Degree Sequences and Poset Structure of Order 9 Graphs

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Abstract

The set $\mathcal{G}(n)$ of unlabelled simple graphs of order n is a poset with partial ordering $G \leq H$ whenever G is a spanning subgraph of H. On the website

`www.maths.uq.edu.au/~pa/research/poset9.html`

we have made available a tabulation of the Hasse diagram for $\mathcal{G}(9)$, a digraph of order 274668 and size 4147388, extending our recent tabulations for $\mathcal{G}(n)$ with $4 \leq n \leq 8$. The present paper is a descriptive summary of features of $\mathcal{G}(9)$ derived from the tabulation, including: the maximum number of graphs in $\mathcal{G}(9)$ with the same degree sequence is 3020, corresponding to $2^1 3^2 4^3 5^2 6^1$; there are 36 self-complementary graphs in $\mathcal{G}(9)$, but 10794 graphs with self-complementary degree sequences; there are 49 graphs in $\mathcal{G}(9)$ that are edge-transitive, and 134996 that have no edge-symmetry; the maximum number of immediate successors of a graph in $\mathcal{G}(9)$ is 28, and 12 graphs attain this maximum; the number of immediate successors of a graph in $\mathcal{G}(9)$ is distributed unimodally, with peak at 16 attained by 25010 graphs. All underlying data are available on the website.

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1 Introduction

The unlabelled simple graphs of finite order are the fundamental structures of graph theory. A natural way to view the structural relationships between the graphs of order n is to regard the set $\mathcal{G}(n)$ of all such graphs as a partially ordered set (poset) with partial ordering $G \leq H$ whenever G is isomorphic to a subgraph of H. It is usual to regard all graphs in $\mathcal{G}(n)$ as having the same vertex set, so $G \leq H$ can be read to mean that G is a spanning subgraph of H.

In his very useful *Field Guide to Simple Graphs* [**7**], Steinbach published the structure of the posets $\mathcal{G}(n)$ for $n \leq 7$. It was natural for him to stop at order 7 because of space considerations: there are just 52 graphs of order 5 or less, but there are 156 graphs of order 6 and 1044 graphs of order 7. The total number of immediate predecessors and successors of any graph in $\mathcal{G}(7)$ averages almost 13, so a substantial amount of space is needed to list the structure of $\mathcal{G}(7)$. In fact, Steinbach listed only the first half of this structure, since the second half can be deduced by simple complementation calculations, yet his tabulation still required 11 pages. There were a few sporadic errors in his tabulations, which were corrected in [**1**].

We recently discussed [**2**] the poset structure of $\mathcal{G}(n)$ as far as $n = 8$. This extends Steinbach's work by a substantial order of magnitude, since there are 12346 graphs in $\mathcal{G}(8)$, and the total number of immediate predecessors and successors of each one averages just over 20. Consequently, our listing of the structure of $\mathcal{G}(8)$ runs to 421 pages. This is far too extensive to warrant hardcopy publication, so we have made it available on the website [**3**]:

`www.maths.uq.edu.au/~pa/research/posets4to8.html`

where we also include the corresponding tabulations for smaller orders. In [**2**] we confined ourselves to summary data and commentary.

In order to explicitly specify the structure of $\mathcal{G}(n)$ it is necessary to identify the graphs of order n. For his work, Steinbach assigned a number $N(G)$ to each graph G of order $n \leq 7$; we call this the *Steinbach number* of G. In their *Atlas of Graphs* [**5**], Reid and Wilson use a different numbering scheme. One advantage of Steinbach numbers over those of Reid and Wilson is that they reflect the structure of $\mathcal{G}(n)$ under complementation. If G^c is the complement of the graph

G, and $G^c \neq G$, the Steinbach numbers (with very few exceptions) satisfy the complementation rule

$$N(G) + N(G^c) = |\mathcal{G}(n)| + 1.$$

They also reflect other poset-related structural properties of the graphs, so are well-suited to describing the poset structure of $\mathcal{G}(n)$.

In **[2]** we introduced "SEAM numbering", a slight adaptation of Steinbach numbering which closely follows the spirit of Steinbach's rules, but extends them so as to provide in principle a unique objective numbering $N^*(G)$ for every finite graph G, no matter what the order. The basis for SEAM numbering is assignment of a unique signature $\Sigma(G)$ to every finite graph G, followed by sorting of graphs in $\mathcal{G}(n)$ based on signature in a way that preserves the complementation rule for all graphs that are not self-complementary. In many cases the SEAM number $N^*(G)$ and Steinbach number $N(G)$ coincide, and in all other cases they are very nearly equal. Our website **[3]** lists all graphs of order $n \leq 8$ by signature and SEAM number, with cross-referencing to Steinbach numbers for orders $n \leq 7$ (as far as Steinbach numbering is in publication).

We have now extended the earlier work to graphs of order 9, with a tabulation of all graphs in $\mathcal{G}(9)$ by signature and SEAM number, and a specification of the poset structure of $\mathcal{G}(9)$ by listing the immediate predecessors and immediate successors of each graph. These tabulations are available on the website **[4]**:

`www.maths.uq.edu.au/~pa/research/poset9.html`

There are 274668 graphs of order 9, so extending our previous work to $\mathcal{G}(9)$ represents yet another substantial increase in order of magnitude. Indeed, a hardcopy of our listing of the poset structure of $\mathcal{G}(9)$ would require some 14000 pages! As a practical solution to making it available, we have placed a text format version on the website **[4]**. Here we shall confine ourselves to some summary data and commentary, first regarding degree sequences of order 9 graphs, and then regarding the poset structure of $\mathcal{G}(9)$.

2 Degree sequences of graphs of order 9

Because the degree sequence of a graph is an important early part of its signature, it is relatively easy to use our tabulation of $\mathcal{G}(9)$, ordered by signature and SEAM number, to derive information about the degree sequences of the graphs in $\mathcal{G}(9)$, extending results for smaller orders in **[2]**.

If there are exactly r graphs in $\mathcal{G}(n)$ with the same degree sequence $\mathbf{d}$, we call r the *multiplicity* of $\mathbf{d}$. Let $f(r)$ denote the number of degree sequences for $\mathcal{G}(n)$ with multiplicity r. In **[2]** we published the sequences $1^{f(1)}2^{f(2)}\dots k^{f(k)}\dots$ for $4 \leq n \leq 8$, each being the *degree sequence multiplicity distribution* for the appropriate $\mathcal{G}(n)$. The corresponding data for $\mathcal{G}(9)$ is included on the website **[4]**. It is too extensive to give here, but the last few terms of the distribution are

$$\cdots\ 1831^2\ 2027^2\ 2218^2\ 2224^2\ 3020^1.$$

What is the average number of graphs of order n with the same degree sequence? The standard measures of central tendency (mean, median and mode) each give interesting information about the degree sequence multiplicity distribution for $\mathcal{G}(n)$, so we give all three in Table 1. (Means are given correct to one decimal place.)

Table 1: Average multiplicity of degree sequences for $\mathcal{G}(n)$

n	1	2	3	4	5	6	7	8	9
mean	1.0	1.0	1.0	1.0	1.1	1.5	3.1	10.2	63.0
median	1	1	1	1	1	1	2	3	6
mode	1	1	1	1	1	1	1	1	1

For example, degree sequences of order 9 have a mean of 63 realizations, whereas half the order 9 degree sequences have 6 or fewer realizations, and there are more order 9 degree sequences with a unique realization than there are with r realizations for any $r > 1$. It seems reasonable to conjecture that for degree sequences of any order $n \geq 1$ the modal multiplicity will be 1, while the median and mean multiplicities will both increase with n, the latter much more rapidly than the former.

The degree sequence multiplicity distribution for $\mathcal{G}(9)$ enables us to extend Theorem 1 of **[2]** as follows:

Theorem 1 *For each $\mathcal{G}(n)$ with $n \geq 1$, the number of distinct degree sequences is*

$$1, 2, 4, 11, 31, 102, 342, 1213, 4361, \cdots$$

The maximum number of graphs with the same degree sequence is

$$1, 1, 1, 1, 2, 5, 20, 184, 3020, \cdots$$

The sequences with maximum multiplicity for $5 \leq n \leq 9$ are

$$\begin{array}{rl} n = 5, f(2) = 3\text{:} & 1^2 2^3,\ 1^1 2^3 3^1,\ 2^3 3^2 \\ n = 6, f(5) = 2\text{:} & 1^2 2^2 3^2,\ 2^2 3^2 4^2 \\ n = 7, f(20) = 2\text{:} & 1^1 2^2 3^3 4^1,\ 2^1 3^3 4^2 5^1 \\ n = 8, f(184) = 3\text{:} & 2^2 3^3 4^2 5^1,\ 2^1 3^3 4^3 5^1,\ 2^1 3^2 4^3 5^2 \\ n = 9, f(3020) = 1\text{:} & 2^1 3^2 4^3 5^2 6^1. \end{array}$$

In particular, the numbers of distinct degree sequences confirm the terms for $n \leq 9$ in sequence A004251 of Sloane's *Encyclopedia of Integer Sequences* [**6**].

It is noteworthy that the degree sequences $1^1 2^3 3^1$, $2^1 3^3 4^3 5^1$ and $2^1 3^2 4^3 5^2 6^1$ appearing in Theorem 1 are all self-complementary. (The other eight sequences in Theorem 1 are complementary pairs.) Any self-complementary graph must have a self-complementary degree sequence, and it is well-known that $\mathcal{G}(n)$ contains self-complementary graphs precisely when $n \equiv 0$ or $1 \pmod 4$. Indeed, a degree sequence of order n can only be self-complementary if $n \equiv 0$ or $1 \pmod 4$, since only in such cases does K_n have an even number of edges. Note that $1^2 2^2$ is a self-complementary sequence of order 4, and all degree sequences of order 4 have multiplicity 1, so Theorem 1 leads us to conjecture that whenever there is a self-complementary degree sequence of order n, there is at least one such sequence that attains maximum multiplicity among all order n degree sequences.

These observations lead us to further examine self-complementary degree sequences. If G is a graph with a self-complementary degree sequence, G need not be self-complementary. Let $\mathbf{d}$ be any self-complementary degree sequence, then the *self-complementary multiplicity* r^* of $\mathbf{d}$ is the number of nonisomorphic self-complementary graphs with degree sequence $\mathbf{d}$. Table 2 lists all self-complementary degree sequences of order $n \leq 9$, their multiplicity r, and their self-complementary multiplicity r^*.

Table 2: Self-complementary degree sequences and their multiplicities

n	Degree sequence	r	r^*	n	Degree sequence	r	r^*
4	$1^2 2^2$	1	1	9	$1^1 2^1 4^5 6^1 7^1$	24	0
5	$1^2 2^1 3^2$	1	1		$1^1 3^3 4^1 5^3 7^1$	84	0
	$1^1 2^3 3^1$	2	0		$1^1 3^2 4^3 5^2 7^1$	220	0
	2^5	1	1		$1^1 3^1 4^5 5^1 7^1$	86	0
8	$1^2 3^2 4^2 6^2$	1	1		$1^1 4^7 7^1$	8	0
	$1^1 2^2 3^1 4^1 5^2 6^1$	8	0		$2^4 4^1 6^4$	2	2
	$1^1 2^1 3^2 4^2 5^1 6^1$	42	0		$2^3 3^1 4^1 5^1 6^3$	44	0
	$1^1 3^3 4^3 6^1$	22	0		$2^3 4^3 6^3$	24	0
	$2^4 5^4$	2	2		$2^2 3^2 4^1 5^2 6^2$	560	6
	$2^3 3^1 4^1 5^3$	20	0		$2^2 3^1 4^3 5^1 6^2$	708	0
	$2^2 3^2 4^2 5^2$	149	3		$2^2 4^5 6^2$	98	4
	$2^1 3^3 4^3 5^1$	184	0		$2^1 3^3 4^1 5^3 6^1$	1234	0
	$3^4 4^4$	50	4		$2^1 3^2 4^3 5^2 6^1$	3020	0
9	$1^2 3^2 4^1 5^2 7^2$	1	1		$2^1 3^1 4^5 5^1 6^1$	1086	0
	$1^2 3^1 4^3 5^1 7^2$	2	0		$2^1 4^7 6^1$	58	0
	$1^2 4^5 7^2$	1	1		$3^4 4^1 5^4$	405	9
	$1^1 2^2 3^1 4^1 5^1 6^2 7^1$	8	0		$3^3 4^3 5^3$	1524	0
	$1^1 2^2 4^3 6^2 7^1$	8	0		$3^2 4^5 5^2$	1147	9
	$1^1 2^1 3^2 4^1 5^2 6^1 7^1$	90	0		$3^1 4^7 5^1$	192	0
	$1^1 2^1 3^1 4^3 5^1 6^1 7^1$	144	0		4^9	16	4

From Table 2, we deduce

Theorem 2 *For $n \geq 1$, the number of graphs of order n with self-complementary degree sequences is*

$$1, 0, 0, 1, 4, 0, 0, 478, 10794, \cdots$$

These results show that a decreasingly small proportion of the graphs with self-complementary degree sequences are actually self-complementary graphs. For example, even though 10794 graphs in $\mathcal{G}(9)$ have self-complementary degree sequences, only 36 of these graphs are self-complementary. The *self-complementary multiplicity distribution* for $\mathcal{G}(n)$ is the sequence $0^{f^*(0)} 1^{f^*(1)} \cdots k^{f^*(k)} \cdots$, where $f^*(k)$ is the number of self-complementary degree sequences of order n with self-complementary multiplicity $r^* = k$, for $k \geq 0$. As usual, in practice we omit terms for which $f^*(k) = 0$. Evidently the number of self-complementary degree sequences for $\mathcal{G}(n)$ is $\Sigma_k f^*(k)$, and the number of self-complementary graphs in $\mathcal{G}(n)$ is $\Sigma_k k f^*(k)$. From Table 2, we have

Theorem 3 *For $n \geq 1$, the number of self-complementary graphs of order n is*

$$1, 0, 0, 1, 2, 0, 0, 10, 36, \cdots$$

and the self-complementary multiplicity distribution for $\mathcal{G}(n)$ is

$$\begin{aligned} n = 4&: \quad 1^1 \\ n = 5&: \quad 0^1 1^2 \\ n = 8&: \quad 0^5 1^1 2^1 3^1 4^1 \\ n = 9&: \quad 0^{19} 1^2 2^1 4^2 6^1 9^2. \end{aligned}$$

In particular, the numbers of self-complementary graphs confirm the terms for $n \leq 9$ in sequence A000171 of Sloane's listing [**6**]. The SEAM numbers of the self-complementary graphs of order $n \leq 9$ are

1:1, 4:6, 5:17..18, 8:6169..6178, 9:137317..137352.

A degree sequence $\mathbf{d}$ is *uniquely graphic* if it has multiplicity $r = 1$, that is, there is a unique graph with degree sequence $\mathbf{d}$. The *size distribution* of uniquely graphic degree sequences of order n is the sequence $0^{a(0)} 1^{a(1)} \cdots k^{a(k)} \cdots$, where $a(k)$ is the number of uniquely graphic degree sequences for graphs in the *level set* $\mathcal{G}(n, k)$, that is, the set of graphs with n vertices and k edges. Since any degree sequence and its complement have equal multiplicity, the size distribution of uniquely graphic degree sequences must be centrally symmetric. We give these distributions for $n \leq 9$ in Table 3, truncating the longer sequences midway.

Table 3: Size distribution of uniquely graphic degree sequences

n	Size distribution
1	0^1
2	$0^1 1^1$
3	$0^1 1^1 2^1 3^1$
4	$0^1 1^1 2^2 3^3 4^2 5^1 6^1$
5	$0^1 1^1 2^2 3^4 4^4 5^4 \cdots$
6	$0^1 1^1 2^2 3^5 4^5 5^7 6^7 7^8 8^8 \cdots$
7	$0^1 1^1 2^2 3^5 4^6 5^8 6^{10} 7^{12} 8^{11} 9^{15} 10^{14} 11^{14} \cdots$
8	$0^1 1^1 2^2 3^5 4^7 5^9 6^{11} 7^{16} 8^{16} 9^{20} 10^{25} 11^{22} 12^{27} 13^{27} 14^{29} \cdots$
9	$0^1 1^1 2^2 3^5 4^7 5^{10} 6^{12} 7^{17} 8^{20} 9^{27} 10^{29} 11^{34} 12^{38} 13^{41} 14^{47} 15^{54} 16^{54} 17^{49} 18^{60} \cdots$

While one expects degree sequences of very small or very large size to have unique realizations, it is of interest to note that there are actually uniquely graphic degree sequences of every size. Indeed, at each $n \geq 1$ with $n \equiv 0$ or 1 (mod 4) there are self-complementary degree sequences which are uniquely graphic: clearly the unique realizations of such sequences are among the self-complementary graphs. Table 3 also yields the following global information:

Theorem 4 *For $n \geq 1$, the number of uniquely graphic degree sequences of order n is*

$$1, 2, 4, 11, 28, 72, 170, 407, 956, \cdots$$

Note that the size distribution of uniquely graphic degree sequences of order 9 is not unimodal. The maximum number of such degree sequences of a given size is 60, and does occur at the central size of 18, but there are 54 such degree sequences at each of the sizes 15 and 16, while at the intermediate size 17 there are slightly fewer, namely 49. The corresponding phenomenon is already apparent at orders 7 and 8.

3 Poset structure of graphs of order 9

The poset structure of $\mathcal{G}(9)$ is conveniently specified by the *Hasse diagram* $\mathcal{HG}(9)$, a digraph with $\mathcal{G}(9)$ as its vertex set, and a directed edge $G \to H$ whenever H is an immediate successor of G (equivalently, whenever G is an immediate predecessor of H), so $H = G + e$ for some edge e. On the website **[4]** we describe $\mathcal{HG}(9)$ by listing, for each $G \in \mathcal{G}(9)$, all the immediate predecessors and immediate successors of G.

For any $G \in \mathcal{G}(9)$, the *outdegree* $d^+(G)$ in $\mathcal{HG}(9)$ is the number of immediate successors of G, the *indegree* $d^-(G)$ is the number of immediate predecessors of G, and $d(G) = d^+(G) + d^-(G)$ is the *full degree*. If $n^+(k, m)$ is the number of graphs $G \in \mathcal{G}(9, m)$ of order 9 and size m with outdegree $d^+ = k$, then the sequence $0^{n^+(0,m)} 1^{n^+(1,m)} \dots k^{n^+(k,m)} \dots$ is the *outdegree sequence for level m* of $\mathcal{HG}(9)$, and if $n^+(k) = \Sigma_m n^+(k, m)$ is the total number of graphs in $\mathcal{HG}(9)$ with outdegree k, then $0^{n^+(0)} 1^{n^+(1)} \dots k^{n^+(k)} \dots$ is the *outdegree sequence* for $\mathcal{HG}(9)$. Similarly we define indegree sequences and full degree sequences for each level of $\mathcal{HG}(9)$, and for the whole digraph.

Let G^c denote the complement of any graph $G \in \mathcal{G}(9)$. If $G \to H$ is a directed edge in $\mathcal{HG}(9)$, so is $H^c \to G^c$. Hence $d^+(G) = d^-(G^c)$ and $d(G) = d(G^c)$. Thus if $m + m' = 36$ then the outdegree sequence for level m of $\mathcal{HG}(9)$ is equal to the indegree sequence for level m', and the two levels have equal full degree sequences. Consequently, on the website **[4]** we list the outdegree sequences, indegree sequences and full degree sequences for each level $m \leq 18$, and for the whole digraph. Table 4 below gives summary data for the degree sequences for the levels of $\mathcal{HG}(9)$, and for the whole digraph.

For example, the outdegree information in line $m = 8$ of Table 4 (the level which contains the trees of order 9) shows that there are 4803 edges in $\mathcal{HG}(9)$ between levels 8 and 9, and the graphs in level 8 have between 1 and 28 immediate successors (1-extensions) in level 9. The indegree information in line $m = 8$ shows that there are 1767 edges in $\mathcal{HG}(9)$ between levels 7 and 8, and the graphs in level 8 have between 1 and 8 immediate predecessors (1-reductions) in level 7. The full degree information in line $m = 8$ shows that there are 6570 edges in $\mathcal{HG}(9)$ with one end in level 8, and the graphs in level 8 have between 2 and 36 neighbours in $\mathcal{HG}(9)$. Clearly, the total outdegree of any level m is equal to the total indegree of level $m+1$. Moreover, the total indegree of level 18 is equal to the total outdegree of that level because if G is any order 9 graph of size 18, its complement G^c is also an order 9 graph of size 18, and $d^-(G) = d^+(G^c)$. The summary line at the foot of Table 4 refers to $\mathcal{HG}(9)$ as a whole.

We can now extend Theorem 2 of **[2]** to include data for $\mathcal{HG}(9)$.

Theorem 5 *For $n \geq 1$, the order of the Hasse diagram $\mathcal{HG}(n)$ for graphs of order n is*

$$1, 2, 4, 11, 34, 156, 1044, 12346, 274668, \cdots$$

and the size of $\mathcal{HG}(n)$ is

$$0, 1, 3, 14, 74, 571, 6558, 125066, 4147388, \cdots$$

If G has indegree 1 in $\mathcal{HG}(9)$, then G is edge-transitive. Since the empty graph K_9^c is trivially edge-transitive, it follows from Table

4 that there is at least one edge-transitive graph of order 9 and size m for every $m < 22$ except for $m \in \{11, 13, 17, 19\}$, and also for $m \in \{24, 27, 28, 36\}$. More explicitly, let $t(m)$ be the number of edge-transitive graphs of order n and size m, and let $0^{t(0)}1^{t(1)} \ldots m^{t(m)} \ldots$ be the *size distribution sequence* for edge-transitive graphs of order n. Table 5 gives these size distribution sequences for $n \leq 9$.

Table 4: Degree sequences for $\mathcal{HG}(9)$

Size	Outdegree			Indegree			Full degree		
m	min	max	total	min	max	total	min	max	total
0	1	1	1	0	0	0	1	1	1
1	2	2	2	1	1	1	3	3	3
2	3	4	7	1	1	2	4	5	9
3	2	7	21	1	2	7	3	9	28
4	2	9	65	1	3	21	3	12	86
5	3	14	200	1	4	65	4	18	265
6	2	23	612	1	6	200	3	29	812
7	3	28	1767	1	7	612	4	35	2379
8	1	28	4803	1	8	1767	2	36	6570
9	1	27	12065	1	9	4803	2	36	16868
10	2	26	27713	1	10	12065	3	36	39778
11	2	25	57770	2	11	27713	4	36	85483
12	1	24	108764	1	12	57770	3	36	166534
13	3	23	184424	2	13	108764	5	36	293188
14	2	22	281454	1	14	184424	3	36	465878
15	1	21	386410	1	15	281454	3	36	667864
16	1	20	477240	1	16	386410	3	36	863650
17	3	19	530376	2	17	477240	5	36	1007616
18	1	18	530376	1	18	530376	2	36	1060752
Whole	0	28	4147388	0	28	4147388	1	36	8294776

Table 5: Size distribution for edge-transitive graphs in $\mathcal{G}(n)$

n	Size distribution sequence
1	0^1
2	$0^1 1^1$
3	$0^1 1^1 2^1 3^1$
4	$0^1\ 1^1\ 2^2\ 3^2\ 4^1\ 6^1$
5	$0^1\ 1^1\ 2^2\ 3^2\ 4^2\ 5^1\ 6^2\ 10^1$
6	$0^1\ 1^1\ 2^2\ 3^3\ 4^3\ 5^2\ 6^4\ 8^1\ 9^1\ 10^1\ 12^1\ 15^1$
7	$0^1\ 1^1\ 2^2\ 3^3\ 4^3\ 5^2\ 6^5\ 7^1\ 8^1\ 9^1\ 10^2\ 12^2\ 15^1\ 21^1$
8	$0^1\ 1^1\ 2^2\ 3^3\ 4^4\ 5^2\ 6^6\ 7^2\ 8^3\ 9^1\ 10^2\ 12^5\ 15^2\ 16^1\ 21^1\ 24^1\ 28^1$
9	$0^1\ 1^1\ 2^2\ 3^3\ 4^4\ 5^2\ 6^7\ 7^2\ 8^4\ 9^3\ 10^2\ 12^6\ 14^1\ 15^2\ 16^1\ 18^2\ 20^1\ 21^1\ 24^1\ 27^1\ 28^1\ 36^1$

Table 5 yields the following extension of Theorem 3 of [**2**].

Theorem 6 *For $n \geq 1$, the number of edge-transitive graphs of order n is*

$$1, 2, 4, 8, 12, 21, 26, 38, 49, \cdots$$

If every automorphism of G fixes every edge, we say that G has *no edge-symmetry*. (Such a graph can have non-identity automorphisms, but every automorphism must fix every vertex in any component of order greater than 2, and there can be at most one component of order 2.) If $G \in \mathcal{G}(9, m)$, it has no edge-symmetry if its indegree in $\mathcal{HG}(9)$ is m. It follows from Table 4 that there is at least one graph of order 9 and size m with no edge-symmetry if $m \leq 1$ or $6 \leq m \leq 28$. More explicitly, let $u(m)$ be the number of graphs in $\mathcal{G}(n, m)$ with no edge-symmetry and let $0^{u(0)} 1^{u(1)} \cdots m^{u(m)} \cdots$ be the *size distribution sequence* for graphs of order n with no edge-symmetry. As usual, in practice we omit terms with zero exponent. Table 6 lists these sequences for $n \leq 9$.

Table 6: Size distribution for graphs in $\mathcal{G}(n)$ with no edge-symmetry

n	Size distribution sequence
1	0^1
2	$0^1 1^1$
3	$0^1 1^1$
4	$0^1\ 1^1$
5	$0^1\ 1^1$
6	$0^1\ 1^1\ 6^1\ 7^3\ 8^2\ 9^1$
7	$0^1\ 1^1\ 6^2\ 7^6\ 8^{12}\ 9^{24}\ 10^{30}\ 11^{28}\ 12^{24}\ 13^{14}\ 14^4\ 15^2$
8	$0^1\ 1^1\ 6^2\ 7^8\ 8^{24}\ 9^{71}\ 10^{160}\ 11^{285}\ 12^{433}\ 13^{559}\ 14^{604}\ 15^{556}\ 16^{434}\ 17^{285}\ 18^{157}$ $19^{69}\ 20^{19}\ 21^2\ 22^1$
9	$0^1\ 1^1\ 6^2\ 7^9\ 8^{30}\ 9^{110}\ 10^{344}\ 11^{900}\ 12^{2074}\ 13^{4140}\ 14^{7182}\ 15^{10986}\ 16^{14816}\ 17^{17677}$ $18^{18764}\ 19^{17694}\ 20^{14794}\ 21^{10979}\ 22^{7171}\ 23^{4088}\ 24^{2021}\ 25^{841}\ 26^{286}\ 27^{75}\ 28^{11}$

Table 6 yields the following companion result to Theorem 6:

Theorem 7 *For $n \geq 1$, the number of order n graphs with no edge-symmetry is*

$$1, 2, 2, 2, 2, 9, 148, 3671, 134996, \cdots$$

In [**2**] we defined the *productivity* of a graph $G \in \mathcal{G}(n)$ as the number of non-isomorphic 1-extensions of G; this is the same as the outdegree of G in $\mathcal{HG}(n)$, since the 1-extensions are precisely the immediate successors of G in the poset. Table 7 extends the productivity information in Table 5 of [**2**].

Table 7: Productivities of order n graphs

n	Outdegree sequence of $\mathcal{HG}(n)$
1	0^1
2	$0^1 1^1$
3	$0^1 1^3$
4	$0^1\, 1^7\, 2^2\, 3^1$
5	$0^1\, 1^{11}\, 2^9\, 3^7\, 4^6$
6	$0^1\, 1^{20}\, 2^{24}\, 3^{33}\, 4^{33}\, 5^{19}\, 6^{11}\, 7^{10}\, 8^4\, 9^1$
7	$0^1\, 1^{25}\, 2^{54}\, 3^{92}\, 4^{133}\, 5^{140}\, 6^{139}\, 7^{130}\, 8^{107}\, 9^{78}\, 10^{58}\, 11^{39}\, 12^{26} 13^{16} 14^4 15^2$
8	$0^1\, 1^{37}\, 2^{110}\, 3^{235}\, 4^{428}\, 5^{600}\, 6^{798}\, 7^{997}\, 8^{1135}\, 9^{1196}\, 10^{1176}\, 11^{1124}$ $12^{1051}\, 13^{967}\, 14^{826}\, 15^{652}\, 16^{467}\, 17^{293}\, 18^{158}\, 19^{71}\, 20^{21}\, 21^2\, 22^1$
9	$0^1\, 1^{48}\, 2^{190}\, 3^{495}\, 4^{1103}\, 5^{1975}\, 6^{3307}\, 7^{5100}\, 8^{7347}\, 9^{9852}\, 10^{12461}\, 11^{14991}\, 12^{17411}$ $13^{19809}\, 14^{22017}\, 15^{23875}\, 16^{25010}\, 17^{24816}\, 18^{23124}\, 19^{19994}\, 20^{15792}\, 21^{11321}$ $22^{7270}\, 23^{4108}\, 24^{2028}\, 25^{847}\, 26^{287}\, 27^{77}\, 28^{12}$.

Hence we can extend Theorem 3 of [**2**] as follows:

Theorem 8 *For $n \geq 1$, the number of maximally productive graphs in $\mathcal{G}(n)$, with their productivity, is*

$$1{:}0,\ 1{:}1,\ 3{:}1,\ 1{:}3,\ 6{:}4,\ 1{:}9,\ 2{:}15,\ 1{:}22,\ 12{:}28,\ \cdots$$

We noted in Theorem 4 of [**2**] that the productivity sequences for each $\mathcal{G}(n)$ with $n \leq 8$ have indices which are unimodal. This extends:

Theorem 9 *Each $\mathcal{G}(n)$ with $n \leq 9$ has an index unimodal productivity sequence. The peak supports and index peaks are*

$$0{:}1,\ 1{:}1,\ 1{:}3,\ 1{:}7,\ 1{:}11,\ \{3,4\}{:}33,\ 5{:}140,\ 9{:}1196,\ 16{:}25010.$$

The total outdegree of level m in $\mathcal{HG}(n)$ corresponds in [**2**] to what we called the *productivity* of level m in $\mathcal{G}(n)$. We noted in Theorem 4 of [**2**] that the productivities of the levels of $\mathcal{G}(n)$ form a unimodal sequence for each $n \leq 8$. Table 4 shows that this continues for $n = 9$:

Theorem 10 *The productivities of the levels of $\mathcal{G}(9)$ form a unimodal sequence, with peak value* 530376 *at levels* 17 *and* 18*.*

This completes our descriptive summary of properties of the poset $\mathcal{G}(9)$. Much more can be deduced from the tables and data on our website [**4**].

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