G' = H$. There are 74 such pairs. By treating the graphs in $\mathcal{G}(5)$ as vertices, and assigning a directed edge $G \rightarrow H$ between those pairs $G, H \in \mathcal{G}(5)$ for which H is an immediate successor of G, we obtain the *Hasse diagram* of the poset $\mathcal{G}(5)$, a directed graph (digraph) of order 34 and size 74.

$1 \rightarrow 2$	$9 \rightarrow 15, 16, 19, 20$	$18 \rightarrow 21, 22, 23, 26$	$27 \rightarrow 31, 32$
$2 \rightarrow 3, 4$	$10 \rightarrow 15, 18$	$19 \rightarrow 21, 23, 26$	$28 \rightarrow 31, 32$
$3 \rightarrow 5, 6, 7, 8$	$11 \rightarrow 20$	$20 \rightarrow 25, 26$	$29 \rightarrow 32$
$4 \rightarrow 7, 8$	$12 \rightarrow 16, 18, 19, 20$	$21 \rightarrow 27, 28$	$30 \rightarrow 32$
$5 \rightarrow 9, 13$	$13 \rightarrow 16$	$22 \rightarrow 27, 30$	$31 \rightarrow 33$
$6 \rightarrow 9, 11, 12$	$14 \rightarrow 16, 17, 18, 19$	$23 \rightarrow 27, 28, 29$	$32 \rightarrow 33$
$7 \rightarrow 9, 10, 12, 14$	$15 \rightarrow 23, 24, 26$	$24 \rightarrow 29$	$33 \rightarrow 34$
$8 \rightarrow 12, 13, 14$	$16 \rightarrow 21, 23, 25$	$25 \rightarrow 28$	$34 \rightarrow \emptyset$
	$17 \rightarrow 21$	$26 \rightarrow 28, 29, 30$	

Table 1: Immediate successors in $\mathcal{G}(5)$.

So far, we have treated the partial order relation $G \leq H$ globally. It is often more helpful to characterise it locally, as follows. We have $G \leq H$ precisely when there is a set of edges E such that $G + E = H$. Emphasising this viewpoint, we then call H an *extension* of G, and G a *reduction* of H. If $|E| > 0$ then the extension (respectively, reduction) is *proper*, and we may write $G < H$. In particular, if $|E| = 1$, then H is a 1-*extension* of G, and G is a 1-*reduction* of H. We noted earlier that there are 74 pairs of graphs $G, H \in \mathcal{G}(5)$ such that H is a 1-extension of G. It can be checked that "r graphs in $\mathcal{G}(5)$ have exactly s 1-extensions" holds for the pairs $(r, s) = (1, 0), (11, 1), (9, 2), (7, 3), (6, 4)$ and $(0, s)$ for $s > 4$. Thus, in summary we have

Theorem 1 *The Hasse diagram of* $\mathcal{G}(5)$ *is a digraph of order* 34 *and size* 74 *with outdegree sequence* $0^1, 1^{11}, 2^9, 3^7, 4^6$.

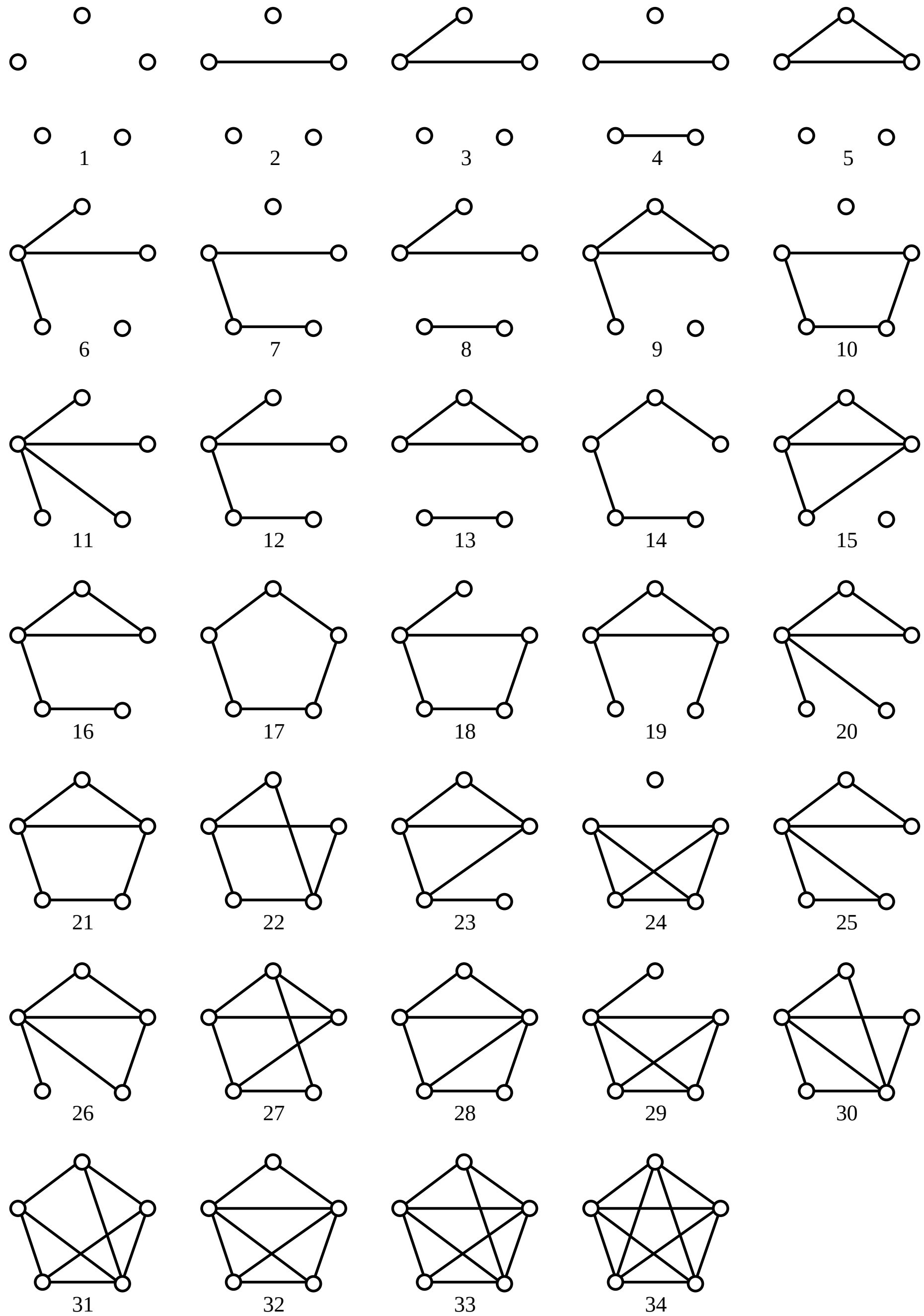

Figure 1: The unlabelled simple graphs of order 5.

A subset $\mathcal{S} \subseteq \mathcal{G}(5)$ is *independent* if none of its members is a proper extension of any other member. Thus, a subset of $\mathcal{G}(5)$ is independent if and only if no two of its members are comparable under the partial ordering $\leq$. The empty subset, and every singleton, are trivial examples of independent subsets of $\mathcal{G}(5)$. Other examples are the *level sets* $\mathcal{G}(5, m)$ for $m \in [0..10]$, comprising all graphs of order 5 and size m (that is, having exactly m edges). It turns out that $\mathcal{G}(5)$ has precisely 862 independent subsets, with "r independent sets of cardinality s" valid for the pairs $(r, s) = (1, 0), (34, 1), (152, 2), (290, 3), (255, 4), (110, 5), (20, 6)$ and $(0, s)$ for $s > 6$. We have listed the independent sets and assigned them reference numbers (rank numbers) in Table 2 (following Section 4). Independent sets (antichains) in the poset $\mathcal{G}(5)$ correspond to *strongly* independent subsets in the Hasse diagram, that is, subsets of vertices no two of which are connected by a directed path. Thus

Theorem 2 *The Hasse diagram of* $\mathcal{G}(5)$ *has* 862 *strongly independent subsets with cardinality sequence* $0^1, 1^{34}, 2^{152}, 3^{290}, 4^{255}, 5^{110}, 6^{20}$.

We shall define two digraph structures on these independent subsets, and study their properties. But first we need to discuss a wider class of subsets of $\mathcal{G}(5)$. Any subset $\mathcal{S} \subseteq \mathcal{G}(5)$ is an *ideal* if it is closed under "betweenness", that is, if $G, H \in \mathcal{S}$ and $G \leq H$, then $\mathcal{S}$ contains every graph $G' \in \mathcal{G}(5)$ such that $G \leq G' \leq H$. In particular, every independent subset of $\mathcal{G}(5)$ is an ideal. An *upper ideal* is any ideal that contains K_5, and a *lower ideal* is any that contains K_5^c. Note that the intersection of two ideals is always an ideal, and the union of two upper (respectively, lower) ideals is an upper (lower) ideal.

For any subset $\mathcal{S} \subseteq \mathcal{G}(5)$, the set $\mathcal{S}\!\uparrow$ ("$\mathcal{S}$-up"), comprising every graph of order 5 which is an extension of some member of $\mathcal{S}$, is the upper ideal *generated by* $\mathcal{S}$. In particular, for a singleton $\mathcal{S} = \{G\}$, we simplify notation to $G\!\uparrow$ for the set of all extensions of G, and call this the *principal* upper ideal generated by G. The lower ideals $\mathcal{S}\!\downarrow$ ("$\mathcal{S}$-down") and $G\!\downarrow$ are defined similarly.

For any subset $\mathcal{S} \subseteq \mathcal{G}(5)$, the *ceiling* $\lceil \mathcal{S} \rceil$ is the upper ideal

$$\lceil \mathcal{S} \rceil := \cap \{G\!\uparrow \,:\, G \in \mathcal{S}\},$$

and the *join* $\mathcal{S}^\vee$ is the set of minimal members of $\lceil \mathcal{S} \rceil$, that is,

$$\mathcal{S}^\vee := \{H \in \lceil \mathcal{S} \rceil \,:\, G < H \Rightarrow G \notin \lceil \mathcal{S} \rceil\}.$$

(This notation was introduced and developed in [2].) Evidently $\mathcal{S}^\vee$ is the independent subset of $\mathcal{G}(5)$ that generates the upper ideal $\lceil \mathcal{S} \rceil$, so $\mathcal{S}^\vee\!\uparrow = \lceil \mathcal{S} \rceil$. Similarly, the *floor* $\lfloor \mathcal{S} \rfloor$ is the lower ideal

$$\lfloor \mathcal{S} \rfloor := \cap \{G\!\downarrow \,:\, G \in \mathcal{S}\},$$

and the *meet* $\mathcal{S}^\wedge$ is the set of maximal members of $\lfloor \mathcal{S} \rfloor$, so

$$\mathcal{S}^\wedge := \{G \in \lfloor \mathcal{S} \rfloor \,:\, G < H \Rightarrow H \notin \lfloor \mathcal{S} \rfloor\}.$$

Clearly $\mathcal{S}^\wedge$ is the independent subset of $\mathcal{G}(5)$ that generates the lower ideal $\lfloor \mathcal{S} \rfloor$, so $\mathcal{S}^\wedge\!\downarrow = \lfloor \mathcal{S} \rfloor$.

Note that for any given set $\mathcal{S} \subseteq \mathcal{G}(5)$ there is an independent subset $\mathcal{R} \subseteq \mathcal{S}$ such that $\mathcal{R}^\vee = \mathcal{S}^\vee$. Indeed, let

$$\mathcal{S}^{\max} := \{G \in \mathcal{S} \,:\, G < H \Rightarrow H \notin \mathcal{S}\}.$$

Clearly, $\mathcal{S}^{\max}$ is an independent subset of $\mathcal{S}$. Again, if $H \in \lceil \mathcal{S}^{\max} \rceil$ then $H \geq G$ for every $G \in \mathcal{S}^{\max}$, so $H \geq G'$ for every $G' \in \mathcal{S}$, and therefore $H \in \lceil \mathcal{S} \rceil$; hence $\lceil \mathcal{S}^{\max} \rceil \subseteq \lceil \mathcal{S} \rceil$. But $\mathcal{S}^{\max} \subseteq \mathcal{S}$, so $\lceil \mathcal{S}^{\max} \rceil \supseteq \lceil \mathcal{S} \rceil$. It follows that $\lceil \mathcal{S}^{\max} \rceil = \lceil \mathcal{S} \rceil$, so $(\mathcal{S}^{\max})^\vee = \mathcal{S}^\vee$.

Similarly, if

$$\mathcal{S}^{\min} := \{H \in \mathcal{S} \,:\, G < H \Rightarrow G \notin \mathcal{S}\},$$

then $\mathcal{S}^{\min}$ is an independent subset of $\mathcal{S}$ and $\lfloor \mathcal{S}^{\min} \rfloor = \lfloor \mathcal{S} \rfloor$, so $(\mathcal{S}^{\min})^\wedge = \mathcal{S}^\wedge$. We have proved

Theorem 3 *For any given subset $\mathcal{S} \subseteq \mathcal{G}(5)$, the independent subsets $\mathcal{S}^{\max}$ and $\mathcal{S}^{\min}$ have the same join and the same meet, respectively, as $\mathcal{S}$.*

Therefore to study the joins and meets of the 2^{34} subsets of $\mathcal{G}(5)$, it suffices to consider the 862 subsets which are independent. Note also that for any $\mathcal{S} \subseteq \mathcal{G}(5)$, we have

$$\mathcal{S}^\vee = \lceil \mathcal{S}^{\max} \rceil^{\min} \quad \text{and} \quad \mathcal{S}^\wedge = \lfloor \mathcal{S}^{\min} \rfloor^{\max}.$$

The join of a given independent set $\mathcal{S} \subseteq \mathcal{G}(5)$ can be a singleton, but typically it contains several graphs. Each graph $H \in \mathcal{S}^\vee$ contains every graph in $\mathcal{S}$ as a spanning subgraph, and is minimal with respect to this property, that is, for any edge $e \in H$ there is at least one graph $G \in \mathcal{S}$ which is not a spanning subgraph of $H - e$. We call any such graph H a *spanning-universal graph* for $\mathcal{S}$ (see [2]). Thus, the set $\mathcal{S}^\vee$ comprises the spanning-universal graphs for $\mathcal{S}$, and this fact makes the determination of the join of each independent subset of $\mathcal{G}(5)$ of considerable interest. Correspondingly, note that each graph $H \in \mathcal{S}^\wedge$ is a spanning subgraph of every graph in $\mathcal{S}$, and is maximal with respect of this property, that is, for any edge $e \notin H$ there is at least one graph $G \in \mathcal{S}$ which does not contain $H + e$ as a subgraph. We shall call any such graph H a *greatest common subgraph* for $\mathcal{S}$. Such graphs seem not to have been extensively studied previously, but are evidently of interest. Since $\mathcal{S}^\wedge$ comprises the greatest common subgraphs for $\mathcal{S}$, the determination of the meet of each independent subset of $\mathcal{G}(5)$ also has intrinsic interest.

It is now time to define the two digraphs which are the objective of our groundwork so far. They will assist greatly in the study of spanning-universal graphs and greatest common subgraphs for subsets of $\mathcal{G}(5)$.

The *join digraph* for independent subsets of $\mathcal{G}(5)$ is the directed graph $\mathcal{JG}(5)$ with the 862 independent subsets of $\mathcal{G}(5)$ as its vertices, and directed edges $\mathcal{S} \to \mathcal{T}$ between precisely those pairs of independent sets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}(5)$ satisfying $\mathcal{T} = \mathcal{S}^\vee$. Likewise, the *meet digraph* is the directed graph $\mathcal{MG}(5)$ on the 862 independent subsets of $\mathcal{G}(5)$, with directed edges $\mathcal{S} \to \mathcal{T}$ for precisely those pairs of independent sets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}(5)$ satisfying $\mathcal{T} = \mathcal{S}^\wedge$. For each independent set $\mathcal{S} \subseteq \mathcal{G}(5)$, we have listed $\mathcal{S}^\vee$ and $\mathcal{S}^\wedge$ in Table 3 (following Section 4). For compactness, these sets are all specified by their rank numbers, assigned in Table 2. For example, if $\mathcal{S} = \{9, 12\}$ then $\mathcal{S}^\vee = \{16, 19, 20\}$ and $\mathcal{S}^\wedge = \{6, 7\}$. Table 2 assigns rank number 142 to $\{9, 12\}$, 591 to $\{16, 19, 20\}$, and 63 to $\{6, 7\}$. Thus Table 3 specifies that the independent set with rank number 142 has join 591 and meet 63.

The *complement* of any graph $G \in \mathcal{G}(5)$ is $G^c := K_5 - E(G)$, where $E(G)$ is the edge set of G. Then $H^c \leq G^c$ if and only if $G \leq H$, so graph complementation $c : \mathcal{G}(5) \to \mathcal{G}(5)$ is a poset anti-automorphism. The Steinbach numbering of graphs in $\mathcal{G}(5)$ is particularly convenient for graph complementation. If x is the Steinbach number of $G \in \mathcal{G}(5)$ and x^c is the number of G^c, then $x + x^c = 35$ holds for all $x \in [1..34] \setminus [16..19]$, while $x + x^c = 34$ holds for $x \in \{16, 17, 18\}$. Note that $17^c = 17$ and $19^c = 19$ are the two self-complementary graphs in $\mathcal{G}(5)$.

For any set $\mathcal{S} \subseteq \mathcal{G}(5)$ we define the elementwise complement of $\mathcal{S}$ to be $\mathcal{S}^c := \{G^c : G \in \mathcal{S}\}$. Evidently $\mathcal{S}^c{\uparrow} = \mathcal{S}{\downarrow}^c$ and $\mathcal{S}^c{\downarrow} = \mathcal{S}{\uparrow}^c$, so $\mathcal{S}^{c\vee} = \mathcal{S}^{\wedge c}$ and $\mathcal{S}^{c\wedge} = \mathcal{S}^{\vee c}$. Each directed edge of the join digraph $\mathcal{JG}(5)$ is of the form $\mathcal{S} \to \mathcal{S}^\vee$, and under elementwise complementation of sets of graphs this directed edge maps into $\mathcal{S}^c \to \mathcal{S}^{\vee c} = \mathcal{S}^{c\wedge}$, which is a directed edge of the meet digraph $\mathcal{MG}(5)$. It easily follows that $c : \mathcal{JG}(5) \to \mathcal{MG}(5)$ is a digraph isomorphism. Indeed, c is a duality, so $\mathcal{JG}(5)$ and $\mathcal{MG}(5)$ are dual digraphs under c. In summary, we have

Theorem 4 *Graph complementation is an anti-automorphism for the poset $\mathcal{G}(5)$, and elementwise complementation of sets of graphs is an isomorphism between the digraphs $\mathcal{JG}(5)$ and $\mathcal{MG}(5)$.*

Note an immediate consequence of Theorem 4 is that the indegree sequence of the Hasse diagram of $\mathcal{G}(5)$ is equal to its outdegree sequence (given in Theorem 1).

2 Structure of the join digraph $\mathcal{JG}(5)$

Every independent subset $\mathcal{S} \subseteq \mathcal{G}(5)$ determines a unique independent set $\mathcal{S}^\vee$, so every vertex of $\mathcal{JG}(5)$ has outdegree 1. The empty set $\emptyset \subseteq \mathcal{G}(5)$ has ceiling $\lceil\emptyset\rceil = \mathcal{G}(5)$, and $K_5^c\!\uparrow = \mathcal{G}(5)$, so $\emptyset^\vee = \{K_5^c\}$. Every graph $G \in \mathcal{G}(5)$ determines a singleton which is equal to its own join, $\{G\}^\vee = \{G\}$, so the 34 vertices of $\mathcal{JG}(5)$ which are singletons have directed loops $\{G\} \to \{G\}$. Every other independent subset $\mathcal{S} \subseteq \mathcal{G}(5)$ contains at least two graphs, so any spanning-universal graph for $\mathcal{S}$ must be a proper extension of each graph in $\mathcal{S}$ and therefore $\mathcal{S} \cap \mathcal{S}^\vee = \emptyset$ when $|\mathcal{S}| \geq 2$. Indeed, it follows that

$$\max\{|E(G)| \,:\, G \in \mathcal{S}\} < \min\{|E(H)| \;:\; H \in \mathcal{S}^\vee\},$$

so when $|\mathcal{S}| \geq 2$ all graphs in $\mathcal{S}^\vee$ lie in higher level sets of $\mathcal{G}(5)$ than any graph in $\mathcal{S}$. This shows that $\mathcal{JG}(5)$ cannot contain directed cycles other than the directed loops on the singletons. Furthermore, the lowest level set in $\mathcal{G}(5)$ that is not a singleton is $\mathcal{G}(5,2)$, with $|\mathcal{G}(5,2)| = 2$, and the highest is its elementwise complement $\mathcal{G}(5,8)$, with $\mathcal{G}(5,8)^\vee = \mathcal{G}(5,9) = \{K_5 - e\}$, so the longest directed path in $\mathcal{JG}(5)$ cannot contain more vertices than the number of level sets from $\mathcal{G}(5,2)$ to $\mathcal{G}(5,9)$ inclusive. In summary, $\mathcal{JG}(5)$ cannot contain a directed path of order greater than 8, and it can contain at most one directed path of order 8. In fact, it turns out that the directed path with initial vertex $\mathcal{G}(5,2)$ does attain order 8.

Because every vertex of $\mathcal{JG}(5)$ has outdegree 1, and there are no directed cycles other than the directed loops, it follows that $\mathcal{JG}(5)$ is a directed forest with exactly one directed loop in each component, on the terminal vertex of every directed path in its component. Hence $\mathcal{JG}(5)$ has 34 components, each of which is naturally characterised by the graph $G \in \mathcal{G}(5)$ with singleton $\{G\}$ which carries the directed loop of the component. We shall refer to the singleton $\{G\}$ as the *root* of its component, and denote the whole component by $<\!G\!>$. Thus we have

Theorem 5 *The join digraph $\mathcal{JG}(5)$ has order 862 and size 862, including 34 directed loops. It is a directed forest with loops. Each of its 34 components is a tree directed towards its root, which is its unique singleton vertex and carries its only directed loop.*

The 34 components of $\mathcal{JG}(5)$ differ widely. We define the *height* of any of its components to be the maximum size attained by directed paths in that component. Computation based on Table 3 establishes that "the component $<\!G\!>$ has order n and height h" holds for the triples $(G, n, h) =$ $(1,2,1)$, $(9,5,1)$, $(12,3,1)$, $(16,13,2)$, $(18,7,1)$, $(20,15,1)$, $(21,43,1)$, $(23,12,1)$, $(26,41,1)$, $(27,30,1)$, $(28,147,2)$, $(29,59,1)$, $(32,309,4)$, $(33,156,7)$ and $(x,1,0)$ for all remaining $x \in [0..34]$. Hence we have

Theorem 6 *The* 34 *components of the join digraph* $\mathcal{JG}(5)$ *have height sequence* $0^{20}, 1^{10}, 2^2, 4, 7$ *and order sequence* $1^{20}, 2, 3, 5, 7, 12, 13, 15, 30, 41, 43, 59, 147, 156, 309$.

It is intriguing to note that there is an approximate "multiples and submultiples of 15" scale distribution among the orders of the components.

For any set $\mathcal{S} \subseteq \mathcal{G}(5)$ the sequence of iterated joins $\mathcal{S}^\vee$, $\mathcal{S}^{\vee\vee}$, $\mathcal{S}^{\vee\vee\vee}$ and so on eventually becomes constant and equal to a singleton $\{G\}$ for some $G \in \mathcal{G}(5)$ uniquely determined by $\mathcal{S}$. We call $\{G\}$ the *join limit* of $\mathcal{S}$. Each singleton is the root of one component of $\mathcal{JG}(5)$, so it is the join limit of every independent set which is a vertex in that component. We shall call $G \in \mathcal{G}(5)$ a *superuniversal* graph if its component $<G>$ has order greater than 1. Equivalently, a graph $G \in \mathcal{G}(5)$ is superuniversal if it is the join of at least one independent set $\mathcal{S} \subseteq \mathcal{G}(5) \setminus \{G\}$. In particular, K_5^c is superuniversal because $\emptyset^\vee = \{K_5^c\}$. Since $\mathcal{S}^\vee = \{G\}$ is equivalent to $\lceil \mathcal{S} \rceil = G\!\uparrow$, we deduce

Theorem 7 *If* $G \in \mathcal{G}(5)$ *is superuniversal and* $\mathcal{S} \subseteq \mathcal{G}(5) \setminus \{G\}$ *has join* $\mathcal{S}^\vee = \{G\}$*, then* $H \in \mathcal{G}(5)$ *is a supergraph of all graphs in* $\mathcal{S}$ *if and only if* H *is a supergraph of* G.

Now that we have seen some of the significance of superuniversality, let us characterize the superuniversal graphs. A graph $G \in \mathcal{G}(5)$ is superuniversal if and only if there is a set $\mathcal{S} \subseteq \mathcal{G}(5) \setminus \{G\}$ such that $\mathcal{S}^\vee = \{G\}$, so $\lceil \mathcal{S} \rceil = G\!\uparrow$. Suppose there is an edge $e \in E(G)$ such that $G - e \notin \mathcal{S}$. Let $\mathcal{S}' := \mathcal{S} \cup \{G - e\}$. Then

$$\lceil \mathcal{S}' \rceil = \lceil \mathcal{S} \rceil \cap (G-e)\!\uparrow = G\!\uparrow \cap (G-e)\!\uparrow = G\!\uparrow = \lceil \mathcal{S} \rceil.$$

By iteration, we have $\lceil \mathcal{T} \rceil = \lceil \mathcal{S} \rceil$ for $\mathcal{T} := \mathcal{S} \cup \{G - e : e \in E(G)\}$. Hence $\lceil \mathcal{T}^{\max} \rceil = \lceil \mathcal{S} \rceil = G\!\uparrow$. But clearly $\mathcal{T}^{\max} = \{G - e : e \in E(G)\}$. It follows that $\{G - e : e \in E(G)\}^\vee = \{G\}$, so we have

Theorem 8 *A graph* $G \in \mathcal{G}(5)$ *is superuniversal if and only if it is the unique spanning-universal graph for its set of* 1*-reductions, that is,*

$$\{G - e : e \in E(G)\}^\vee = \{G\}.$$

No nonempty graph which is edge-transitive can be superuniversal, since it has a singleton set of 1-reductions, so that set is its own join. This explains why graphs $2, 3, 4, 5, 6, 10, 11, 17, 22, 24$ and 34 are not superuniversal. Any other graph in $\mathcal{G}(5)$ is a spanning-universal graph for its set of 1-reductions, but not necessarily the only spanning-universal graph for that set. For example, graphs 7 and 8 both have $\{3, 4\}$ as their set of 1-reductions, and $\{3, 4\}^\vee = \{7, 8\}$, so neither of $7, 8$ is superuniversal. Applying Theorem 8, simple checking from Tables 1, 2 and 3 identifies the superuniversal graphs. Let us partition the graphs in $\mathcal{G}(5)$ into three classes, as follows: $G \in \mathcal{G}(5)$ belongs to *class* 0, 1 or 2, respectively, according as the join of the set of 1-reductions $\{G - e : e \in E(G)\}$ is itself, or the singleton $\{G\}$, or a set which contains G and at least one other graph. Thus, the members of class 1 are the superuniversal graphs, and the members of class 0 are the nonempty edge-transitive graphs. In summary we have

Theorem 9 *The poset $\mathcal{G}(5)$ has the following class structure:*
Class $0 = \{2, 3, 4, 5, 6, 10, 11, 17, 22, 24, 34\}$*;*
Class $1 = \{1, 9, 12, 16, 18, 20, 21, 23, 26, 27, 28, 29, 32, 33\}$*;*
Class $2 = \{7, 8, 13, 14, 15, 19, 25, 30, 31\}$.

3 The digraphs $\mathcal{MG}(5)$ and $\mathcal{JMG}(5)$

We have already noted that the meet digraph $\mathcal{MG}(5)$ is isomorphic to the join digraph $\mathcal{JG}(5)$, under elementwise complementation of sets of graphs. Hence, in principle we can deduce whatever we wish to know about $\mathcal{MG}(5)$ provided we know the corresponding information about $\mathcal{JG}(5)$. For example, if $\mathcal{S} \subseteq \mathcal{G}(5)$ is an independent set, its successor $\mathcal{S}^\wedge$ in $\mathcal{MG}(5)$ is $\mathcal{S}^\wedge = \mathcal{S}^{cc\wedge} = \mathcal{S}^{c\vee c}$, so we can determine $\mathcal{S}^\wedge$ from $\mathcal{JG}(5)$ by computing the elementwise complement $\mathcal{S}^c$, finding its successor $\mathcal{S}^{c\vee}$ in $\mathcal{JG}(5)$, and then computing the elementwise complement of this latter set. This is a simple three step computation, but it is time-consuming to do many such computations by hand, so for convenience in Table 3 we have listed both $\mathcal{S}^\vee$ and $\mathcal{S}^\wedge$ for each independent subset $\mathcal{S} \subseteq \mathcal{G}(5)$.

The interaction of the join and meet operations can be studied by superposing the digraphs $\mathcal{JG}(5)$ and $\mathcal{MG}(5)$, to produce $\mathcal{JMG}(5)$, the *join and meet digraph* for $\mathcal{G}(5)$. Specifically, the vertices of $\mathcal{JMG}(5)$ are the 862 independent subsets of $\mathcal{G}(5)$, and there is a directed edge $\mathcal{S} \to \mathcal{T}$ if and only if $\mathcal{T} = \mathcal{S}^\vee$ or $\mathcal{T} = \mathcal{S}^\wedge$. Note that we assign only one directed loop to any singleton $\mathcal{S} = \{G\}$. So long as the sets $\mathcal{S}$ and $\mathcal{T}$ are known explicitly, if $|\mathcal{S}| \geq 2$ and $\mathcal{S} \to \mathcal{T}$ is a directed edge of $\mathcal{JMG}(5)$, we can deduce whether $\mathcal{T} = \mathcal{S}^\vee$ or $\mathcal{T} = \mathcal{S}^\wedge$ from the fact that the smallest graph in $\mathcal{S}^\vee$ is larger than the largest graph in $\mathcal{S}$, while the largest graph in $\mathcal{S}^\wedge$ is smaller than the smallest graph in $\mathcal{S}$. The Steinbach numbering of graphs in $\mathcal{G}(5)$ increases monotonically with the size of the graphs, so if $|\mathcal{S}| \geq 2$, all Steinbach numbers for graphs in $\mathcal{S}^\vee$ are greater than all Steinbach numbers for graphs in $\mathcal{S}$, which in turn are greater than all Steinbach numbers for graphs in $\mathcal{S}^\wedge$.

Perhaps the feature of most obvious interest in $\mathcal{JMG}(5)$ is the occurrence of directed 2-cycles. If $\mathcal{S} \to \mathcal{T} \to \mathcal{S}$ is such a 2-cycle, with $\mathcal{T} = \mathcal{S}^\vee$, then $\mathcal{S} = \mathcal{S}^{\vee\wedge}$, with $\mathcal{S}^\vee \neq \mathcal{S}$. This corresponds to the fact that $|\mathcal{S}| \geq 2$ and the greatest common subgraphs for the set of spanning-universal graphs of $\mathcal{S}$ are precisely the graphs in the original set $\mathcal{S}$. One might have expected that this would commonly occur, but in fact it is relatively rare: there are just 20 such 2-cycles in $\mathcal{JMG}(5)$, gathered in Table 4. The 2-cycle $\mathcal{S} \to \mathcal{T} \to \mathcal{S}$, with $\mathcal{T} = \mathcal{S}^{\vee}$, is the same as the 2-cycle $\mathcal{T} \to \mathcal{S} \to \mathcal{T}$, so $\mathcal{S} = \mathcal{T}^{\wedge}$ and $\mathcal{T} = \mathcal{T}^{\wedge\vee}$. Thus the spanning-universal graphs for the set of greatest common subgraphs of $\mathcal{T}$ are precisely the graphs of the original set. We shall write $\mathcal{S} \longleftrightarrow \mathcal{T}$ to compactly denote the 2-cycle $\mathcal{S} \to \mathcal{T} \to \mathcal{S}$.

Note that when $\mathcal{S} = \mathcal{S}^{\vee\wedge}$, it does *not* follow in general that $\mathcal{S} = \mathcal{S}^{\wedge\vee}$. When the latter does happen to occur, $\mathcal{JMG}(5)$ has a chain of two 2-cycles linked at $\mathcal{S}$: if $\mathcal{S}^{\wedge} = \mathcal{R}$ and $\mathcal{S}^{\vee} = \mathcal{T}$ then $\mathcal{R} \longleftrightarrow \mathcal{S}$ and $\mathcal{S} \longleftrightarrow \mathcal{T}$ are the linked 2-cycles, and we write $\mathcal{R} \longleftrightarrow \mathcal{S} \longleftrightarrow \mathcal{T}$. It turns out that ten of the 2-cycles of $\mathcal{JMG}(5)$ are disjoint, while the remaining ten form two chains of two 2-cycles and one chain of six 2-cycles, as listed in Table 4.

N	Chains of 2-cycles	N^c
C1	$\{3,4\} \longleftrightarrow \{7,8\} \longleftrightarrow \{12,14\} \longleftrightarrow \{16,18,19\}$	
	$\longleftrightarrow \{21,23\} \longleftrightarrow \{27,28\} \longleftrightarrow \{31,32\}$	C1
C2	$\{4,5\} \longleftrightarrow \{9,13\}$	C13
C3	$\{5,8\} \longleftrightarrow \{13,19,20\}$	C10
C4	$\{6,7\} \longleftrightarrow \{9,12\} \longleftrightarrow \{16,19,20\}$	C9
C5	$\{6,10\} \longleftrightarrow \{15,18\}$	C11
C6	$\{9,10\} \longleftrightarrow \{15,21\}$	C8
C7	$\{9,12,14\} \longleftrightarrow \{16,19\}$	C12
C8	$\{14,20\} \longleftrightarrow \{25,26\}$	C6
C9	$\{15,18,19\} \longleftrightarrow \{23,26\} \longleftrightarrow \{28,29\}$	C4
C10	$\{15,19,22\} \longleftrightarrow \{27,30\}$	C3
C11	$\{16,20\} \longleftrightarrow \{25,29\}$	C5
C12	$\{18,19\} \longleftrightarrow \{21,23,26\}$	C7
C13	$\{22,26\} \longleftrightarrow \{30,31\}$	C2

Table 4: Chains of directed 2-cycles in $\mathcal{JG}(5)$.

Under elementwise complementation of sets of graphs, $c : \mathcal{JMG}(5) \to \mathcal{JMG}(5)$ is a period 2 digraph automorphism, and under this map the ten disjoint 2-cycles form five pairs, the two chains of two 2-cycles form a pair, and the chain of six 2-cycles maps end-for-end onto itself. These structural properties are noted in Table 4: for example, the chain C1 is mapped onto itself by c, the chain C2 is mapped onto C13, and so on.

The possibility of larger directed cycles in $\mathcal{JMG}(5)$ is also of interest. For example, if there were a directed 3-cycle this would correspond to the existence of an independent subset $\mathcal{S} \subseteq \mathcal{G}(5)$ for which $\mathcal{S}^{\vee\vee\wedge} = \mathcal{S}$. Computer search of $\mathcal{JMG}(5)$ shows that in fact there are no directed 3-cycles, no directed 5-cycles and no proper directed 4-cycles. There are independent sets $\mathcal{S} \subseteq \mathcal{G}(5)$ with the properties $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\vee\wedge\wedge} = \mathcal{S}$, namely sets like $\{6,7\}$ from the first of two linked 2-cycles. However, we regard these as degenerate solutions since they also satisfy $\mathcal{S}^{\vee\vee\wedge} = \mathcal{S}^{\vee}$. The absence of proper directed 4-cycles implies the absence of independent sets which satisfy $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\vee\wedge\wedge} = \mathcal{S}$, with $\mathcal{S}^{\vee\vee\wedge} \neq \mathcal{S}^{\vee}$. Again, there are no independent sets which satisfy $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\wedge\vee\wedge} = \mathcal{S}$, with $\mathcal{S}^{\vee\wedge} \neq \mathcal{S}$, and so on. Given that there are no proper directed cycles of orders 3,4 or 5, one might expect that there is no such cycle of order greater than 2. However, it turns out that there is precisely one proper directed 6-cycle in $\mathcal{JMG}(5)$:

$$\{6,7\} \to \{9,12\} \to \{16,19,20\} \to \{28,29\} \\ \to \{23,26\} \to \{15,18,19\} \to \{6,7\}.$$

Of course, this directed 6-cycle is mapped onto itself (with a 3-step shift) by elementwise complementation. Thus, $\{6,7\}$ is the unique independent set $\mathcal{S} \subseteq \mathcal{G}(5)$ with the properties $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\vee\vee\wedge\wedge\wedge} = \mathcal{S}$, with $\mathcal{S}^{\vee\vee\vee\wedge} \neq \mathcal{S}^{\vee\vee}$. The 6-cycle includes a pair of edges from each of the 2-cycle chains $\{6,7\} \longleftrightarrow \{9,12\} \longleftrightarrow \{16,19,20\}$ and $\{15,18,19\} \longleftrightarrow \{23,26\} \longleftrightarrow \{28,29\}$. There are no independent sets corresponding to other types of proper directed 6-cycle, such as $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\vee\wedge\vee\vee\wedge} = \mathcal{S}$, or $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\vee\vee\vee\wedge\wedge} = \mathcal{S}$, or $|\mathcal{S}| \geq 2$ and $\mathcal{S}^{\vee\wedge\vee\wedge\vee\wedge} = \mathcal{S}$ with $\mathcal{S}^{\vee\wedge} \neq \mathcal{S}$. To summarise, we have

Theorem 10 *Elementwise complementation of sets of graphs is an automorphism for the join and meet digraph $\mathcal{JMG}(5)$, which has order* 862 *and size* 1690*. It has* 34 *directed loops,* 20 *proper directed cycles of order* 2*, none of orders* 3,4 *or* 5*, and a unique proper directed cycle of order* 6*. The proper* 2*-cycles occur as a chain of six, two chains of two, and ten disjoint individuals.*

4 Closing remarks

As a service to fellow graph-theorists, we have made electronic versions of the data in Tables 1 to 4 available at the website

`http://www.maths.uq.edu.au/~pa/research/joinmeetG5.html`

where freedom from space limitations allows us to present the data in a more explicit form convenient for computations. We also include additional information, in particular: explicit tabulations of all directed paths in $\mathcal{JG}(5)$ and $\mathcal{MG}(5)$, and data for the poset $\mathcal{G}(4)$ of 11 order 4 graphs and the order 24 digraphs $\mathcal{JG}(4)$ and $\mathcal{MG}(4)$.

Acknowledgement. The second author gratefully acknowledges the hospitality and support of the Department of Mathematics at The University of Queensland while this paper was being finalised.

r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$
1	$\emptyset$	**59**	5,17,22	**117**	9,10,13	**175**	10,11,16
2	1	**60**	5,18	**118**	9,10,13,14	**176**	10,11,16,17
3	2	**61**	5,22	**119**	9,10,13,17	**177**	10,11,16,17,19
4	3	**62**	6	**120**	9,10,14	**178**	10,11,16,19
5	3,4	**63**	6,7	**121**	9,10,17	**179**	10,11,17
6	4	**64**	6,7,8	**122**	9,11	**180**	10,11,17,19
7	4,5	**65**	6,7,13	**123**	9,11,12	**181**	10,11,19
8	4,5,6	**66**	6,8	**124**	9,11,12,13	**182**	10,12
9	4,5,11	**67**	6,8,10	**125**	9,11,12,13,14	**183**	10,12,13
10	4,6	**68**	6,10	**126**	9,11,12,13,17	**184**	10,12,13,14
11	4,11	**69**	6,10,13	**127**	9,11,12,14	**185**	10,12,13,17
12	5	**70**	6,10,13,14	**128**	9,11,12,17	**186**	10,12,14
13	5,6	**71**	6,10,13,17	**129**	9,11,13	**187**	10,12,17
14	5,6,7	**72**	6,10,14	**130**	9,11,13,14	**188**	10,13
15	5,6,7,8	**73**	6,10,17	**131**	9,11,13,17	**189**	10,13,14
16	5,6,8	**74**	6,13	**132**	9,11,13,17,18	**190**	10,13,14,20
17	5,6,8,10	**75**	6,13,14	**133**	9,11,13,17,22	**191**	10,13,17
18	5,6,10	**76**	6,13,17	**134**	9,11,13,18	**192**	10,13,17,19
19	5,6,10,14	**77**	6,14	**135**	9,11,13,22	**193**	10,13,17,19,20
20	5,6,10,17	**78**	6,17	**136**	9,11,14	**194**	10,13,17,20
21	5,6,14	**79**	7	**137**	9,11,17	**195**	10,13,19
22	5,6,17	**80**	7,8	**138**	9,11,17,18	**196**	10,13,19,20
23	5,7	**81**	7,8,11	**139**	9,11,17,22	**197**	10,13,20
24	5,7,8	**82**	7,11	**140**	9,11,18	**198**	10,14
25	5,7,8,11	**83**	7,11,13	**141**	9,11,22	**199**	10,14,20
26	5,7,11	**84**	7,13	**142**	9,12	**200**	10,16
27	5,8	**85**	8	**143**	9,12,13	**201**	10,16,17
28	5,8,10	**86**	8,9	**144**	9,12,13,14	**202**	10,16,17,19
29	5,8,10,11	**87**	8,9,10	**145**	9,12,13,17	**203**	10,16,17,19,20
30	5,8,11	**88**	8,9,10,11	**146**	9,12,14	**204**	10,16,17,20
31	5,10	**89**	8,9,11	**147**	9,12,17	**205**	10,16,19
32	5,10,11	**90**	8,10	**148**	9,13	**206**	10,16,19,20
33	5,10,11,12	**91**	8,10,11	**149**	9,13,14	**207**	10,16,20
34	5,10,11,12,14	**92**	8,11	**150**	9,13,17	**208**	10,17
35	5,10,11,12,17	**93**	8,11,15	**151**	9,13,17,18	**209**	10,17,19
36	5,10,11,14	**94**	8,11,24	**152**	9,13,17,22	**210**	10,17,19,20
37	5,10,11,17	**95**	8,15	**153**	9,13,18	**211**	10,17,19,25
38	5,10,12	**96**	8,24	**154**	9,13,22	**212**	10,17,20
39	5,10,12,14	**97**	9	**155**	9,14	**213**	10,17,25
40	5,10,12,17	**98**	9,10	**156**	9,17	**214**	10,19
41	5,10,14	**99**	9,10,11	**157**	9,17,18	**215**	10,19,20
42	5,10,17	**100**	9,10,11,12	**158**	9,17,22	**216**	10,19,25
43	5,11	**101**	9,10,11,12,13	**159**	9,18	**217**	10,20
44	5,11,12	**102**	9,10,11,12,13,14	**160**	9,22	**218**	10,25
45	5,11,12,14	**103**	9,10,11,12,13,17	**161**	10	**219**	11
46	5,11,12,17	**104**	9,10,11,12,14	**162**	10,11	**220**	11,12
47	5,11,14	**105**	9,10,11,12,17	**163**	10,11,12	**221**	11,12,13
48	5,11,17	**106**	9,10,11,13	**164**	10,11,12,13	**222**	11,12,13,14
49	5,11,17,18	**107**	9,10,11,13,14	**165**	10,11,12,13,14	**223**	11,12,13,14,15
50	5,11,17,22	**108**	9,10,11,13,17	**166**	10,11,12,13,17	**224**	11,12,13,14,24
51	5,11,18	**109**	9,10,11,14	**167**	10,11,12,14	**225**	11,12,13,15
52	5,11,22	**110**	9,10,11,17	**168**	10,11,12,17	**226**	11,12,13,15,17
53	5,12	**111**	9,10,12	**169**	10,11,13	**227**	11,12,13,17
54	5,12,14	**112**	9,10,12,13	**170**	10,11,13,14	**228**	11,12,13,17,24
55	5,12,17	**113**	9,10,12,13,14	**171**	10,11,13,17	**229**	11,12,13,24
56	5,14	**114**	9,10,12,13,17	**172**	10,11,13,17,19	**230**	11,12,14
57	5,17	**115**	9,10,12,14	**173**	10,11,13,19	**231**	11,12,14,15
58	5,17,18	**116**	9,10,12,17	**174**	10,11,14	**232**	11,12,14,24

Table 2: Rank numbering of the independent subsets of $\mathcal{G}(5)$.
(Example: independent set $\mathcal{S} = \{9, 12\}$ has rank number $r = 142$.)

r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$
233	11,12,15	**291**	11,15,16,19,22	**349**	11,18,24	**407**	13,15,17,20,22
234	11,12,15,17	**292**	11,15,16,22	**350**	11,19	**408**	13,15,17,22
235	11,12,17	**293**	11,15,17	**351**	11,19,22	**409**	13,15,18
236	11,12,17,24	**294**	11,15,17,18	**352**	11,19,22,24	**410**	13,15,18,19
237	11,12,24	**295**	11,15,17,18,19	**353**	11,19,24	**411**	13,15,18,19,20
238	11,13	**296**	11,15,17,19	**354**	11,21	**412**	13,15,18,20
239	11,13,14	**297**	11,15,17,19,22	**355**	11,21,22	**413**	13,15,19
240	11,13,14,15	**298**	11,15,17,22	**356**	11,21,22,23	**414**	13,15,19,20
241	11,13,14,24	**299**	11,15,18	**357**	11,21,22,23,24	**415**	13,15,19,20,22
242	11,13,15	**300**	11,15,18,19	**358**	11,21,22,24	**416**	13,15,19,22
243	11,13,15,17	**301**	11,15,19	**359**	11,21,23	**417**	13,15,20
244	11,13,15,17,18	**302**	11,15,19,22	**360**	11,21,23,24	**418**	13,15,20,22
245	11,13,15,17,18,19	**303**	11,15,21	**361**	11,21,24	**419**	13,15,22
246	11,13,15,17,19	**304**	11,15,21,22	**362**	11,22	**420**	13,17
247	11,13,15,17,19,22	**305**	11,15,22	**363**	11,22,23	**421**	13,17,18
248	11,13,15,17,22	**306**	11,16	**364**	11,22,23,24	**422**	13,17,18,19
249	11,13,15,18	**307**	11,16,17	**365**	11,22,24	**423**	13,17,18,19,20
250	11,13,15,18,19	**308**	11,16,17,18	**366**	11,23	**424**	13,17,18,19,20,24
251	11,13,15,19	**309**	11,16,17,18,19	**367**	11,23,24	**425**	13,17,18,19,24
252	11,13,15,19,22	**310**	11,16,17,18,19,24	**368**	11,24	**426**	13,17,18,20
253	11,13,15,22	**311**	11,16,17,18,24	**369**	11,24,27	**427**	13,17,18,20,24
254	11,13,17	**312**	11,16,17,19	**370**	11,27	**428**	13,17,18,24
255	11,13,17,18	**313**	11,16,17,19,22	**371**	12	**429**	13,17,19
256	11,13,17,18,19	**314**	11,16,17,19,22,24	**372**	12,13	**430**	13,17,19,20
257	11,13,17,18,19,24	**315**	11,16,17,19,24	**373**	12,13,14	**431**	13,17,19,20,22
258	11,13,17,18,24	**316**	11,16,17,22	**374**	12,13,14,15	**432**	13,17,19,20,22,24
259	11,13,17,19	**317**	11,16,17,22,24	**375**	12,13,14,24	**433**	13,17,19,20,24
260	11,13,17,19,22	**318**	11,16,17,24	**376**	12,13,15	**434**	13,17,19,22
261	11,13,17,19,22,24	**319**	11,16,18	**377**	12,13,15,17	**435**	13,17,19,22,24
262	11,13,17,19,24	**320**	11,16,18,19	**378**	12,13,17	**436**	13,17,19,24
263	11,13,17,22	**321**	11,16,18,19,24	**379**	12,13,17,24	**437**	13,17,20
264	11,13,17,22,24	**322**	11,16,18,24	**380**	12,13,24	**438**	13,17,20,22
265	11,13,17,24	**323**	11,16,19	**381**	12,14	**439**	13,17,20,22,24
266	11,13,18	**324**	11,16,19,22	**382**	12,14,15	**440**	13,17,20,24
267	11,13,18,19	**325**	11,16,19,22,24	**383**	12,14,24	**441**	13,17,22
268	11,13,18,19,24	**326**	11,16,19,24	**384**	12,15	**442**	13,17,22,24
269	11,13,18,24	**327**	11,16,22	**385**	12,15,17	**443**	13,17,22,24,26
270	11,13,19	**328**	11,16,22,24	**386**	12,17	**444**	13,17,22,26
271	11,13,19,22	**329**	11,16,24	**387**	12,17,24	**445**	13,17,24
272	11,13,19,22,24	**330**	11,17	**388**	12,24	**446**	13,17,24,26
273	11,13,19,24	**331**	11,17,18	**389**	13	**447**	13,17,24,30
274	11,13,22	**332**	11,17,18,19	**390**	13,14	**448**	13,17,26
275	11,13,22,24	**333**	11,17,18,19,24	**391**	13,14,15	**449**	13,17,30
276	11,13,24	**334**	11,17,18,24	**392**	13,14,15,20	**450**	13,18
277	11,14	**335**	11,17,19	**393**	13,14,20	**451**	13,18,19
278	11,14,15	**336**	11,17,19,22	**394**	13,14,20,24	**452**	13,18,19,20
279	11,14,24	**337**	11,17,19,22,24	**395**	13,14,24	**453**	13,18,19,20,24
280	11,15	**338**	11,17,19,24	**396**	13,15	**454**	13,18,19,24
281	11,15,16	**339**	11,17,22	**397**	13,15,17	**455**	13,18,20
282	11,15,16,17	**340**	11,17,22,23	**398**	13,15,17,18	**456**	13,18,20,24
283	11,15,16,17,18	**341**	11,17,22,23,24	**399**	13,15,17,18,19	**457**	13,18,24
284	11,15,16,17,18,19	**342**	11,17,22,24	**400**	13,15,17,18,19,20	**458**	13,19
285	11,15,16,17,19	**343**	11,17,23	**401**	13,15,17,18,20	**459**	13,19,20
286	11,15,16,17,19,22	**344**	11,17,23,24	**402**	13,15,17,19	**460**	13,19,20,22
287	11,15,16,17,22	**345**	11,17,24	**403**	13,15,17,19,20	**461**	13,19,20,22,24
288	11,15,16,18	**346**	11,18	**404**	13,15,17,19,20,22	**462**	13,19,20,24
289	11,15,16,18,19	**347**	11,18,19	**405**	13,15,17,19,22	**463**	13,19,22
290	11,15,16,19	**348**	11,18,19,24	**406**	13,15,17,20	**464**	13,19,22,24

Table 2 (continued)

r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$
465	13,19,24	**523**	15,17,20	**581**	16,17,30	**639**	17,20,22,23,24
466	13,20	**524**	15,17,20,22	**582**	16,18	**640**	17,20,22,24
467	13,20,22	**525**	15,17,22	**583**	16,18,19	**641**	17,20,23
468	13,20,22,24	**526**	15,17,22,25	**584**	16,18,19,20	**642**	17,20,23,24
469	13,20,24	**527**	15,17,25	**585**	16,18,19,20,24	**643**	17,20,24
470	13,22	**528**	15,18	**586**	16,18,19,24	**644**	17,22
471	13,22,24	**529**	15,18,19	**587**	16,18,20	**645**	17,22,23
472	13,22,24,26	**530**	15,18,19,20	**588**	16,18,20,24	**646**	17,22,23,24
473	13,22,26	**531**	15,18,19,25	**589**	16,18,24	**647**	17,22,23,24,25
474	13,24	**532**	15,18,20	**590**	16,19	**648**	17,22,23,24,25,26
475	13,24,26	**533**	15,18,25	**591**	16,19,20	**649**	17,22,23,24,26
476	13,24,30	**534**	15,19	**592**	16,19,20,22	**650**	17,22,23,25
477	13,26	**535**	15,19,20	**593**	16,19,20,22,24	**651**	17,22,23,25,26
478	13,30	**536**	15,19,20,22	**594**	16,19,20,24	**652**	17,22,23,26
479	14	**537**	15,19,22	**595**	16,19,22	**653**	17,22,24
480	14,15	**538**	15,19,22,25	**596**	16,19,22,24	**654**	17,22,24,25
481	14,15,20	**539**	15,19,25	**597**	16,19,24	**655**	17,22,24,25,26
482	14,20	**540**	15,20	**598**	16,20	**656**	17,22,24,26
483	14,20,24	**541**	15,20,21	**599**	16,20,22	**657**	17,22,25
484	14,24	**542**	15,20,21,22	**600**	16,20,22,24	**658**	17,22,25,26
485	15	**543**	15,20,22	**601**	16,20,24	**659**	17,22,25,29
486	15,16	**544**	15,21	**602**	16,22	**660**	17,22,26
487	15,16,17	**545**	15,21,22	**603**	16,22,24	**661**	17,22,29
488	15,16,17,18	**546**	15,21,22,25	**604**	16,22,24,26	**662**	17,23
489	15,16,17,18,19	**547**	15,21,25	**605**	16,22,26	**663**	17,23,24
490	15,16,17,18,19,20	**548**	15,22	**606**	16,24	**664**	17,23,24,25
491	15,16,17,18,20	**549**	15,22,25	**607**	16,24,26	**665**	17,23,24,25,26
492	15,16,17,19	**550**	15,25	**608**	16,24,30	**666**	17,23,24,25,30
493	15,16,17,19,20	**551**	16	**609**	16,26	**667**	17,23,24,26
494	15,16,17,19,20,22	**552**	16,17	**610**	16,30	**668**	17,23,24,30
495	15,16,17,19,22	**553**	16,17,18	**611**	17	**669**	17,23,25
496	15,16,17,20	**554**	16,17,18,19	**612**	17,18	**670**	17,23,25,26
497	15,16,17,20,22	**555**	16,17,18,19,20	**613**	17,18,19	**671**	17,23,25,30
498	15,16,17,22	**556**	16,17,18,19,20,24	**614**	17,18,19,20	**672**	17,23,26
499	15,16,18	**557**	16,17,18,19,24	**615**	17,18,19,20,24	**673**	17,23,30
500	15,16,18,19	**558**	16,17,18,20	**616**	17,18,19,24	**674**	17,24
501	15,16,18,19,20	**559**	16,17,18,20,24	**617**	17,18,19,24,25	**675**	17,24,25
502	15,16,18,20	**560**	16,17,18,24	**618**	17,18,19,25	**676**	17,24,25,26
503	15,16,19	**561**	16,17,19	**619**	17,18,20	**677**	17,24,25,30
504	15,16,19,20	**562**	16,17,19,20	**620**	17,18,20,24	**678**	17,24,26
505	15,16,19,20,22	**563**	16,17,19,20,22	**621**	17,18,24	**679**	17,24,30
506	15,16,19,22	**564**	16,17,19,20,22,24	**622**	17,18,24,25	**680**	17,25
507	15,16,20	**565**	16,17,19,20,24	**623**	17,18,25	**681**	17,25,26
508	15,16,20,22	**566**	16,17,19,22	**624**	17,19	**682**	17,25,29
509	15,16,22	**567**	16,17,19,22,24	**625**	17,19,20	**683**	17,25,29,30
510	15,17	**568**	16,17,19,24	**626**	17,19,20,22	**684**	17,25,30
511	15,17,18	**569**	16,17,20	**627**	17,19,20,22,24	**685**	17,26
512	15,17,18,19	**570**	16,17,20,22	**628**	17,19,20,24	**686**	17,29
513	15,17,18,19,20	**571**	16,17,20,22,24	**629**	17,19,22	**687**	17,29,30
514	15,17,18,19,25	**572**	16,17,20,24	**630**	17,19,22,24	**688**	17,30
515	15,17,18,20	**573**	16,17,22	**631**	17,19,22,24,25	**689**	18
516	15,17,18,25	**574**	16,17,22,24	**632**	17,19,22,25	**690**	18,19
517	15,17,19	**575**	16,17,22,24,26	**633**	17,19,24	**691**	18,19,20
518	15,17,19,20	**576**	16,17,22,26	**634**	17,19,24,25	**692**	18,19,20,24
519	15,17,19,20,22	**577**	16,17,24	**635**	17,19,25	**693**	18,19,24
520	15,17,19,22	**578**	16,17,24,26	**636**	17,20	**694**	18,19,24,25
521	15,17,19,22,25	**579**	16,17,24,30	**637**	17,20,22	**695**	18,19,25
522	15,17,19,25	**580**	16,17,26	**638**	17,20,22,23	**696**	18,20

Table 2 (continued)

r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$	r	$\mathcal{S}$
697	18,20,24	**739**	21,22,23,25,26	**781**	22,23,24,25,26	**823**	24,28
698	18,24	**740**	21,22,23,26	**782**	22,23,24,26	**824**	24,28,30
699	18,24,25	**741**	21,22,24	**783**	22,23,25	**825**	24,30
700	18,25	**742**	21,22,24,25	**784**	22,23,25,26	**826**	24,30,31
701	19	**743**	21,22,24,25,26	**785**	22,23,26	**827**	24,31
702	19,20	**744**	21,22,24,26	**786**	22,24	**828**	25
703	19,20,22	**745**	21,22,25	**787**	22,24,25	**829**	25,26
704	19,20,22,24	**746**	21,22,25,26	**788**	22,24,25,26	**830**	25,26,27
705	19,20,24	**747**	21,22,25,29	**789**	22,24,26	**831**	25,27
706	19,22	**748**	21,22,26	**790**	22,24,28	**832**	25,27,29
707	19,22,24	**749**	21,22,29	**791**	22,25	**833**	25,27,29,30
708	19,22,24,25	**750**	21,23	**792**	22,25,26	**834**	25,27,30
709	19,22,25	**751**	21,23,24	**793**	22,25,29	**835**	25,29
710	19,24	**752**	21,23,24,25	**794**	22,26	**836**	25,29,30
711	19,24,25	**753**	21,23,24,25,26	**795**	22,28	**837**	25,30
712	19,25	**754**	21,23,24,25,30	**796**	22,28,29	**838**	26
713	20	**755**	21,23,24,26	**797**	22,29	**839**	26,27
714	20,21	**756**	21,23,24,30	**798**	23	**840**	27
715	20,21,22	**757**	21,23,25	**799**	23,24	**841**	27,28
716	20,21,22,23	**758**	21,23,25,26	**800**	23,24,25	**842**	27,28,29
717	20,21,22,23,24	**759**	21,23,25,30	**801**	23,24,25,26	**843**	27,28,29,30
718	20,21,22,24	**760**	21,23,26	**802**	23,24,25,30	**844**	27,28,30
719	20,21,23	**761**	21,23,30	**803**	23,24,26	**845**	27,29
720	20,21,23,24	**762**	21,24	**804**	23,24,30	**846**	27,29,30
721	20,21,24	**763**	21,24,25	**805**	23,25	**847**	27,30
722	20,22	**764**	21,24,25,26	**806**	23,25,26	**848**	28
723	20,22,23	**765**	21,24,25,30	**807**	23,25,30	**849**	28,29
724	20,22,23,24	**766**	21,24,26	**808**	23,26	**850**	28,29,30
725	20,22,24	**767**	21,24,30	**809**	23,30	**851**	28,30
726	20,23	**768**	21,25	**810**	24	**852**	29
727	20,23,24	**769**	21,25,26	**811**	24,25	**853**	29,30
728	20,24	**770**	21,25,29	**812**	24,25,26	**854**	29,30,31
729	20,24,27	**771**	21,25,29,30	**813**	24,25,26,27	**855**	29,31
730	20,27	**772**	21,25,30	**814**	24,25,27	**856**	30
731	21	**773**	21,26	**815**	24,25,27,30	**857**	30,31
732	21,22	**774**	21,29	**816**	24,25,30	**858**	31
733	21,22,23	**775**	21,29,30	**817**	24,26	**859**	31,32
734	21,22,23,24	**776**	21,30	**818**	24,26,27	**860**	32
735	21,22,23,24,25	**777**	22	**819**	24,27	**861**	33
736	21,22,23,24,25,26	**778**	22,23	**820**	24,27,28	**862**	34
737	21,22,23,24,26	**779**	22,23,24	**821**	24,27,28,30		
738	21,22,23,25	**780**	22,23,24,25	**822**	24,27,30		

Table 2 (concluded).

$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$
862	**1**	2	4	**13**	97	4	**25**	713	4	**37**	848	4	**49**	848
2	**2**	2	4	**14**	97	4	**26**	713	4	**38**	760	4	**50**	859
3	**3**	3	4	**15**	591	4	**27**	459	4	**39**	760	4	**51**	838
4	**4**	4	4	**16**	591	4	**28**	760	4	**40**	731	4	**52**	857
3	**5**	80	4	**17**	760	4	**29**	838	4	**41**	760	4	**53**	591
6	**6**	6	4	**18**	544	4	**30**	713	4	**42**	731	4	**54**	590
3	**7**	148	4	**19**	760	4	**31**	544	4	**43**	713	4	**55**	731
3	**8**	97	4	**20**	731	4	**32**	838	4	**44**	713	4	**56**	590
3	**9**	713	4	**21**	590	4	**33**	838	4	**45**	829	4	**57**	731
3	**10**	142	4	**22**	731	4	**34**	838	4	**46**	848	4	**58**	731
3	**11**	713	4	**23**	97	4	**35**	848	4	**47**	829	4	**59**	840
12	**12**	12	4	**24**	591	4	**36**	838	4	**48**	848	4	**60**	760

Table 3: Joins and meets of independent subsets of $\mathcal{G}(5)$.
(Example: join and meet of set number 10 are sets numbered 142 and 3.)

$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$
4	**61**	847	79	**120**	760	4	**179**	848	4	**238**	835	4	**297**	859
62	**62**	62	79	**121**	731	4	**180**	848	4	**239**	835	4	**298**	859
4	**63**	142	62	**122**	713	4	**181**	838	4	**240**	849	62	**299**	838
4	**64**	371	62	**123**	713	79	**182**	689	4	**241**	852	62	**300**	838
4	**65**	551	4	**124**	835	5	**183**	750	4	**242**	849	62	**301**	838
4	**66**	371	4	**125**	835	5	**184**	750	4	**243**	848	62	**302**	857
4	**67**	689	4	**126**	848	5	**185**	731	4	**244**	848	62	**303**	848
4	**68**	528	4	**127**	829	79	**186**	689	4	**245**	848	62	**304**	859
4	**69**	750	4	**128**	848	79	**187**	731	4	**246**	848	62	**305**	857
4	**70**	750	4	**129**	835	5	**188**	750	4	**247**	859	62	**306**	835
4	**71**	731	4	**130**	835	5	**189**	750	4	**248**	859	4	**307**	848
4	**72**	689	4	**131**	848	5	**190**	849	4	**249**	849	4	**308**	848
4	**73**	731	4	**132**	848	5	**191**	731	4	**250**	849	4	**309**	848
4	**74**	551	4	**133**	859	5	**192**	731	4	**251**	849	4	**310**	860
4	**75**	551	4	**134**	849	5	**193**	848	4	**252**	859	4	**311**	860
4	**76**	731	4	**135**	859	5	**194**	848	4	**253**	859	4	**312**	848
4	**77**	583	4	**136**	829	5	**195**	750	4	**254**	848	4	**313**	859
4	**78**	731	4	**137**	848	5	**196**	849	4	**255**	848	4	**314**	860
79	**79**	79	4	**138**	848	5	**197**	849	4	**256**	848	4	**315**	860
5	**80**	381	4	**139**	859	79	**198**	689	4	**257**	860	4	**316**	859
4	**81**	713	62	**140**	838	79	**199**	838	4	**258**	860	4	**317**	860
4	**82**	713	62	**141**	857	79	**200**	750	4	**259**	848	4	**318**	860
4	**83**	835	63	**142**	591	79	**201**	731	4	**260**	859	62	**319**	849
5	**84**	551	5	**143**	551	79	**202**	731	4	**261**	860	62	**320**	849
85	**85**	85	5	**144**	551	79	**203**	848	4	**262**	860	62	**321**	852
5	**86**	591	5	**145**	731	79	**204**	848	4	**263**	859	62	**322**	852
5	**87**	760	79	**146**	590	79	**205**	750	4	**264**	860	62	**323**	849
4	**88**	838	79	**147**	731	79	**206**	849	4	**265**	860	62	**324**	859
4	**89**	713	7	**148**	551	79	**207**	849	4	**266**	849	62	**325**	860
5	**90**	689	5	**149**	551	79	**208**	731	4	**267**	849	62	**326**	852
4	**91**	838	5	**150**	731	79	**209**	731	4	**268**	852	62	**327**	859
4	**92**	713	5	**151**	731	79	**210**	848	4	**269**	852	62	**328**	860
4	**93**	838	5	**152**	840	79	**211**	848	4	**270**	849	62	**329**	852
4	**94**	852	5	**153**	750	79	**212**	848	4	**271**	859	4	**330**	848
5	**95**	808	5	**154**	840	79	**213**	848	4	**272**	860	4	**331**	848
5	**96**	852	79	**155**	590	79	**214**	760	4	**273**	852	4	**332**	848
97	**97**	97	79	**156**	731	79	**215**	838	4	**274**	859	4	**333**	860
79	**98**	544	79	**157**	731	79	**216**	848	4	**275**	860	4	**334**	860
4	**99**	838	79	**158**	840	79	**217**	838	4	**276**	852	4	**335**	848
4	**100**	838	63	**159**	760	79	**218**	848	4	**277**	829	4	**336**	859
4	**101**	849	63	**160**	847	219	**219**	219	4	**278**	838	4	**337**	860
4	**102**	849	161	**161**	161	62	**220**	713	4	**279**	852	4	**338**	860
4	**103**	848	4	**162**	838	4	**221**	835	62	**280**	838	4	**339**	859
4	**104**	838	4	**163**	838	4	**222**	835	62	**281**	849	4	**340**	859
4	**105**	848	4	**164**	849	4	**223**	849	4	**282**	848	4	**341**	860
4	**106**	849	4	**165**	849	4	**224**	852	4	**283**	848	4	**342**	860
4	**107**	849	4	**166**	848	4	**225**	849	4	**284**	848	4	**343**	848
4	**108**	848	4	**167**	838	4	**226**	848	4	**285**	848	4	**344**	860
4	**109**	838	4	**168**	848	4	**227**	848	4	**286**	859	4	**345**	860
4	**110**	848	4	**169**	849	4	**228**	860	4	**287**	859	62	**346**	838
79	**111**	760	4	**170**	849	4	**229**	852	62	**288**	849	62	**347**	838
5	**112**	750	4	**171**	848	4	**230**	829	62	**289**	849	62	**348**	852
5	**113**	750	4	**172**	848	4	**231**	838	62	**290**	849	62	**349**	852
5	**114**	731	4	**173**	849	4	**232**	852	62	**291**	859	62	**350**	838
79	**115**	760	4	**174**	838	62	**233**	838	62	**292**	859	62	**351**	857
79	**116**	731	4	**175**	849	4	**234**	848	4	**293**	848	62	**352**	860
5	**117**	750	4	**176**	848	4	**235**	848	4	**294**	848	62	**353**	852
5	**118**	750	4	**177**	848	4	**236**	860	4	**295**	848	62	**354**	848
5	**119**	731	4	**178**	849	62	**237**	852	4	**296**	848	62	**355**	859

Table 3 (continued)

$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$
62	**356**	859	5	**415**	859	7	**474**	852	63	**533**	848	371	**592**	859
62	**357**	860	5	**416**	840	7	**475**	852	97	**534**	808	63	**593**	860
62	**358**	860	7	**417**	849	7	**476**	860	97	**535**	838	97	**594**	852
62	**359**	848	5	**418**	859	27	**477**	849	63	**536**	857	381	**595**	840
62	**360**	860	5	**419**	840	27	**478**	860	63	**537**	847	63	**596**	860
62	**361**	860	85	**420**	731	479	**479**	479	63	**538**	859	97	**597**	852
62	**362**	857	85	**421**	731	79	**480**	808	97	**539**	848	142	**598**	835
62	**363**	859	85	**422**	731	79	**481**	838	97	**540**	838	371	**599**	859
62	**364**	860	85	**423**	848	80	**482**	829	97	**541**	848	63	**600**	860
62	**365**	860	5	**424**	860	79	**483**	852	63	**542**	859	97	**601**	852
62	**366**	849	5	**425**	860	79	**484**	852	63	**543**	857	381	**602**	840
62	**367**	852	85	**426**	848	485	**485**	485	98	**544**	841	63	**603**	860
62	**368**	852	5	**427**	860	97	**486**	798	68	**545**	840	63	**604**	860
62	**369**	860	5	**428**	860	79	**487**	841	63	**546**	859	381	**605**	859
62	**370**	859	85	**429**	731	79	**488**	841	97	**547**	848	97	**606**	852
371	**371**	371	85	**430**	848	79	**489**	841	68	**548**	847	97	**607**	852
85	**372**	551	85	**431**	859	79	**490**	848	63	**549**	859	97	**608**	860
85	**373**	551	5	**432**	860	79	**491**	848	97	**550**	848	146	**609**	849
5	**374**	798	5	**433**	860	79	**492**	841	551	**551**	551	146	**610**	860
5	**375**	852	85	**434**	840	79	**493**	848	479	**552**	731	611	**611**	611
5	**376**	798	5	**435**	860	79	**494**	859	479	**553**	731	479	**612**	731
5	**377**	841	5	**436**	860	79	**495**	840	479	**554**	731	479	**613**	731
85	**378**	731	85	**437**	848	79	**496**	848	80	**555**	848	80	**614**	848
5	**379**	860	85	**438**	859	79	**497**	859	79	**556**	860	79	**615**	860
5	**380**	852	5	**439**	860	79	**498**	840	79	**557**	860	79	**616**	860
80	**381**	583	5	**440**	860	63	**499**	798	80	**558**	848	79	**617**	860
79	**382**	808	85	**441**	840	63	**500**	798	79	**559**	860	479	**618**	848
79	**383**	852	5	**442**	860	63	**501**	849	79	**560**	860	80	**619**	848
63	**384**	808	5	**443**	860	63	**502**	849	479	**561**	731	79	**620**	860
79	**385**	841	85	**444**	859	97	**503**	798	80	**562**	848	79	**621**	860
80	**386**	731	5	**445**	860	97	**504**	849	80	**563**	859	79	**622**	860
79	**387**	860	5	**446**	860	63	**505**	859	79	**564**	860	479	**623**	848
63	**388**	852	5	**447**	860	63	**506**	840	79	**565**	860	479	**624**	731
389	**389**	389	85	**448**	848	97	**507**	849	479	**566**	840	80	**625**	848
85	**390**	551	85	**449**	860	63	**508**	859	79	**567**	860	80	**626**	859
5	**391**	798	85	**450**	750	63	**509**	840	79	**568**	860	79	**627**	860
5	**392**	849	85	**451**	750	79	**510**	841	80	**569**	848	79	**628**	860
85	**393**	835	85	**452**	849	79	**511**	841	80	**570**	859	479	**629**	840
5	**394**	852	5	**453**	852	79	**512**	841	79	**571**	860	79	**630**	860
5	**395**	852	5	**454**	852	79	**513**	848	79	**572**	860	79	**631**	860
7	**396**	798	85	**455**	849	79	**514**	848	479	**573**	840	479	**632**	859
5	**397**	841	5	**456**	852	79	**515**	848	79	**574**	860	79	**633**	860
5	**398**	841	5	**457**	852	79	**516**	848	79	**575**	860	79	**634**	860
5	**399**	841	27	**458**	750	79	**517**	841	479	**576**	859	479	**635**	848
5	**400**	848	27	**459**	849	79	**518**	848	79	**577**	860	80	**636**	848
5	**401**	848	85	**460**	859	79	**519**	859	79	**578**	860	80	**637**	859
5	**402**	841	5	**461**	860	79	**520**	840	79	**579**	860	80	**638**	859
5	**403**	848	7	**462**	852	79	**521**	859	479	**580**	848	79	**639**	860
5	**404**	859	85	**463**	840	79	**522**	848	479	**581**	860	79	**640**	860
5	**405**	840	5	**464**	860	79	**523**	848	381	**582**	750	80	**641**	848
5	**406**	848	7	**465**	852	79	**524**	859	381	**583**	750	79	**642**	860
5	**407**	859	27	**466**	835	79	**525**	840	371	**584**	849	79	**643**	860
5	**408**	840	85	**467**	859	79	**526**	859	63	**585**	852	479	**644**	840
5	**409**	798	5	**468**	860	79	**527**	848	63	**586**	852	479	**645**	840
5	**410**	798	7	**469**	852	68	**528**	808	371	**587**	849	79	**646**	860
5	**411**	849	85	**470**	840	63	**529**	808	63	**588**	852	79	**647**	860
5	**412**	849	5	**471**	860	63	**530**	838	63	**589**	852	79	**648**	860
7	**413**	798	5	**472**	860	63	**531**	848	146	**590**	750	79	**649**	860
7	**414**	849	85	**473**	859	63	**532**	838	142	**591**	849	479	**650**	859

Table 3 (continued)

$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$	$\mathcal{S}^\wedge$	$\mathcal{S}$	$\mathcal{S}^\vee$
479	**651**	859	63	**694**	860	68	**737**	860	63	**780**	860	485	**823**	860
479	**652**	859	381	**695**	848	381	**738**	859	63	**781**	860	485	**824**	860
79	**653**	860	371	**696**	838	381	**739**	859	68	**782**	860	485	**825**	860
79	**654**	860	63	**697**	852	689	**740**	859	381	**783**	859	485	**826**	861
79	**655**	860	68	**698**	852	68	**741**	860	381	**784**	859	485	**827**	861
79	**656**	860	63	**699**	860	63	**742**	860	689	**785**	859	828	**828**	828
479	**657**	859	381	**700**	848	63	**743**	860	68	**786**	860	482	**829**	848
479	**658**	859	701	**701**	701	68	**744**	860	63	**787**	860	146	**830**	859
479	**659**	860	142	**702**	838	381	**745**	859	63	**788**	860	551	**831**	859
479	**660**	859	371	**703**	857	381	**746**	859	68	**789**	860	551	**832**	860
479	**661**	860	63	**704**	860	381	**747**	860	68	**790**	860	146	**833**	860
479	**662**	841	97	**705**	852	689	**748**	859	381	**791**	859	146	**834**	860
79	**663**	860	381	**706**	847	689	**749**	860	381	**792**	859	598	**835**	860
79	**664**	860	63	**707**	860	583	**750**	841	381	**793**	860	482	**836**	860
79	**665**	860	63	**708**	860	98	**751**	860	689	**794**	857	482	**837**	860
79	**666**	860	381	**709**	859	97	**752**	860	689	**795**	859	838	**838**	838
79	**667**	860	97	**710**	852	97	**753**	860	689	**796**	860	529	**839**	859
79	**668**	860	97	**711**	860	97	**754**	860	689	**797**	860	840	**840**	840
479	**669**	848	146	**712**	848	98	**755**	860	798	**798**	798	750	**841**	859
479	**670**	848	713	**713**	713	98	**756**	860	485	**799**	852	798	**842**	860
479	**671**	860	142	**714**	848	551	**757**	848	97	**800**	860	529	**843**	860
479	**672**	848	371	**715**	859	146	**758**	848	97	**801**	860	529	**844**	860
479	**673**	860	371	**716**	859	146	**759**	860	97	**802**	860	798	**845**	860
79	**674**	860	63	**717**	860	690	**760**	848	485	**803**	852	529	**846**	860
79	**675**	860	63	**718**	860	690	**761**	860	485	**804**	860	537	**847**	860
79	**676**	860	142	**719**	848	98	**762**	860	551	**805**	848	848	**848**	848
79	**677**	860	97	**720**	860	97	**763**	860	146	**806**	848	808	**849**	860
79	**678**	860	97	**721**	860	97	**764**	860	146	**807**	860	838	**850**	860
79	**679**	860	371	**722**	857	97	**765**	860	529	**808**	849	838	**851**	860
479	**680**	848	371	**723**	859	98	**766**	860	529	**809**	860	852	**852**	852
479	**681**	848	63	**724**	860	98	**767**	860	810	**810**	810	838	**853**	860
479	**682**	860	63	**725**	860	551	**768**	848	97	**811**	860	838	**854**	861
479	**683**	860	142	**726**	849	146	**769**	848	97	**812**	860	808	**855**	861
479	**684**	860	97	**727**	852	551	**770**	860	97	**813**	860	856	**856**	856
479	**685**	848	97	**728**	852	146	**771**	860	97	**814**	860	794	**857**	861
479	**686**	860	97	**729**	860	146	**772**	860	97	**815**	860	858	**858**	858
479	**687**	860	142	**730**	859	690	**773**	848	97	**816**	860	841	**859**	861
479	**688**	860	731	**731**	731	583	**774**	860	485	**817**	852	860	**860**	860
689	**689**	689	689	**732**	840	690	**775**	860	485	**818**	860	861	**861**	861
381	**690**	760	689	**733**	840	690	**776**	860	485	**819**	860	862	**862**	862
371	**691**	838	68	**734**	860	777	**777**	777	485	**820**	860			
63	**692**	852	63	**735**	860	689	**778**	840	485	**821**	860			
63	**693**	852	63	**736**	860	68	**779**	860	485	**822**	860			

Table 3 (concluded).

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