Taxonomy of Graphs of Order 10

Peter Adams*† Roger B. Eggleton‡

James A. MacDougall§

July 17, 2006

Abstract

Extending earlier data summaries for graphs of order $n \leq 9$, this paper describes structural characteristics and relationships for the 12005168 graphs of order 10. It summarises data for their degree sequences, their component structure, their cycle structure, and their poset structure under the subgraph partial order. A standardized listing of the order 10 graphs, along with related data, is provided on the website

`www.maths.uq.edu/~pa/research/poset10.html`

2000 *Mathematics subject classification*: primary 05C99, secondary 05C07, 06A07.

1 Introduction

The simple graphs of small order are the fundamental objects of graph theory. Their basic structural characteristics and relationships, which we regard as their taxonomy, are of central importance to graph theory. Here we report descriptive data for the taxonomy of simple graphs of order 10. This extends earlier studies covering simple graphs of orders $n \leq 8$ [1] and order 9 [3], with associated websites [2] and [4] providing more extensive data and full lists.

Let $\mathcal{G}(n)$ be the set of all unlabelled simple graphs ("graphs" for brevity) of order n. The basic structural characteristics of any graph $G \in \mathcal{G}(n)$ include its size, its degree sequence, its number of components and their orders, the order of its smallest cycle and the number of such cycles, and its canonical edge list. For each $G \in \mathcal{G}(n)$ we represent these structural characteristics by a numerical sequence $\Sigma(G)$, called the *signature* of G.

*Research supported by the Australian Research Council

†School of Phys ical Sciences, Univ. of Queensland, QLD 4072, Australia

‡Mathematics Department, Illinois State Univ., Normal, IL 61790-4520, USA

§School of Mathematical and Physical Sciences, The Univ. of Newcastle, NSW 2308, Australia

For the signatures of two graphs G and H we define the ordering $\Sigma(G) < \Sigma(H)$ to hold provided the first place s in which the signatures differ satisfies $s(G) < s(H)$, with the single exception (for historical reasons [7]) that we require $s(G) > s(H)$ if s corresponds to the number of cycles of smallest order (girth multiplicity). For brevity, we call this the *lexicographic* order of signatures.

A linear ordering of $\mathcal{G}(n)$ results from sorting the graphs to match the lexicographic order of their signatures, modified by centrally symmetric placement of each complementary pair of graphs, and relocation of any self-complementary graphs to the middle of the list while preserving their relative order. Each $G \in \mathcal{G}(n)$ is assigned a unique number $N^*(G)$ according to its rank in this linear ordering. The extreme graphs in the linear ordering are K_n, the complete graph of order n, and its complement K_n^c, the empty graph of order n, with $N^*(K_n^c) = 1$ and $N^*(K_n) = |\mathcal{G}(n)|$. The SEAM numbering

$$N^* : \mathcal{G}(n) \to [1..|\mathcal{G}(n)|]$$

preserves the main characteristics of the numbering N introduced by Peter Steinbach [10] for graphs of order $n \leq 7$, but removes the subjectivity present in the fine structure of Steinbach's numbering. A full specification of the SEAM numbering N^* is given in [1].

In addition to structural characteristics of each graph $G \in \mathcal{G}(n)$ we are interested in structural relationships between graphs in $\mathcal{G}(n)$. Key structural relationships in $\mathcal{G}(n)$ are captured by the *subgraph partial ordering* $G \leq H$, holding for any pair $G, H \in \mathcal{G}(n)$ such that G is isomorphic to a subgraph of H. If we regard all graphs in $\mathcal{G}(n)$ as having the same vertex set, $G \leq H$ may be read as "G is a subgraph of H" or, more explicitly, "G is a spanning subgraph of H". It is convenient, and in practice not confusing, to also use $\mathcal{G}(n)$ to denote the partially ordered set (poset) of graphs of order n, under the subgraph partial ordering.

If $G, H \in \mathcal{G}(n)$ satisfy $G \leq H$, we say that G is an *immediate predecessor* of H, and H is an *immediate successor* of G, if H has an edge e such that $G = H - e$. The structure of the poset $\mathcal{G}(n)$ is represented by its Hasse diagram $\mathcal{HG}(n)$, which is the directed graph with vertex set $\mathcal{G}(n)$, and a directed edge $G \to H$ for every pair $G, H \in \mathcal{G}(n)$ such that H is an immediate successor of G. In practice, a convenient representation of the structure of $\mathcal{HG}(n)$ is a listing of immediate predecessors and immediate successors for each $G \in \mathcal{G}(n)$, the graphs themselves being specified by their SEAM numbers.

Graph signatures, the SEAM numbering and the Hasse diagram of $\mathcal{G}(n)$ are useful tools for studying the taxonomy of graphs of order n, so long as n is small. However, data sets such as these are subject to practical limitations imposed by the space needed to store them and the time needed to search and extract information from them. Steinbach's methods made hardcopy publication of his lists [10] practical only as far as $n = 7$. The total number of graphs of all orders $n \leq 7$ is 1252, while the number of graphs of order 8 is 12346, a tenfold increase, and the number of order 9 is 274668, a further twentyfold increase. To circumvent the space limitations of hardcopy, our lists for orders $n \leq 8$ [2] and order 9 [4] are in electronic form. We confined ourselves to summary data and commentary in the associated hardcopy publications [1], [3]. Significant computational improvements were needed to bring the order 9 graphs within practical reach, and another quantum leap was required for the current study of order 10 graphs, which number 12005168, a fortyfold increase in magnitude.

Although we have computed the list of signatures for $\mathcal{G}(10)$, and used it to produce the summary data in this paper, it is too large (over 1.4 Gb) to make available in text format on our website. As a compromise, we have placed on the website [5] a compact file (about 19Mb) encoding the graphs of order 10 in SEAM order. Only the first 6002584 graphs are given, as the remaining graphs are readily deduced by centrally-symmetric complementation: if G is any one of the first 6002584 graphs of order 10, and its SEAM number is $N^*(G)$, then its complement G^c has SEAM number $N^*(G^c) = 12005169 - N^*(G)$. For each G we simply give the canonical edge sequence, not the full signature. From that the size is immediate, and it is simple to determine the degree sequence and the component sequence of G; if desired, it is also straightforward to determine the order of the smallest cycle (girth) and its multiplicity, so the signature $\Sigma(G)$ can be fully recovered.

We call any graph $G \in \mathcal{G}(n)$ a *sentinel* if it comes first in the SEAM ordering of all graphs with the same degree sequence as G. The SEAM numbers and full signatures of all 16016 sentinels of order 10 are on the website [5], as are the SEAM numbers and full signatures of all sentinels of orders $n \leq 9$.

In what follows we present summary data describing the taxonomy of order 10 graphs derived from their full signatures. More extensive data and lists are provided on the website [5]:

`www.maths.uq.edu/~pa/research/poset10.html`

2 Degree sequences of graphs of order 10

We have computed the signatures of all graphs of order 10, and sorted them into SEAM order. Thus for every $G \in \mathcal{G}(10)$ the signature $\Sigma(G)$ and the SEAM number $N^*(G)$ are known. From the list of signatures we can readily derive descriptive data about many structural properties of all graphs of order 10.

In particular, for each $G \in \mathcal{G}(10)$ the signature $\Sigma(G)$ begins with the size of G followed by its degree sequence, so we can readily derive distribution data for degree sequences of graphs of order 10. Where appropriate, we shall report this data in the context of corresponding data for graphs of smaller orders.

Let $d(n,m)$ be the number of distinct degree sequences of graphs of order n and size m, so $d(n) = \Sigma_m d(n,m)$ is the number of distinct degree sequences of graphs of order n. The sequence $0^{d(n,0)}1^{d(n,1)} \dots m^{d(n,m)} \dots$ is the *degree sequence size distribution* for $\mathcal{G}(n)$. Since any graph and its complement have complementary degree sequences, the degree sequence size distribution for $\mathcal{G}(n)$ is centrally symmetric. Table 1 presents these distributions for $n \le 10$, with truncation midway for the longer sequences.

n	$d(n)$	Distribution
1	1	0^1
2	2	$0^1\,1^1$
3	4	$0^1\,1^1\,2^1\,3^1$
4	11	$0^1\,1^1\,2^2\,3^3\,4^2\,5^1\,6^1$
5	31	$0^1\,1^1\,2^2\,3^4\,4^5\,5^5\,6^5\,7^4\,8^2\,9^1\,10^1$
6	102	$0^1\,1^1\,2^2\,3^5\,4^7\,5^{10}\,6^{12}\,7^{13}\,8^{13}\,9^{12}\,10^{10}\,11^7\,12^5\,13^2\,14^1\,15^1$
7	342	$0^1\,1^1\,2^2\,3^5\,4^8\,5^{13}\,6^{19}\,7^{24}\,8^{28}\,9^{34}\,10^{36}\,11^{36}\ \dots$
8	1213	$0^1\,1^1\,2^2\,3^5\,4^9\,5^{15}\,6^{24}\,7^{35}\,8^{45}\,9^{59}\,10^{74}\,11^{85}\,12^{97}\,13^{102}\,14^{105}\ \dots$
9	4361	$0^1\,1^1\,2^2\,3^5\,4^9\,5^{16}\,6^{27}\,7^{42}\,8^{60}\,9^{84}\,10^{111}\,11^{143}\,12^{178}\,13^{212}\,14^{247}$ $15^{276}\,16^{298}\,17^{309}\,18^{319}\ \dots$
10	16016	$0^1\,1^1\,2^2\,3^5\,4^9\,5^{17}\,6^{29}\,7^{47}\,8^{71}\,9^{106}\,10^{147}\,11^{198}\,12^{265}\,13^{336}\,14^{423}$ $15^{515}\,16^{609}\,17^{696}\,18^{789}\,19^{857}\,20^{925}\,21^{968}\,22^{992}\ \dots$

Table 1: Degree sequence size distribution for $\mathcal{G}(n)$.

The definition of an index unimodal sequence, its index peak and its peak support are given in [1], but these notions are probably contextually clear in the following theorem, based on Table 1:

Theorem 1 *For each $n \ge 1$, the number of distinct degree sequences of $\mathcal{G}(n)$ is*

$$1, 2, 4, 11, 31, 102, 342, 1213, 4361, 16016, \dots$$

The degree sequence size distribution for each $\mathcal{G}(n)$ is centrally symmetric. For $n \le 10$ it is also index unimodal: the index peaks and peak supports are

1:0, 1:[0..1], 1:[0..3], 3:3, 5:[4..6], 13:[7..8], 36:[10..11],
105:14, 319:18, 992:[22..23].

The numbers of distinct degree sequences in Theorem 1 confirm the first 10 terms of sequence A004251 in Sloane's list of integer sequences [9].

The *multiplicity* of a degree sequence $\mathbf{d}$ is the number of distinct graphs with the same degree sequence $\mathbf{d}$. Let $m(n,r)$ be the number of degree sequences of multiplicity r for $\mathcal{G}(n)$, so $m(n) = \Sigma_r\, m(n,r)$ is the total number of degree sequences of order n. The sequence $1^{m(n,1)}2^{m(n,2)}\ldots r^{m(n,r)}\ldots$ is the *degree sequence multiplicity distribution* of $\mathcal{G}(n)$. These sequences for $4 \le n \le 8$ were tabulated in [1]. For $n = 9$, in view of the length, just the last few terms of the distribution were given explicitly in [3], while the full sequence is given on the website [4]. The sequence for $n = 10$ is given on the website [5]; the last few terms of this distribution are

$$\ldots, 41091^2, 44637^2, 46909^2, 49177^2, 51433^2, 52996^2, 53453^2, \\ 55382^2, 57180^2, 57498^2, 61903^2, 62771^2, 65500^2.$$

Also $r = 8645, 1523$ and 713 are the last multiplicities with $m(10,r) = 4, 6$ and 8, respectively. The degree sequence multiplicities give us the following extension of Theorem 1 of [3]:

Theorem 2 *For each $\mathcal{G}(n)$ with $n \ge 1$, the maximum number of graphs with the same degree sequence is*

$$1, 1, 1, 1, 2, 5, 20, 184, 3020, 65500, \ \ldots$$

The degree sequences with maximum multiplicity for $5 \le n \le 10$ are

$$\begin{aligned}
m(5,2) = 3:&\quad 1^2 2^3,\ 1^2 2^3 3^1,\ 2^3 3^2\\
m(6,5) = 2:&\quad 1^2 2^2 3^2,\ 2^2 3^2 4^2\\
m(7,20) = 2:&\quad 1^1 2^2 3^3 4^1,\ 2^1 3^3 4^2 5^1\\
m(8,184) = 3:&\quad 2^2 3^3 4^2 5^1,\ 2^1 3^3 4^3 5^1,\ 2^1 3^2 4^3 5^2\\
m(9,3020) = 1:&\quad 2^1 3^2 4^3 5^2 6^1\\
m(10,65500) = 2:&\quad 2^1 3^2 4^3 5^2 6^2,\ 3^2 4^2 5^3 6^2 7^1.
\end{aligned}$$

Theorem 2 is concerned with the high end of the degree sequence multiplicity distribution. The low end is also of interest. As in [3], we say a degree sequence $\mathbf{d}$ is *uniquely graphic* if its multiplicity is 1, that is, there is only one graph with degree sequence $\mathbf{d}$. The *size distribution* of uniquely graphic degree sequences of order n is the sequence $0^{a(n,0)}1^{a(n,1)}\ldots m^{a(n,m)}\ldots$ where $a(n,m)$ is the number of uniquely graphic degree sequences for graphs of order n and size m, so $a(n) = \Sigma_m\, a(n,m)$ is the total number of uniquely graphic degree sequences of order n. This distribution is centrally symmetric, because any graph and its complement have complementary degree sequences. Table 3 of [3] gives the size distribution of uniquely graphic degree sequences of orders $n \le 9$; the first half of the distribution for order 10 is

$$0^1 1^1 2^2 3^5 4^7 5^{11} 6^{13} 7^{18} 8^{21} 9^{32} 10^{37} 11^{39} 12^{53} 13^{53} 14^{65} 15^{79} 16^{82} 17^{84} \\ 18^{105} 19^{95} 20^{107} 21^{111} 22^{105} \ldots$$

Theorem 3 *For $n \geq 1$, the number of uniquely graphic degree sequences of order n is*

$$1, 2, 4, 11, 28, 72, 170, 407, 956, 2252, \ \ldots$$

The size distribution of uniquely graphic degree sequences of order n is centrally symmetric, but for $7 \leq n \leq 10$ it is not unimodal, and for $n = 7$ and $n = 10$ its maximum value is not central.

In [3] we remarked on the phenomenon that for all orders $n \leq 9$ there are graphs of every possible size which are the unique realizations of their degree sequences. In a forthcoming paper [7] we supersede this remark by proving that for every order $n \geq 1$ and every possible size m there is a uniquely graphic degree sequence of order n and size m. Specifically, for given n and m, let r be the smallest positive integer such that $m \leq \binom{r}{2}$, and let s be the difference $s = \binom{r}{2} - m$. Then the graph

$$G(r, s) = (K_r - E(K_{1,s})) \cup (n - r)K_1$$

has order n and size m, and is the unique realization of its degree sequence. Indeed, let $\mathcal{G}(n, m)$ be the sequence of all graphs of order n and size m in SEAM order, and let $m + m' = \binom{n}{2}$. Then $G(r, s)$ is the first graph in the sequence $\mathcal{G}(n, m)$ if $m \leq m'$, and the last graph in the sequence if $m > m'$. Moreover, if r' and s' are the corresponding integers determined by m', then $G(r', s')^c$ is also in the sequence $\mathcal{G}(n, m)$, is the unique realization of its degree sequence, and is distinct from $G(r, s)$ if $n \geq 4$ and $3 \leq m \leq \binom{n}{2} - 3$.

In Table 1 of [3] we noted the average multiplicity of graphic degree sequences of orders $n \leq 9$. All three standard measures of central tendency (mean, median and mode) were reported. We conjectured that for degree sequences of any order $n \geq 1$ the modal multiplicity will be 1, while the median and mean multiplicities will both increase with n, the latter more rapidly than the former. In Table 2 below we include the data for $n = 10$, which is consistent with our conjecture. (Means and standard deviations are given correct to one decimal place.) In particular, degree sequences of order 10 have a mean of almost 750 realizations, whereas half have 19 or fewer realizations, and most commonly a sequence has a unique realization.

n	1	2	3	4	5	6	7	8	9	10
mean	1.0	1.0	1.0	1.0	1.1	1.5	3.1	10.2	63.0	749.6
s.d.	–	0.0	0.0	0.0	0.3	1.0	3.6	20.0	194.2	3482.9
median	1	1	1	1	1	1	2	3	6	19
mode	1	1	1	1	1	1	1	1	1	1

Table 2: Average multiplicity of degree sequences for $\mathcal{G}(n)$.

3 Components of graphs of order 10

Each graph signature $\Sigma(G)$ contains the number of components of G and their orders. We now report descriptive data for $\mathcal{G}(10)$ based on this information, in the context of corresponding data for graphs of smaller orders.

Let $c(n,m,k)$ be the number of graphs of order n and size m with exactly k components, so $c(n,k) = \Sigma_m\, c(n,m,k)$ is the total number of graphs of order n with exactly k components, and $c(n) = \Sigma_k\, kc(n,k)$ is the total number of components possessed by all graphs of order n. The sequence $1^{c(n,1)}2^{c(n,2)}\ldots k^{c(n,k)}\ldots$ is the *component distribution* of $\mathcal{G}(n)$. Note that

$$\Sigma_k\, c(n,k) = |\mathcal{G}(n)|,$$

so $\overline{c}(n) = \Sigma_k\, kc(n,k)/\Sigma_k\, c(n,k) = c(n)/|\mathcal{G}(n)|$ is the mean number of components possessed by graphs of order n. Also $\overline{c}(n,1) = c(n,1)/|\mathcal{G}(n)|$ is the fraction of all graphs of order n that are connected, or equivalently, the probability that a graph of order n is connected (assuming uniform distribution).

The tabulation of $c(n,m,k)$ for $n \leq 10$ is too large to include here, so we have placed it on the website [5]. From it we derive Table 3, giving $c(n)$ and the component distribution of $\mathcal{G}(n)$, together with $\overline{c}(n)$ correct to 3 decimal places and $\overline{c}(n,1)$ correct to 4 decimal places, for $n \leq 10$.

n	$c(n,k)$ sequence	$c(n)$	$\overline{c}(n)$	$\overline{c}(n,1)$
1	1^1	1	1.000	1.0000
2	$1^1\,2^1$	3	1.500	0.5000
3	$1^2\,2^1\,3^1$	7	1.750	0.5000
4	$1^6\,2^3\,3^1\,4^1$	19	1.727	0.5455
5	$1^{21}\,2^8\,3^3\,4^1\,5^1$	55	1.618	0.6176
6	$1^{112}\,2^{30}\,3^9\,4^3\,5^1\,6^1$	222	1.423	0.7179
7	$1^{853}\,2^{145}\,3^{32}\,4^9\,5^3\,6^1\,7^1$	1303	1.248	0.8170
8	$1^{11117}\,2^{1028}\,3^{154}\,4^{33}\,5^9\,6^3\,7^1\,8^1$	13845	1.121	0.9005
9	$1^{261080}\,2^{12320}\,3^{1065}\,4^{156}\,5^{33}\,6^9\,7^3\,8^1\,9^1$	289796	1.055	0.9505
10	$1^{11716571}\,2^{274806}\,3^{12513}\,4^{1074}\,5^{157}$ $6^{33}\,7^9\,8^3\,9^1\,10^1$	12309107	1.025	0.9760

Table 3: Number of components of graphs in $\mathcal{G}(n)$.

Summarizing global information that we obtain from the component distribution of $\mathcal{G}(n)$, we have

Theorem 4 *For $n \geq 1$, the number of connected graphs of order n is*

$$1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571,\ \ldots$$

and the total number of components possessed by all graphs of order n is

$$1, 3, 7, 19, 55, 222, 1303, 13845, 289796, 12309107,\ \ldots$$

The numbers of connected graphs in Theorem 6 confirm the first 10 terms of sequence A001349 in Sloane's list [9].

Let $b(n,m,k)$ be the total number of components of order k possessed by all graphs of order n and size m, so $b(n,k) = \Sigma_m\, b(n,m,k)$ is the total number of components of order k possessed by all graphs of order n. The sequence $1^{b(n,1)}2^{b(n,2)} \dots k^{b(n,k)} \dots$ is the *component order distribution* of $\mathcal{G}(n)$. Since

$$\Sigma_k\, b(n,k) = c(n) \text{ and } \Sigma_k\, kb(n,k) = n|\mathcal{G}(n)|,$$

then $\overline{b}(n) = \Sigma_k\, kb(n,k)/\Sigma_k\, b(n,k) = n|\mathcal{G}(n)|/c(n)$ is the mean component order for graphs of order n, that is, the mean number of vertices per component in graphs of order n. Similarly, let $b^*(n,m,k)$ be the total number of graphs of order n and size m with largest component of order k, so $b^*(n,k) = \Sigma_m\, b^*(n,m,k)$ is the total number of graphs of order n with largest component of order k. The sequence $1^{b^*(n,1)}2^{b^*(n,2)} \dots k^{b^*(n,k)} \dots$ is the *largest component order distribution* of $\mathcal{G}(n)$. Since

$$\Sigma_k\, b^*(n,k) = |\mathcal{G}(n)|,$$

the mean order of the largest component for graphs of order n is given by $\overline{b^*}(n) = \Sigma_k\, kb^*(n,k)/\Sigma_k\, b^*(n,k) = \Sigma_k\, kb^*(n,k)/|\mathcal{G}(n)|$.

The tabulations of $b(n,m,k)$ and $b^*(n,m,k)$ for $n \le 10$ are placed on the website [5]. From them we have derived Tables 4 and 5, giving the component order distribution and the largest component order distribution of $\mathcal{G}(n)$, together with $\overline{b}(n)$ and $\overline{b^*}(n)$ correct to 3 decimal places, for $n \le 10$.

n	$b(n,k)$ sequence	$\overline{b}(n)$
1	1^{1}	1.000
2	$1^{2}2^{1}$	1.333
3	$1^{4}2^{1}3^{2}$	1.714
4	$1^{8}2^{3}3^{2}4^{6}$	2.316
5	$1^{19}2^{5}3^{4}4^{6}5^{21}$	3.091
6	$1^{53}2^{14}3^{10}4^{12}5^{21}6^{112}$	4.216
7	$1^{209}2^{39}3^{24}4^{24}5^{42}6^{112}7^{853}$	5.609
8	$1^{1253}2^{170}3^{72}4^{72}5^{84}6^{224}7^{853}8^{11117}$	7.134
9	$1^{13599}2^{1083}3^{322}4^{210}5^{231}6^{448}7^{1706}8^{11117}9^{261080}$	8.530
10	$1^{288267}2^{12516}3^{2112}4^{948}5^{735}6^{1232}7^{3412}8^{22234}9^{261080}10^{11716571}$	9.753

Table 4: Component order distribution of $\mathcal{G}(n)$.

We say that an edge in G is *isolated* if it forms a component of order 2 in G, just as isolated vertices coincide with components of order 1. A connected graph of order n and size m is *sparse* if $m \le n$, and any $G \in \mathcal{G}(n)$

n	$b^*(n,k)$ sequence	$\overline{b^*}(n)$
1	1^1	1.000
2	$1^2\,2^1$	1.500
3	$1^1\;2^1\;3^2$	2.250
4	$1^1\,2^2\,3^2\,4^6$	3.182
5	$1^1\,2^2\,3^4\,4^6\,5^{21}$	4.294
6	$1^1\,2^3\,3^7\,4^{12}\,5^{21}\,6^{112}$	5.468
7	$1^1\,2^3\,3^9\,4^{24}\,5^{42}\,6^{112}\,7^{853}$	6.689
8	$1^1\,2^4\,3^{12}\,4^{51}\,5^{84}\,6^{224}\,7^{853}\,8^{11117}$	7.850
9	$1^1\,2^4\,3^{18}\,4^{63}\,5^{231}\,6^{448}\,7^{1706}\,8^{11117}\,9^{261080}$	8.937
10	$1^1\,2^5\,3^{21}\,4^{108}\,5^{504}\,6^{1232}\,7^{3412}\,8^{22234}\,9^{261080}\,10^{11716571}$	9.973

Table 5: Largest component order distribution of $\mathcal{G}(n)$.

is *sparse* if every component is sparse. Sparse graphs have average degree at most 2, and at most one cycle in any component.

Let $f(n,c)$ and $s(n,c)$ be, respectively, the number of forests and number of sparse graphs of order n with exactly c components. The sequences $1^{f(n,1)}2^{f(n,2)}\dots c^{f(n,c)}\dots$ and $1^{s(n,1)}2^{s(n,2)}\dots c^{s(n,c)}\dots$ are, respectively, the *component number distributions* for forests and sparse graphs in $\mathcal{G}(n)$. The numbers of forests and sparse graphs of order n are $f(n) = \Sigma_c\, f(n,c)$ and $s(n) = \Sigma_c\, s(n,c)$ respectively, and the mean numbers of components for forests and sparse graphs of order n are $\overline{f}(n) = \Sigma_c\, cf(n,c)/f(n)$ and $\overline{s}(n) = \Sigma_c\, cs(n,c)/s(n)$ (see Table 6).

n	$f(n,c)$ sequence	$\overline{f}(n)$	$s(n,c)$ sequence	$\overline{s}(n)$
1	1^1	1.000	1^1	1.000
2	$1^1 2^1$	1.500	$1^1 2^1$	1.500
3	$1^1 2^1 3^1$	2.000	$1^2 2^1 3^1$	1.750
4	$1^2 2^2 3^1 4^1$	2.167	$1^4 2^3 3^1 4^1$	1.889
5	$1^3 2^3 3^2 4^1 5^1$	2.400	$1^8 2^6 3^3 4^1 5^1$	2.000
6	$1^6 2^6 3^4 4^2 5^1 6^1$	2.450	$1^{19} 2^{15} 3^7 4^3 5^1 6^1$	2.022
7	$1^{11} 2^{11} 3^7 4^4 5^2 6^1 7^1$	2.514	$1^{44} 2^{35} 3^{17} 4^7 5^3 6^1 7^1$	2.046
8	$1^{23} 2^{23} 3^{14} 4^8 5^4 6^2 7^1 8^1$	2.500	$1^{112} 2^{89} 3^{42} 4^{18} 5^7 6^3 7^1 8^1$	2.037
9	$1^{47} 2^{46} 3^{29} 4^{15} 5^8 6^4 7^2 8^1 9^1$	2.490	$1^{287} 2^{226} 3^{109} 4^{44} 5^{18} 6^7 7^3 8^1 9^1$	2.029
10	$1^{106} 2^{99} 3^{60} 4^{32} 5^{16} 6^8$ $7^4 8^2 9^1 10^1$	2.441	$1^{763} 2^{599} 3^{283} 4^{116} 5^{45} 6^{18}$ $7^7 8^3 9^1 10^1$	2.015

Table 6: Forest and sparse graph component number distributions in $\mathcal{G}(n)$.

The following global information is derived from Tables 4, 5 and 6.

Theorem 5 *For $n \geq 1$, the total number of isolated vertices possessed by all graphs of order n is*

$$1, 2, 4, 8, 19, 53, 209, 1253, 13599, 288267, \ \ldots$$

the total number of isolated edges possessed by all graphs of order n is

$$0, 1, 1, 3, 5, 14, 39, 170, 1083, 12516, \ \ldots$$

and the total number of sparse components of size at least 2 possessed by all graphs of order n is

$$0, 0, 2, 6, 16, 45, 119, 346, 1113, 4683, \ \ldots$$

The number of forests of order n is

$$1, 2, 3, 6, 10, 20, 37, 76, 153, 329, \ \ldots$$

and the number of sparse graphs of order n is

$$1, 2, 4, 9, 19, 46, 108, 273, 696, 1836, \ \ldots$$

In Theorem 7 the total numbers of isolated vertices in graphs of order n agree with the first 10 terms of sequence A006897, and the numbers of forests of order n agree with the first 10 terms of sequence A005195, in Sloane's list [9].

4 Smallest cycles in graphs of order 10

Each graph signature $\Sigma(G)$ contains the order of the smallest cycle in G (the *girth* of G, defined to be 0 if G is acyclic), and the number of such cycles. In this section we report descriptive data for $\mathcal{G}(10)$ based on this information, in the context of corresponding data for graphs of smaller orders.

Let $g(n, m, k)$ be the number of graphs of order n and size m with girth k, so $g(n, k) = \Sigma_m\, g(n, m, k)$ is the total number of graphs of order n with girth k. The sequence $0^{g(n,0)}3^{g(n,3)} \ldots k^{g(n,k)} \ldots$ is the *girth distribution* of $\mathcal{G}(n)$. Among the graphs of order n with at least one cycle, the mean girth is $\overline{g}(n) = \Sigma_{k\geq 3}\, kg(n, k)/\Sigma_{k\geq 3}\, g(n, k)$. Note that the number of graphs of order n with at least one cycle is

$$\Sigma_{k\geq 3}\, g(n, k) = |\mathcal{G}(n)| - g(n, 0)$$

where $g(n, 0)$ is the number of acyclic graphs of order n, so $g(n, 0) = f(n)$.

Similarly, for $k \geq 3$ let $g^*(n, m, k)$ be the total number of *girth cycles* (cycles of order k) possessed by graphs of order n, size m and girth k. Then the total number of girth cycles possessed by graphs of order n and girth k is $g^*(n, k) = \Sigma_m\, g^*(n, m, k)$. The sequence $3^{g^*(n,3)} \ldots k^{g^*(n,k)} \ldots$ is the *girth* *cycle distribution* of $\mathcal{G}(n)$. Among the graphs of order n with at least one cycle, the mean number of girth cycles is $\overline{g^*}(n) = \Sigma_{k\geq 3}\, g^*(n,k)/\Sigma_{k\geq 3}\, g(n,k)$.

As for $c(n,m,k)$, the tabulations of $g(n,m,k)$ and $g^*(n,m,k)$ for $n \leq 10$ are placed on the website [5]. From them we have derived Tables 7 and 8, giving the girth distribution and the girth cycle distribution of $\mathcal{G}(n)$, together with $\overline{g}(n)$ and $\overline{g^*}(n)$ correct to 3 decimal places, for $n \leq 10$.

n	$g(n,k)$ sequence	$\overline{g}(n)$
1	0^1	–
2	0^2	–
3	$0^3\, 3^1$	3.000
4	$0^6\, 3^4\, 4^1$	3.200
5	$0^{10}\, 3^{20}\, 4^3\, 5^1$	3.208
6	$0^{20}\, 3^{118}\, 4^{15}\, 5^2\, 6^1$	3.162
7	$0^{37}\, 3^{937}\, 4^{59}\, 5^8\, 6^2\, 7^1$	3.084
8	$0^{76}\, 3^{11936}\, 4^{296}\, 5^{26}\, 6^9\, 7^2\, 8^1$	3.032
9	$0^{153}\, 3^{272771}\, 4^{1604}\, 5^{101}\, 6^{28}\, 7^8\, 8^2\, 9^1$	3.007
10	$0^{329}\, 3^{11992996}\, 4^{11303}\, 5^{396}\, 6^{107}\, 7^{25}\, 8^9\, 9^2\, 10^1$	3.001

Table 7: Girth distribution of $\mathcal{G}(n)$.

n	$g^*(n,k)$ sequence	$\overline{g^*}(n)$
1	–	–
2	–	–
3	3^1	1.000
4	$3^8\, 4^1$	1.800
5	$3^{57}\, 4^5\, 5^1$	2.625
6	$3^{468}\, 4^{39}\, 5^2\, 6^1$	3.750
7	$3^{5126}\, 4^{193}\, 5^9\, 6^2\, 7^1$	5.294
8	$3^{92020}\, 4^{1291}\, 5^{34}\, 6^{11}\, 7^2\, 8^1$	7.609
9	$3^{2976692}\, 4^{8813}\, 5^{149}\, 6^{35}\, 7^8\, 8^2\, 9^1$	10.876
10	$3^{182783912}\, 4^{80391}\, 5^{676}\, 6^{153}\, 7^{26}\, 8^9\, 9^2\, 10^1$	15.233

Table 8: Girth cycle distribution of $\mathcal{G}(n)$.

Summarizing global information derived from Tables 7 and 8, we have

Theorem 6 *The number of graphs of order n with girth 3 is*

$$0, 0, 1, 4, 20, 118, 937, 11936, 272771, 11992996, \ \ldots$$

the number of graphs of order n with girth greater than 3 is

$$0, 0, 0, 1, 4, 18, 70, 334, 1744, 11843, \ldots$$

and the total number of girth cycles possessed by all graphs of order n is

$$0, 0, 1, 9, 63, 510, 5331, 93359, 2985700, 182865170, \ldots$$

5 Poset structure of graphs of order 10

The poset structure of $\mathcal{G}(10)$ is represented by the Hasse diagram $\mathcal{HG}(10)$, so we shall now describe some features of $\mathcal{HG}(10)$.

For each $G \in \mathcal{G}(n)$, the *outdegree* $d^+(G)$ in $\mathcal{HG}(n)$ is the number of immediate successors of G, the *indegree* $d^-(G)$ is the number of immediate predecessors of G, and $d(G) = d^+(G) + d^-(G)$ is the *full degree* of G. Let $\alpha(n, m, k)$ be the number of graphs of order n and size m with outdegree k, so $\alpha(n, k) = \Sigma_m \alpha(n, m, k)$ is the total number of graphs of order n with outdegree k. Similarly, let $\beta(n, m, k)$ and $\beta(n, k)$ be the corresponding indegree counts, and let $\gamma(n, m, k)$ and $\gamma(n, k)$ be the corresponding full degree counts. The sequence $0^{\alpha(n,m,0)}1^{\alpha(n,m,1)} \ldots k^{\alpha(n,m,k)} \ldots$ is the *outdegree sequence for level* m of $\mathcal{HG}(n)$, while $0^{\alpha(n,0)}1^{\alpha(n,1)} \ldots k^{\alpha(n,k)} \ldots$ is the *outdegree sequence* of $\mathcal{HG}(n)$. We define the indegree sequences and the full degree sequences similarly.

Let $p(n, m) = |\mathcal{G}(n, m)|$ be the number of graphs of order n and size m, which is the cardinality of *level* m of $\mathcal{H}G(n)$, the set of graphs at distance m from the empty graph nK_1. Let $q(n, m) = \Sigma_k \alpha(n, m, k) = \Sigma_k \beta(n, m+1, k)$ be the number of directed edges of $\mathcal{H}G(n)$ between level m and level $m+1$. Then $p(n) = \Sigma_m p(n, m)$ is the order of $\mathcal{H}G(n)$, and $q(n) = \Sigma_m q(n, m)$ is the size of $\mathcal{H}G(n)$.

Let $m + m' = \binom{n}{2}$. Since any graph $G \in \mathcal{G}(n)$ and its complement G^c satisfy $d^+(G) = d^-(G^c)$ and $d(G) = d(G^c)$, the outdegree sequence for level m of $\mathcal{HG}(n)$ and the indegree sequence for level m' are equal, and the full degree sequences for levels m and m' are equal. Consequently the outdegree sequence and the indegree sequence of $\mathcal{HG}(n)$ are equal.

On the website [5] we have listed, for each level $0 \leq m \leq 22$ of $\mathcal{HG}(10)$, the outdegree sequence, the indegree sequence and the full degree sequence. In Table 9 below we give summary data derived from these sequences. The last two columns report $p(10, m)$ and $q(10, m)$.

Table 9 allows us to extend Theorem 5 of [3] to include data for $\mathcal{HG}(10)$:

Theorem 7 *For $n \geq 1$, the order of the Hasse diagram $\mathcal{H}G(n)$ is*

$$1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, \ldots$$

and the size of $\mathcal{H}G(n)$ is

$$0, 1, 3, 14, 74, 571, 6558, 125066, 4147388, 247179594, \ldots$$

The order data is confirmed by sequence A000088 in [9].

Size m	Outdegree min	max	Indegree min	max	Full degree min	max	$p(10,m)$	$q(10,m)$
0	1	1	0	0	1	1	1	1
1	2	2	1	1	3	3	1	2
2	3	4	1	1	4	5	2	7
3	2	7	1	2	3	9	5	21
4	3	9	1	3	4	12	11	66
5	1	14	1	4	2	18	26	207
6	2	23	1	6	3	29	66	669
7	3	30	1	7	5	37	165	2135
8	3	37	1	8	4	45	428	6703
9	1	36	1	9	2	45	1103	20156
10	2	35	1	10	3	45	2769	57419
11	3	34	2	11	5	45	6759	153085
12	2	33	1	12	4	45	15772	378495
13	2	32	2	13	4	45	34663	862766
14	3	31	1	14	5	45	71318	1807074
15	1	30	1	15	2	45	136433	3470298
16	2	29	1	16	3	45	241577	6105746
17	1	28	2	17	3	45	395166	9841158
18	1	27	1	18	3	45	596191	14532332
19	2	26	2	19	4	45	828728	19668044
20	1	25	1	20	2	45	1061159	24406445
21	1	24	1	21	3	45	1251389	27776990
22	2	23	2	22	5	45	1358852	28999956
Whole	0	37	0	37	1	45	12005168	247179594

Table 9: Minimum and maximum degrees, order and size in $\mathcal{HG}(10)$.

6 Edge-symmetry of graphs of order 10

In this section we discuss information concerning the extreme values of the degrees in $\mathcal{HG}(n)$. In a related paper [6] we establish some relevant theory and report computational results on several classes of graphs of interest here.

We define a graph G to be *edge-transitive* if for every pair (e, e') of edges of G there is a graph automorphism $\theta : G \to G$ such that $\theta(e) = e'$. There is a subtle question of whether one takes a broad or narrow interpretation of this definition. We use the broad interpretation that the definition is vacuously satisfied by the identity automorphism if G has no edges, and is trivially satisfied if G has exactly one edge, so we regard all graphs of size

$m \leq 1$ as edge-transitive. Others take the narrow interpretation that G must have at least two edges to be a candidate, and then G is edge-transitive just when the definition is satisfied for every pair of *distinct* edges of G. A different definition, based on maps which are permutations of the edge-set of G, and not necessarily induced by permutations of the vertex-set of G, allows a wider class of graphs to qualify as edge-transitive, but is more group-theoretic than graph-theoretic (see [6]).

A graph G of order n and size $m \geq 1$ is edge-transitive if and only if it has indegree 1 in $\mathcal{HG}(n)$. Intuitively this is not surprising, and in Theorem 6 of [3] we reported data which assumed this characterization. However, we have since realised that it is not a trivial fact, so we give a proof in [6]. Applying the same characterization here, we deduce from Table 9 that there is at least one edge-transitive graph of order 10 and size m for every $m < 26$ except for $m \in \{11, 13, 17, 19, 22, 23\}$, and such graphs also exist for $m \in \{27, 28, 30, 36, 40, 45\}$. Continuing with a phenomenon noted in [3] for orders $n \leq 9$, it is intriguing to note for order 10 that primes greater than 10 are the predominant exceptions for sizes of edge-transitive graphs.

As in [3], we can give more explicit information about edge-transitive graphs as follows. Let $t(n, m)$ be the number of edge-transitive graphs of order n and size m, so $t(n) = \Sigma_m \, t(n, m)$ is the total number of edge-transitive graphs of order n and the sequence $0^{t(n,0)} 1^{t(n,1)} \ldots m^{t(n,m)} \ldots$ is the *size distribution of edge-transitive graphs* in $\mathcal{G}(n)$. Table 5 of [3] gives the size distributions of edge-transitive graphs for orders $n \leq 9$. For $\mathcal{G}(10)$ the distribution is

$$0^1, 1^1, 2^2, 3^3, 4^4, 5^3, 6^7, 7^2, 8^5, 9^4, 10^4, 12^8, 14^1, 15^3, 16^2, 18^2, 20^4, 21^2, 24^2, 25^1, 27^1, 28^1, 30^1, 36^1, 40^1, 45^1.$$

Our counts for edge-transitive graphs differ from A095352 of [9] because, as indicated above, our definition of edge-transitive differs from that used for A095352. Extending Theorem 6 of [3], we now have

Theorem 8 *For $n \geq 1$, the number of edge-transitive graphs of order n is*

$$1, 2, 4, 8, 12, 21, 26, 38, 49, 67, \ \ldots$$

We say that a graph G has *no edge-symmetry* if G has no automorphism which is edge-moving, that is, which maps an edge onto a different edge. If the indegree of G in $\mathcal{H}G(n)$ is equal to the size of G, it is easy to see that G has no edge-symmetry, and it is tempting to suppose that this property characterizes graphs with no edge-symmetry. In [3] we yielded to this temptation, claiming to report on graphs of order $n \leq 9$ with no edge-symmetry, when we were actually reporting data for graphs with no isomorphic 1-reductions. We are indebted to Neil Calkin [8] for querying whether these two properties are indeed equivalent. Counter-intuitively, it turns out that they are not. We discuss this in some detail in [6], and there we give the correct data for graphs of order $n \leq 10$ with no edge-symmetry.

Thus, Table 6 and Theorem 7 of [3] actually refer to graphs with no isomorphic 1-reductions. We now extend this to order 10 graphs. Let $u(n, m)$ be the number of graphs of order n and size m with no isomorphic 1-reductions, so $u(n) = \Sigma_m u(n, m)$ is the total number of order n graphs with no isomorphic 1-reductions, and the sequence $0^{u(n,0)} 1^{u(n,1)} \ldots m^{u(n,m)} \ldots$ is the *size distribution of graphs with no isomorphic* 1*-reductions* in $\mathcal{G}(n)$. With the corrected description, Table 6 of [3] gives these sequences for $n \leq 9$, and the corresponding distribution for $\mathcal{G}(10)$ is

$$0^{1}1^{1}6^{2}7^{9}8^{31}9^{125}10^{465}11^{1553}12^{4732}13^{12904}14^{31470}15^{68897}16^{135156}17^{238729}$$
$$18^{381632}19^{553102}20^{728820}21^{875447}22^{958988}23^{958743}24^{875412}25^{728918}26^{552665}$$
$$27^{381171}28^{238114}29^{134035}30^{67719}31^{30337}32^{11867}33^{4010}34^{1109}35^{229}36^{34}37^{3}.$$

Now the data in Theorem 7 of [3] can be extended to give

Theorem 9 *For $n \geq 1$, the number of graphs of order n with no isomorphic* 1*-reductions is*

$$1, 2, 2, 2, 2, 9, 148, 3671, 134996, 7976430, \ldots$$

In [1] we defined the *productivity* of any graph G to be the number of non-isomorphic graphs which can be produced by adding a single edge to G, that is, the number of isomorphism classes of 1-extensions of G. This is the outdegree of G in $\mathcal{HG}(n)$, so the outdegree sequence of $\mathcal{HG}(n)$ is the productivity distribution of $\mathcal{G}(n)$. Table 7 of [3] gives this information for $n \leq 9$; the outdegree sequence of $\mathcal{HG}(10)$ is

$$0^{1}1^{66}2^{322}3^{1019}4^{2692}5^{5768}6^{11307}7^{20749}8^{35519}9^{57369}10^{87894}11^{128381}12^{179756}$$
$$13^{243535}14^{321727}15^{417279}16^{532155}17^{664542}18^{807818}19^{947717}20^{1063426}$$
$$21^{1133415}22^{1138467}23^{1070033}24^{936567}25^{758431}26^{564731}27^{385341}28^{239388}$$
$$29^{134290}30^{67778}31^{30396}32^{11885}33^{4018}34^{1119}35^{230}36^{34}37^{3}.$$

Then Theorem 8 of [3] extends to give

Theorem 10 *For $n \geq 1$, the maximum outdegree of $\mathcal{HG}(n)$, and the number of graphs achieving that outdegree, is*

0:1, 1:1, 1:3, 3:1, 4:6, 9:1, 15:2, 22:1, 28:12, 37:3, ...

We also note that the indices of the outdegree sequence of $\mathcal{HG}(10)$ are unimodal, so Theorem 9 of [3] extends to give

Theorem 11 *Each $\mathcal{H}G(n)$ with $n \leq 10$ has an index unimodal outdegree sequence. The index peaks and peak supports are*

1:0, 1:1, 3:1, 7:1, 11:1, 33:[3..4], 140:5, 1196:9, 25010:16, 1138467:22.

This completes our descriptive summary of the taxonomy of graphs of order 10. The tables and data on the website [5] provide access to source data used to prepare this paper. The compact listing of order 10 graphs in SEAM order is provided for those wishing to derive other information of interest.

Acknowledgement *The second author wishes to thank the School of Physical Sciences at The University of Queensland for hospitality and support during the time most of the research for this paper was carried out. He also thanks the School of Mathematical and Physical Sciences at The University of Newcastle, where he holds a conjoint professorship, for hospitality and support during the completion of this paper.*

References

[1] P. Adams, R.B. Eggleton and J.A. MacDougall, Structure of graph posets of orders 4 to 8, *Congressus Numerantium*, **166** (2004), 63-81.

[2] P. Adams, R.B. Eggleton and J.A. MacDougall, The posets of graphs of orders 4 to 8, website:
`www.maths.uq.edu.au/~pa/research/posets4to8.html`

[3] P. Adams, R.B. Eggleton and J.A. MacDougall, Degree sequences and poset structure of order 9 graphs, *Congressus Numerantium*, **166** (2004), 83-95.

[4] P. Adams, R.B. Eggleton and J.A. MacDougall, The poset of graphs of order 9, website: `www.maths.uq.edu.au/~pa/research/poset9.html`

[5] P. Adams, R.B. Eggleton and J.A. MacDougall, The poset of graphs of order 10, website:
`www.maths.uq.edu.au/~pa/research/poset10.html`

[6] P. Adams, R.B. Eggleton and J.A. MacDougall, Graphs with isomorphic 1-reductions, website:
`www.maths.uq.edu.au/~pa/research/1reductions.html`

[7] P. Adams, R.B. Eggleton and J.A. MacDougall, Graphs with a given degree sequence, website:
`www.maths.uq.edu.au/~pa/research/degseq.html`

[8] N. Calkin, oral communication (March 2006).

[9] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, website:
`www.research.att.com/~njas/sequences/`

[10] P. Steinbach, *Field Guide to Simple Graphs*, second ed. (1995) or corrected second ed. (1999), published by Design Lab, Albuquerque Technical-Vocational Institute, Albuquerque, NM.