Structure of Graph Posets for Orders 4 to 8

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Abstract

The poset $\mathcal{G}(n)$ comprises the unlabelled simple graphs of order n, with partial ordering $G \leq H$ whenever G is a spanning subgraph of H. We define a modified Steinbach numbering of the graphs in $\mathcal{G}(n)$, apply this numbering to each $\mathcal{G}(n)$ with $n \leq 8$, and use it to tabulate the Hasse diagram structure of the posets with $4 \leq n \leq 8$ together with key aspects of the independence structure of these posets. In particular, the Hasse diagram of $\mathcal{G}(8)$ is a directed graph of order 12346 and size 125066; the poset $\mathcal{G}(8)$ has 51952895 independent pairs of graphs, and 96775426396 independent triples. We present 14 tables of descriptive data for $\mathcal{G}(n)$ with $4 \leq n \leq 8$. All of the underlying data can be found on our webpage

`www.maths.uq.edu.au/~pa/research/posets4to8.html`

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1 Introduction

Let $\mathcal{G}(n)$ be the set of unlabelled simple graphs of order n, or "the set of graphs of order n" for simplicity. The set $\mathcal{G}(n)$ is a partially ordered set (poset) under the relation $G \leq H$, defined whenever $G, H \in \mathcal{G}(n)$ and G is isomorphic to a subgraph of H. For convenience, we shall regard all graphs in $\mathcal{G}(n)$ as having the same vertex set, and so interpret $G \leq H$ as meaning that G is a spanning subgraph of H. Experience shows that using the set notation $\mathcal{G}(n)$ to also denote the poset does not cause confusion, so we shall follow that practice. Since the graphs of small order are the fundamental structures of graph theory, their structural relationships are of great interest. Motivated by this fact, our objective here is to present precise computational data on various structural characteristics of the posets $\mathcal{G}(n)$ for small values of n.

The posets $\mathcal{G}(1)$, $\mathcal{G}(2)$ and $\mathcal{G}(3)$ have orders 1, 2 and 4 respectively, and each is linearly ordered, so their structure is completely transparent. The first member of the family that is not linearly ordered is the poset $\mathcal{G}(4)$, of order 11. Subsequent members have orders

$$|\mathcal{G}(5)| = 34, |\mathcal{G}(6)| = 156, |\mathcal{G}(7)| = 1044, |\mathcal{G}(8)| = 12346, \cdots.$$

Indeed, the orders $|\mathcal{G}(n)|$ for $n \geq 1$ are the terms of sequence A000088 in Sloane's *Encyclopedia of Integer Sequences* **[5]**. The structural complexity of $\mathcal{G}(n)$ grows correspondingly rapidly, and properties which are easily determined by inspection for $\mathcal{G}(4)$ may be computationally challenging to determine for $\mathcal{G}(n)$ when $n = 8$, or even sooner.

In each poset $\mathcal{G}(n)$, the complete graph K_n is the unique maximal element. The complement G^c of any graph $G \in \mathcal{G}(n)$ is the graph $G^c := K_n - E(G)$, where $E(G)$ is the edge set of G. In particular, the empty graph K_n^c is the unique minimal element of $\mathcal{G}(n)$. If $G, H \in \mathcal{G}(n)$ and $G \leq H$, then $H^c \leq G^c$, so the complementation map $c : \mathcal{G}(n) \to \mathcal{G}(n)$, mapping each graph G to its complement G^c, is an anti-automorphism of the poset $\mathcal{G}(n)$. Thus $\mathcal{G}(n)$ has a type of central symmetry.

2 Numbering the graphs of order n

To discuss the poset structure of $\mathcal{G}(n)$ we must first have a method of specifying the graphs of order n. In his *Field Guide to Simple Graphs* [**6**], Steinbach illustrates every graph of order $n \leq 7$ (a total of 1252 graphs), and to each such graph G he assigns a number $N(G)$, which we call the *Steinbach number* of G. With four exceptions, Steinbach numbers satisfy the complementation rule

$$N(G) + N(G^c) = |\mathcal{G}(n)| + 1,$$

so a graph and its complement occupy complementary positions within the listing in [**6**]. Because Steinbach numbers reflect the anti-automorphism of $\mathcal{G}(n)$ produced by complementation, they are well suited to our purposes. (In their *Atlas of Graphs* [**4**], Read and Wilson give a different numbering system and set of illustrations for the members of $\mathcal{G}(n)$ with $n \leq 7$, but their numbering has no simple relationship with complementation, so is not as well suited to poset description.)

For each $n \leq 7$, Steinbach's numbering of the graphs in $\mathcal{G}(n)$ first sorts them by increasing size (number of edges). Once a graph G in the "first half" of $\mathcal{G}(n)$ has been assigned a Steinbach number, if $G^c \neq G$ then G^c lies in the "second half" of $\mathcal{G}(n)$, and the Steinbach number of G^c follows from the complementation rule. Let $\mathcal{G}(n,m)$ denote the *level set* comprising all graphs of size m in $\mathcal{G}(n)$. If $m < n(n-1)/4$, then all graphs in $\mathcal{G}(n,m)$ are clearly in the "first half" of $\mathcal{G}(n)$ and none is self-complementary. The Steinbach numbering of graphs in $\mathcal{G}(n,m)$ sorts them by lexicographic order of increasing degree sequence, and for those with the same degree sequence, lists the disconnected graphs before the connected graphs. (Steinbach's final listing within these classes is rather subjective.) When $n = 4$ or 5 and $m = n(n-1)/4$, the middle level set $\mathcal{G}(n,m)$ is split between the "first half" and the "second half" of $\mathcal{G}(n)$, and the self-complementary graphs present an added complication. It seems natural to place the self-complementary graphs in the "middle" of $\mathcal{G}(n,m)$, and to place the other pairs of graphs G, G^c (with $G^c \neq G$) symmetrically about the middle, assigning G to the "first half" if its increasing degree sequence lexicographically precedes that of G^c and making some "tie-breaking" assignment if G and G^c have the same degree sequence, then sorting the "first half" of $\mathcal{G}(n,m)$ by lexicographic order of increasing degree sequence and ordering the "second half" of $\mathcal{G}(n,m)$ in accordance with the complementation rule. This procedure yields Steinbach's numbering of $\mathcal{G}(4,3)$. However for $\mathcal{G}(5,5)$ Steinbach uses lexicographic order of increasing degree sequence before placing the self-complementary graphs in the "middle", so the two self-complementary graphs end up being separated by one graph which is not self-complementary (though it does have the same degree sequence as its complement). Thus Steinbach's numbering of $\mathcal{G}(5,5)$ has four exceptions to the complementation rule, whereas the alternative procedure would have had the two self-complementary graphs as its only exceptions.

We now propose a modified Steinbach numbering system for any $\mathcal{G}(n)$, in the spirit of Steinbach's system but eliminating quirks arising from the subjectivity of its "fine structure". Our modified numbering system ("*SEAM numbering*") slightly adjusts Steinbach's rules and extends them so that it objectively assigns a number $N^*(G)$ to each graph $G \in \mathcal{G}(n)$ and preserves the complementation rule for every graph that is not self-complementary.

Given $\mathcal{G}(n)$, the SEAM numbering of the graphs in $\mathcal{G}(n)$ is a bijection $N^* : \mathcal{G}(n) \rightarrow [1..|\mathcal{G}(n)|]$ defined implicitly by the following rules. The "first half" $\mathcal{F}$, the "middle" $\mathcal{M}$ and the "second half" $\mathcal{S}$ of $\mathcal{G}(n)$ are disjoint subsets with union equal to $\mathcal{G}(n)$, such that if $G \in \mathcal{M}$ then $G^c = G$, and $G \in \mathcal{F}$ if and only if $G^c \in \mathcal{S}$. Note that $\mathcal{M}$ can be empty, so technically this may not be a partition of $\mathcal{G}(n)$; also note that if $\mathcal{M}$ is nonempty then $\mathcal{F}$ and $\mathcal{S}$ are not strictly halves of $\mathcal{G}(n)$.

With each graph $G \in \mathcal{G}(n)$ associate the *signature*, defined to be the sequence

$$\begin{aligned}\Sigma(G) :=\ & m; 0^{d(0)}, 1^{d(1)}, \cdots, (n-1)^{d(n-1)}; n-c; \\ & 1^{c(1)}, 2^{c(2)}, \cdots, n^{c(n)}; g, f; e(1), e(2), \cdots, e(m)\end{aligned}$$

where

(a) m is the size of G;

(b) $d(r)$ is the number of vertices of degree r, so the increasing degree sequence for G is $0^{d(0)}, 1^{d(1)}, \cdots, (n-1)^{d(n-1)}$ where the indices denote multiplicities (terms with 0 multiplicity are omitted in practice);

(c) c is the number of components of G, so $n-c$ is the size of any spanning forest in G;

(d) $c(r)$ is the number of components of order r, so the increasing component order sequence for G is $1^{c(1)}, 2^{c(2)}, \cdots, n^{c(n)}$ where again the indices denote multiplicities (terms with 0 multiplicity are omitted in practice);

(e) g is the girth (order of the smallest cycle) of G, and f is the frequency (number) of cycles of order g in G (if G is acyclic then $g = f = 0$);

(f) $e(1), e(2), \cdots, e(m)$ is the canonical edge sequence of G, namely, the lexicographically earliest sequence specifying the edge set of G under all possible bijections from the vertex set of G to $[1..n]$, and we write ij to denote an edge between vertices labelled i and j, with $i < j$.

The signatures determine which graphs belong to $\mathcal{F}$ and which belong to $\mathcal{S}$. If $G \in \mathcal{G}(n)\backslash\mathcal{M}$ then $G \in \mathcal{F}$ precisely when the signature $\Sigma(G)$ is lexicographically earlier than the signature $\Sigma(G^c)$. If $G \in \mathcal{F}$ then $N^*(G) < N^*(G^c)$ and the complementation rule

$$N^*(G) + N^*(G^c) = |\mathcal{G}(n)| + 1$$

holds. If $G \in \mathcal{F}$ and $H \in \mathcal{M}$ then $N^*(G) < N^*(H)$. Finally, if $G, H \in \mathcal{F}$ or $G, H \in \mathcal{M}$ then $N^*(G) < N^*(H)$ precisely when the signature $\Sigma(G)$ is lexicographically earlier than the signature $\Sigma(H)$.

To clarify these rules, let us note how lexicographic order applies to sequences such as those of the form $0^{d(0)}, 1^{d(1)}, \cdots, (n-1)^{d(n-1)}$. The indices denote multiplicities so this represents a sequence in which the first $d(0)$ terms are 0, then the next $d(1)$ terms are 1, and so on. Let $a(0), a(1), \cdots, a(n-1)$ and $b(0), b(1), \cdots, b(n-1)$ be two given sequences of non-negative integers. Then the sequence $0^{a(0)}, 1^{a(1)}, \cdots, (n-1)^{a(n-1)}$ lexicographically precedes the sequence $0^{b(0)}, 1^{b(1)}, \cdots, (n-1)^{b(n-1)}$ precisely when there is some non-negative integer $k \leq n-1$ such that $a(k) > b(k)$ and $a(r) = b(r)$ for every non-negative integer $r < k$.

Note that we use $n-c$ rather than c in the signature of any graph since, following Steinbach, we want the disconnected realizations of a given degree sequence to precede its connected realizations in the SEAM listing.

Clearly SEAM numbering is definitive, since every graph has a unique canonical edge sequence and non-isomorphic graphs cannot have the same edge sequence. In fact, we could use just the canonical edge sequence of each graph to linearly order $\mathcal{G}(n)$. However, the resultant ordering would be not nearly as "natural" as the SEAM ordering. For example, under canonical edge sequence ordering, when $n \geq 4$ the complete graph K_n would precede the path P_n, which in turn would precede any linear forest of two or more components.

Under the rules for SEAM numbering of $\mathcal{G}(n)$, self-complementary graphs of order n (if there are any) are ordered by signature, and assigned middle SEAM numbers. The graphs which are not self-complementary are assigned SEAM numbers which satisfy the complementation rule, and are arranged so that each complementary pair is ordered by signature, and all graphs with SEAM numbers preceding those of their complements are ordered by signature.

As a service to fellow graph-theorists, on the web page **[3]** we tabulate the signature $\Sigma(G)$ and the SEAM number $N^*(G)$ of each graph $G \in \mathcal{G}(n)$ for $4 \leq n \leq 8$, cross-referenced with the Steinbach number $N(G)$ when $4 \leq n \leq 7$ (which is as far as Steinbach's tables go). In $\mathcal{G}(5)$ only three graphs have Steinbach number different from their SEAM number, while there are 20 displacements in $\mathcal{G}(6)$ and 563 in $\mathcal{G}(7)$. The largest displacement is a shift by two places in $\mathcal{G}(5)$, a shift by one place in $\mathcal{G}(6)$ and a shift by 15 places in $\mathcal{G}(7)$.

Is the full signature ever needed to find the relative ranking of two graphs in order to assign their SEAM numbers? Certainly! Let $A, B \in \mathcal{G}(6)$ be the two trees with degree sequence $1^3 2^2 3^1$. The signatures $\Sigma(A)$ and $\Sigma(B)$ differ only at the very last edge in their canonical edge sequences, so the full signatures are needed to decide the SEAM numbers $N^*(A)$ and $N^*(B)$. (These SEAM numbers turn out to be 6:29 and 6:30. Note that for definiteness we can use the order as prefix, but in contexts where the order is understood we usually omit the prefix.) In fact A, B is the first pair of graphs requiring the full signature for SEAM discrimination, and it is easy to see that such pairs exist for every $n \geq 6$.

From our signature tabulations for all graphs in $\mathcal{G}(n)$ with $4 \leq n \leq 8$ we can deduce in particular many facts about degree sequences. If there are exactly r graphs in $\mathcal{G}(n)$ with the same degree sequence, we say that r is the *multiplicity* of the degree sequence they have in common. Let $f(r)$ denote the number of degree sequences for $\mathcal{G}(n)$ that have multiplicity r. Then

$$1^{f(1)}, 2^{f(2)}, \cdots, r^{f(r)}, \cdots$$

is the *degree sequence multiplicity distribution* for $\mathcal{G}(n)$. Table 1 gives the number of distinct degree sequences and their multiplicity distributions for $4 \leq n \leq 8$. For each order n, the column "First max" specifies the first degree sequence that attains maximum multiplicity. Our tabulations independently confirm the listing of degree sequences of order $n \leq 8$, with multiplicities, given in **[4]**. (Many features not apparent in that format are highlighted by the frequency distributions in Table 1.) The column "Total" indicates the number of distinct graphic degree sequences of order n. This confirms the terms with $n \leq 8$ in sequence A004251 of Sloane's listing **[5]**.

Table 1: Degree sequence multiplicity distribution for $\mathcal{G}(n)$

n	Total	Multiplicity distribution	First max
4	11	1^{11}	0^4
5	31	$1^{28}\ 2^3$	$1^2 2^3$
6	102	$1^{72}\ 2^{16}\ 3^6\ 4^6\ 5^2$	$1^2 2^2 3^2$
7	342	$1^{170}\ 2^{56}\ 3^{28}\ 4^{26}\ 5^{16}\ 6^6\ 7^{10}\ 8^4\ 9^2\ 11^8\ 12^4\ 13^2$	
		$14^2\ 17^2\ 18^2\ 19^2\ 20^2$	$1^1 2^2 3^3 4^1$
8	1213	$1^{407}\ 2^{165}\ 3^{86}\ 4^{88}\ 5^{60}\ 6^{30}\ 7^{40}\ 8^{21}\ 9^{24}\ 10^8\ 11^{28}\ 12^{22}\ 13^{16}$	
		$14^{16}\ 15^8\ 16^4\ 17^6\ 18^8\ 19^{10}\ 20^{13}\ 21^2\ 22^5\ 23^6\ 24^4\ 25^4$	
		$26^4\ 27^8\ 28^8\ 29^{12}\ 30^4\ 31^6\ 35^2\ 36^2\ 37^2\ 39^4\ 40^6\ 41^2$	
		$42^5\ 43^6\ 45^2\ 46^2\ 50^1\ 51^2\ 56^2\ 57^2\ 58^2\ 60^2\ 61^2\ 64^2$	
		$65^2\ 66^2\ 67^2\ 69^2\ 71^4\ 75^2\ 79^4\ 86^2\ 87^2\ 94^4\ 96^2\ 99^2$	
		$109^2\ 110^2\ 115^2\ 117^2\ 149^1\ 184^3$	$2^2 3^3 4^2 5^1$

From Table 1 we have

Theorem 1 *For each $\mathcal{G}(n)$ with $n \geq 1$, the maximum number of graphs with the same degree sequence is*

$$1, 1, 1, 1, 2, 5, 20, 184, \cdots$$

and the number of distinct degree sequences is

$$1, 2, 4, 11, 31, 102, 342, 1213, \cdots$$

For $n \leq 6$, Table 1 shows that the vast majority of degree sequences of order n have unique realizations, but by order 8 the majority of sequences have at least 3 realizations. At order 7 the fine structure of the multiplicity distribution begins to become apparent, and at order 8 it is both surprising and striking how irregularly the multiplicities are distributed. For $\mathcal{G}(8)$ the maximum multiplicity of 184 is attained by the complementary sequences $2^2 3^3 4^2 5^1$ and $2^1 3^2 4^3 5^2$, and the self-complementary sequence $2^1 3^3 4^3 5^1$, while the second highest multiplicity of 149 is attained only by the self-complementary sequence $2^2 3^2 4^2 5^2$. None of the 184 realizations of $2^1 3^3 4^3 5^1$ is self-complementary, and exactly three of the 149 realizations of $2^2 3^2 4^2 5^2$ are self-complementary.

Note that a degree sequence and its complement always have the same multiplicity, so for any order n the number $f(r)$ is odd just when there is an odd number of self-complementary degree sequences with multiplicity r. When $n \equiv 2$ or $3 \pmod 4$ no degree sequence of order n can be self-complementary (since the size of K_n is odd), so in these cases $f(r)$ is even for every r. When $n \equiv 0$ or $1 \pmod 4$ there always exist self-complementary graphs of order n, so there certainly are self-complementary degree sequences of order n. In fact, for such n it appears that there are increasingly many self-complementary degree sequences which have no self-complementary realizations. We note that $f(r)$ is odd if $n = 4$ and $r = 1$, if $n = 5$ and $r = 2$, and if $n = 8$ and $r \in \{1, 2, 8, 20, 22, 42, 50, 149, 184\}$. For $n \equiv 0$ or 1 (mod 4), we conjecture that there is always an r for which $f(r)$ is odd.

3 Hasse diagrams

Suppose $G, H \in \mathcal{G}(n)$. If $G \leq H$, then $G + E = H$ for some set of edges E. If $|E| = r$, we call H an *r-extension* of G, and G an *r-reduction* of H. In particular, the *Hasse diagram* of $\mathcal{G}(n)$ is a digraph $\mathcal{HG}(n)$ in which $\mathcal{G}(n)$ is the vertex set, and $G \to H$ is a directed edge precisely when H is a 1-extension of G. Evidently $\mathcal{HG}(n)$ fully represents the poset $\mathcal{G}(n)$, since there is a directed path from G to H in $\mathcal{HG}(n)$ if and only if $G \leq H$ in $\mathcal{G}(n)$. The Hasse diagram $\mathcal{HG}(n)$ has the empty graph K_n^c as its unique source vertex and the complete graph K_n as its unique sink vertex. The vertices at distance m from K_n^c in $\mathcal{HG}(n)$ comprise the level set $\mathcal{G}(n, m)$. The complementation map is a digraph anti-automorphism of $\mathcal{HG}(n)$.

Steinbach [**6**] specified the structure of $\mathcal{HG}(n)$ for $n \leq 7$ by listing the in-neighbours (that is, the immediate predecessors or 1-reductions) and the out-neighbours (that is, the immediate successors or 1-extensions) of each $G \in \mathcal{G}(n)$ for $n \leq 6$, and of each G in the "first half" of $\mathcal{G}(7)$, that is, each $G \in \mathcal{G}(7)$ with $N(G) \leq 522$, the structure of the "second half" of $\mathcal{G}(7)$ being implied by the complementation rule. Steinbach's tables contained some sporadic errors, all of which were corrected in our paper [**1**]. Full tables of the in-neighbours and out-neighbours of each $G \in \mathcal{G}(n)$ for $n \leq 7$, utilizing Steinbach numbers, are given on the webpage for that paper.

Our new webpage [**3**] includes tables of the in-neighbours and out-neighbours of each $G \in \mathcal{G}(n)$ for $n \leq 8$, utilizing SEAM numbers. The inclusion of $\mathcal{HG}(8)$ takes us significantly beyond earlier tabulations, since $\mathcal{G}(8)$ contains almost ten times as many graphs as belong to all $\mathcal{HG}(n)$ with $n \leq 7$. Clearly the table for $\mathcal{HG}(8)$ is too extensive for publication in a hardcopy journal article. However, outdegree sequences allow us to include here summary information about $\mathcal{HG}(n)$. The *outdegree sequence at level* m for $\mathcal{HG}(n)$ is the sequence

$$0^{d(m,0)}, 1^{d(m,1)}, \ldots, r^{d(m,r)}, \ldots$$

where $d(m, r)$ is the number of graphs $G \in \mathcal{G}(n, m)$ with r 1-extensions, and the *outdegree sequence* for $\mathcal{HG}(n)$ is the corresponding sequence in which the index of r is $d(r) = \Sigma_m d(m, r)$. Tables 2–4 present these outdegree sequences for $\mathcal{HG}(n)$ when $4 \leq n \leq 6$, and Table 5 presents summary information for $4 \leq n \leq 8$. In Tables 2–4 the order and size at level m are the order of $\mathcal{G}(n, m)$ and the size of the digraph between $\mathcal{G}(n, m)$ and $\mathcal{G}(n, m + 1)$, equal to $\Sigma_r r d(m, r)$. In Table 5 the order and size at n are the order and size of $\mathcal{HG}(n)$, equal to $\Sigma_r d(r)$ and $\Sigma_r r d(r)$, respectively. In Tables 2–4, the "All min" column records by SEAM number each graph which achieves the minimum outdegree within its level, and "First max" records by SEAM number the first graph which achieves the maximum outdegree within its level. The complement of any graph with outdegree 1 is edge-transitive, so these graphs can be found in the "All min" column; every other entry in this column is recorded in square brackets.

Because of the complementation anti-automorphism of $\mathcal{HG}(n)$, each indegree sequence for $\mathcal{HG}(n)$, specifying numbers of 1-reductions,

Table 2: Outdegree sequences for $\mathcal{HG}(4)$

m	Order	Size	Outdegree sequence	All min	First max
0	1	1	1^1	1	1
1	1	2	$1^0\,2^1$	[2]	2
2	2	4	$1^1\,2^0\,3^1$	4	3
3	3	4	$1^2\,2^1$	5 7	6
4	2	2	1^2	8 9	8
5	1	1	1^1	10	10
6	1	0	0^1	[11]	11

Table 3: Outdegree sequences for $\mathcal{HG}(5)$

m	Order	Size	Outdegree sequence	All min	First max
0	1	1	1^1	1	1
1	1	2	$1^0\,2^1$	[2]	2
2	2	6	$1^0\,2^1\,3^0\,4^1$	[4]	3
3	4	12	$1^0\,2^1\,3^2\,4^1$	[5]	7
4	6	16	$1^2\,2^1\,3^0\,4^3$	11 13	9
5	6	16	$1^1\,2^1\,3^3\,4^1$	18	19
6	6	12	$1^2\,2^2\,3^2$	24 25	23
7	4	6	$1^2\,2^2$	29 30	27
8	2	2	1^2	31 32	31
9	1	1	1^1	33	33
10	1	0	0^1	[34]	34

Table 4: Outdegree sequences for $\mathcal{HG}(6)$

m	Order	Size	Outdegree sequence	All min	First max
0	1	1	1^1	1	1
1	1	2	$1^0\,2^1$	[2]	2
2	2	7	$1^0\,2^0\,3^1\,4^1$	[4]	3
3	5	18	$1^1\,2^1\,3^0\,4^1\,5^1\,6^1$	9	8
4	9	40	$1^0\,2^0\,3^4\,4^2\,5^1\,6^0\,7^1\,8^1$	[11 12 14 16]	13
5	15	68	$1^1\,2^2\,3^2\,4^3\,5^1\,6^3\,7^2\,8^1$	25	23
6	21	96	$1^1\,2^4\,3^2\,4^2\,5^5\,6^3\,7^3\,8^0\,9^1$	53	46
7	24	107	$1^1\,2^2\,3^2\,4^9\,5^5\,6^2\,7^1\,8^2$	63	68
8	24	96	$1^0\,2^2\,3^{10}\,4^5\,5^3\,6^1\,7^3$	[94 101]	85
9	21	68	$1^4\,2^3\,3^3\,4^7\,5^3\,6^1$	103 104 118 123	111
10	15	40	$1^2\,2^4\,3^6\,4^3$	132 133	127
11	9	18	$1^3\,2^3\,3^3$	139 145 146	140
12	5	7	$1^3\,2^2$	148 151 152	149
13	2	2	1^2	153 154	153
14	1	1	1^1	155	155
15	1	0	0^1	[156]	156

Table 5: The outdegree sequence for $\mathcal{HG}(n)$

n	Order	Size	Outdegree sequence
4	11	14	$0^1\ 1^7\ 2^2\ 3^1$
5	34	74	$0^1\ 1^{11}\ 2^9\ 3^7\ 4^6$
6	156	571	$0^1\ 1^{20}\ 2^{24}\ 3^{33}\ 4^{33}\ 5^{19}\ 6^{11}\ 7^{10}\ 8^4\ 9^1$
7	1044	6558	$0^1\ 1^{25}\ 2^{54}\ 3^{92}\ 4^{133}\ 5^{140}\ 6^{139}\ 7^{130}\ 8^{107}$ $9^{78}\ 10^{58}\ 11^{39}\ 12^{26} 13^{16} 14^4 15^2$
8	12346	125066	$0^1\ 1^{37}\ 2^{110}\ 3^{235}\ 4^{428}\ 5^{600}\ 6^{798}\ 7^{997}\ 8^{1135}\ 9^{1196}\ 10^{1176}$ $11^{1124}\ 12^{1051}\ 13^{967}\ 14^{826}\ 15^{652}\ 16^{467}\ 17^{293}\ 18^{158}$ $19^{71}\ 20^{21}\ 21^2\ 22^1$

is equal to an appropriate outdegree sequence, so we need not list indegree sequences explicitly.

Tables 2–5 tell us much about the Hasse diagrams of the posets of graphs of orders $n \leq 8$. Theorems 2, 3 and 4 summarize some of this information.

Theorem 2 *For $n \geq 1$, the Hasse diagram $\mathcal{HG}(n)$ has size*

$$0, 1, 3, 14, 74, 571, 6558, 125066, \cdots$$

We say that any graph $G \in \mathcal{G}(n)$ has *productivity* d if it has exactly d distinct 1-extensions: evidently d is the outdegree of G as a vertex of $\mathcal{HG}(n)$. A graph G has productivity 1 precisely when its complement G^c is nonempty and edge-transitive, so the number of edge-transitive graphs in $\mathcal{G}(n)$ is one greater than the number of vertices of outdegree 1 in $\mathcal{HG}(n)$, because the empty graph K_n^c is trivially edge-transitive.

Theorem 3 *For $n \geq 1$, the number of edge-transitive graphs in $\mathcal{G}(n)$ is*

$$1, 2, 4, 8, 12, 21, 26, 38, \cdots$$

For each $n \geq 1$, the number of maximally productive graphs, with their productivity, is

1:0, 1:1, 3:1, 1:3, 6:4, 1:9, 2:15, 1:22, $\cdots$

The data in Theorem 3 suggest that the productivity of the maximally productive graphs grows quadratically: is $O(n^2)$ the correct order of magnitude?

It is natural to define the *productivity* of $\mathcal{G}(n,m)$ to be the total number of edges from $\mathcal{G}(n,m)$ to $\mathcal{G}(n,m+1)$ in $\mathcal{HG}(n)$. The *productivity sequence* for $\mathcal{G}(n)$ is the sequence with dth term equal to the total number of graphs $G \in \mathcal{G}(n)$ having productivity d. Theorem 4 gives descriptive information about these quantities, which follows from the outdegree sequences for $\mathcal{HG}(n)$, but first we introduce some terminology. We say that a sequence $a(0), a(1), \cdots, a(k)$ is *unimodal* if

$$a(0) \le a(1) \le \cdots \le a(r) \ge a(r+1) \ge \cdots \ge a(k)$$

for some integer r such that $0 \le r \le k$. In this case the *peak* of the sequence is its maximum value, and its *peak support* is the set (interval) of all values of r for which $a(r)$ is equal to the peak. If $a(0), a(1), \cdots, a(k)$ is unimodal, with peak A and peak support $\{r_0, \cdots, r_1\}$, then the sequence $0^{a(0)}, 1^{a(1)}, \cdots, k^{a(k)}$ is *index unimodal*, with *index peak* A and *peak support* $\{r_0, \cdots, r_1\}$. In each case, if the peak support is a singleton, we simply specify it as an integer.

Theorem 4 *Each $\mathcal{G}(n)$ with $n \le 8$ has an index unimodal productivity sequence. The peak supports and index peaks are*

0:1, 1:1, 1:3, 1:7, 1:11, {3,4}:33, 5:140, 9:1196.

The productivities of the level sets of each $\mathcal{G}(n)$ form a unimodal sequence for each $n \le 8$. The peak supports and peak values are

0:0, 0:1, {0,1,2}:1, {2,3}:4, {4,5}:16, 7:107, 10:1066, {13,14}:17739.

It is natural to conjecture that every $\mathcal{G}(n)$ has an index unimodal productivity sequence and that the sequence of productivities of its level sets is unimodal.

4 Independence digraphs

A subset $\mathcal{S} \subseteq \mathcal{G}(n)$ is an *independent set* (or *antichain*) if none of its members is a subgraph of any other member. Trivially any singleton is independent. The number of independent subsets in $\mathcal{G}(n)$ grows rapidly with n. For example, $\mathcal{G}(3)$ has only the trivial independent subsets, but $\mathcal{G}(4)$ has 24 independent subsets, and $\mathcal{G}(5)$ has 862 independent subsets **[2]**. Certain natural operations on independent subsets of $\mathcal{G}(n)$ produce other independent subsets of $\mathcal{G}(n)$. Such operations may be studied as digraphs on the family $\mathcal{A}$ of all independent subsets of $\mathcal{G}(n)$, so $\mathcal{A}$ is of considerable interest. In **[2]** we studied two such operations on the independent subsets of $\mathcal{G}(5)$, but the proliferation of independent sets makes it impractical to extend such comprehensive studies to $\mathcal{G}(6)$. However, we shall now introduce a structure that gives some insights into the family of all independent subsets of $\mathcal{G}(n)$ without explicitly listing them all.

The *independence digraph* $\mathcal{IG}(n)$ is a directed graph with $\mathcal{G}(n)$ as its vertex set, and a directed edge $G{\rightarrow}H$ precisely when $N^*(G) < N^*(H)$ and $\{G, H\}$ is an independent subset of $\mathcal{G}(n)$. Because N^* imposes a linear ordering on the graphs of order n, every directed clique in $\mathcal{IG}(n)$ is transitively directed. Any subset $\mathcal{S} \subseteq \mathcal{G}(n)$ is independent precisely when it induces a complete subgraph in $\mathcal{IG}(n)$, and the cliques of $\mathcal{IG}(n)$ correspond to the maximal independent subsets of $\mathcal{G}(n)$.

As for $\mathcal{HG}(n)$, the *outdegree sequence at level m* for $\mathcal{IG}(n)$ is the sequence

$$0^{d(m,0)}, 1^{d(m,1)}, \cdots, r^{d(m,r)}, \cdots$$

where now $d(m, r)$ is the number of graphs $G \in \mathcal{G}(n, m)$ for which there are exactly r graphs with larger SEAM number than G that are not extensions of G, that is, the number of vertices with outdegree r in $\mathcal{IG}(n)$ that lie in $\mathcal{G}(n, m)$. The *outdegree sequence* for $\mathcal{IG}(n)$ is the corresponding sequence in which the index of r is $d(r) = \Sigma_m d(m, r)$. Also $\Sigma_r d(m, r)$ is the *level m outsize* of $\mathcal{IG}(n)$. Tables 6−8 present summary information about these descriptors for $\mathcal{IG}(n)$ when $4 \leq n \leq 6$, and Table 9 presents an even more condensed summary for $\mathcal{IG}(n)$ with $4 \leq n \leq 8$. In particular, the outdegree sequences are too lengthy for full presentation, so just the minimum and maximum terms from these sequences, and the mean term (to one decimal place) are given in Tables 6−8. In each case, the column "First max" records by SEAM number the first graph to achieve maximum outdegree within its level.

Since the complementation map $c : \mathcal{G}(n){\rightarrow}\mathcal{G}(n)$ maps independent subsets of $\mathcal{G}(n)$ to independent subsets, it readily follows that if $G{\rightarrow}H$ is a directed edge in $\mathcal{IG}(n)$, then $H^c{\rightarrow}G^c$ is also a directed edge, and therefore complementation is an anti-automorphism of $\mathcal{IG}(n)$.

Table 6: Outdegrees in $\mathcal{IG}(4)$

m	Outsize	Min	Max	Mean	First min	First max
0	0	0	0	0.0	1	1
1	0	0	0	0.0	2	2
2	3	1	2	1.5	3	4
3	5	1	3	1.7	6	5
4	1	0	1	0.5	9	8
5	0	0	0	0.0	10	10
6	0	0	0	0.0	11	11

Table 7: Outdegrees in $\mathcal{IG}(5)$

m	Outsize	Min	Max	Mean	First min	First max
0	0	0	0	0.0	1	1
1	0	0	0	0.0	2	2
2	4	1	3	2.0	3	4
3	24	3	10	6.0	7	5
4	48	3	13	8.0	14	11
5	39	3	9	6.5	19	18
6	28	1	7	4.7	26	21
7	8	1	3	2.0	30	27
8	1	0	1	0.5	32	31
9	0	0	0	0.0	33	33
10	0	0	0	0.0	34	34

Table 8: Outdegrees in $\mathcal{IG}(6)$

m	Outsize	Min	Max	Mean	First min	First max
0	0	0	0	0.0	1	1
1	0	0	0	0.0	2	2
2	6	2	4	3.0	3	4
3	111	7	47	22.2	8	9
4	299	16	51	33.2	15	12
5	701	28	98	46.7	23	25
6	1084	33	87	51.6	46	34
7	1142	30	73	47.6	72	63
8	928	24	55	38.7	96	80
9	567	17	43	27.0	122	104
10	229	8	23	15.3	137	125
11	64	2	12	7.1	147	139
12	13	1	5	2.6	152	148
13	1	0	1	0.5	154	153
14	0	0	0	0.0	155	155
15	0	0	0	0.0	156	156

Table 9: Outdegrees in $\mathcal{IG}(n)$

n	Outsize	Mean	Max	First max
4	9	0.8	3	5
5	152	4.5	13	11
6	5145	33.0	98	25
7	303191	290.4	827	62
8	51952895	4208.1	11137	166

It follows that any indegree information about $\mathcal{IG}(n)$ can be deduced from the corresponding outdegree information, so we do not tabulate it explicitly. Tables 6–9, or the calculations used to produce them, yield the following structural theorems about the independence digraphs $\mathcal{IG}(n)$ for $n \leq 8$.

Theorem 5 *For $n \geq 1$, the size of $\mathcal{IG}(n)$ is*

$$0, 0, 0, 9, 152, 5145, 303191, 51952895, \cdots$$

and its maximum outdegree is

$$0, 0, 0, 3, 13, 98, 827, 11137, \cdots$$

Theorem 6 *The level sets of each $\mathcal{IG}(n)$ with $n \leq 8$ have a unimodal sequence of outsizes. The peak supports and peak values are*

0:0, {0,1}:0, {0,1,2,3}:0, 3:5, 4:48, 7:1142, 10:59563, 13:9356461.

We conjecture that the level sets of every $\mathcal{IG}(n)$ have a unimodal sequence of outsizes.

Let us now introduce some terminology that gives a different perspective on the structure of $\mathcal{IG}(n)$ from that given by outdegree. If $G \to H$ is a directed edge of $\mathcal{IG}(n)$, with $\text{size}(G) = m \leq \text{size}(H) = m'$, we define the *height* of $G \to H$ to be $m' - m$. The level sets $\mathcal{G}(n, m)$ are independent subsets of $\mathcal{G}(n)$, so any two graphs $G, H \in \mathcal{G}(n, m)$ determine a directed edge of height 0 in $\mathcal{IG}(n)$. If $G \to H$ is a directed edge of height h in $\mathcal{IG}(n)$, then $H^c \to G^c$ is also a directed edge of height h, so complementation is a height-preserving anti-automorphism of $\mathcal{IG}(n)$. For any integers $h, m \geq 0$, let $e(m, h)$ be the number of edges in $\mathcal{IG}(n)$ with one vertex in the level set

$\mathcal{G}(n,m)$ and the other in the level set $\mathcal{G}(n,m+h)$. We call the sequence

$$0^{e(m,0)}, 1^{e(m,1)}, \cdots, h^{e(m,h)}, \cdots$$

the *edge height sequence at level* m for $\mathcal{IG}(n)$. Note that the complementation anti-automorphism of $\mathcal{IG}(n)$ results in the identity

$$e(m,h) = e(m',h) \text{ whenever } m + m' + h = \binom{n}{2}.$$

The *edge height sequence* for $\mathcal{IG}(n)$ is the corresponding sequence in which the index of h is $e(h) = \Sigma_m e(m,h)$. Note that $\Sigma_h e(m,h)$ is equal to the level m outsize of $\mathcal{IG}(n)$. In Tables 10–12 we specify these height sequences for $\mathcal{IG}(n)$ when $4 \leq n \leq 6$, and in Table 13 we present summary information for $\mathcal{IG}(n)$ when $4 \leq n \leq 8$. In each table, the column "First max" records the SEAM numbers of the graphs at the start and end of the first directed edge of maximum height originating at level m.

Table 10: Edge height sequences for $\mathcal{IG}(4)$

m	Size	Edge height sequence	First max
0	0	0^0	
1	0	0^0	
2	3	$0^1\,1^2$	4,5
3	5	$0^3\,1^2$	5,8
4	1	0^1	8,9
5	0	0^0	
6	0	0^0	

Table 11: Edge height sequences for $\mathcal{IG}(5)$

m	Size	Edge height sequence	First max
0	0	0^0	
1	0	0^0	
2	4	$0^1\,1^2\,2^1$	4,11
3	24	$0^6\,1^{12}\,2^4\,3^2$	5,22
4	48	$0^{15}\,1^{20}\,2^{11}\,3^2$	11,27
5	39	$0^{15}\,1^{20}\,2^4$	16,30
6	28	$0^{15}\,1^{12}\,2^1$	24,31
7	8	$0^6\,1^2$	29,31
8	1	0^1	31,32
9	0	0^0	
10	0	0^0	

Table 12: Edge height sequences for $\mathcal{IG}(6)$

m	Size	Edge height sequence	First max
0	0	0^0	
1	0	0^0	
2	6	$0^1\,1^3\,2^1\,3^1$	4,25
3	111	$0^{10}\,1^{27}\,2^{28}\,3^{23}\,4^{12}\,5^7\,6^3\,7^1$	9,132
4	299	$0^{36}\,1^{95}\,2^{87}\,3^{48}\,4^{20}\,5^{10}\,6^3$	16,132
5	701	$0^{105}\,1^{247}\,2^{184}\,3^{98}\,4^{46}\,5^{17}\,6^3\,7^1$	25,148
6	1084	$0^{210}\,1^{408}\,2^{272}\,3^{135}\,4^{46}\,5^{10}\,6^3$	34,148
7	1142	$0^{276}\,1^{469}\,2^{272}\,3^{98}\,4^{20}\,5^7$	55,148
8	928	$0^{276}\,1^{408}\,2^{184}\,3^{48}\,4^{12}$	79,152
9	567	$0^{210}\,1^{247}\,2^{87}\,3^{23}$	103,151
10	229	$0^{105}\,1^{95}\,2^{28}\,3^1$	132,153
11	64	$0^{36}\,1^{27}\,2^1$	145,153
12	13	$0^{10}\,1^3$	148,154
13	1	0^1	153,154
14	0	0^0	
15	0	0^0	

Table 13: The edge height sequence for $\mathcal{IG}(n)$.

n	Size	Edge height sequence	First max
4	9	$0^5\,1^4$	4,5
5	152	$0^{59}\,1^{68}\,2^{21}\,3^4$	5,22
6	5145	$0^{1276}\,1^{2029}\,2^{1144}\,3^{475}\,4^{156}\,5^{51}\,6^{12}\,7^2$	9,132
7	303191	$0^{54430}\,1^{99744}\,2^{75160}\,3^{43365}\,4^{19346}\,5^{7380}\,6^{2564}\,7^{848}$ $8^{260}\,9^{76}\,10^{16}\,11^2$	19,983
8	51952895	$0^{7121581}\,1^{13734944}\,2^{12102437}\,3^{9163174}\,4^{5536492}$ $5^{2635630}\,6^{1057576}\,7^{389458}\,8^{138233}\,9^{48308}\,10^{16700}$ $11^{5692}\,12^{1864}\,13^{582}\,14^{172}\,15^{42}\,16^8\,17^2$	20,12181

Theorem 7 *Each $\mathcal{IG}(n)$ with $n \leq 8$ has an index unimodal edge height sequence. The peak supports and the index peaks are*

$$0{:}0,\ 0{:}0,\ 0{:}0,\ 0{:}5,\ 1{:}68,\ 1{:}2029,\ 1{:}99744,\ 1{:}13734944.$$

We conjecture that every $\mathcal{IG}(n)$ has an index unimodal edge height sequence, and perhaps the peak support is 1 for every $n \geq 5$.

Again, for any integer $h \geq 0$, let $t(h)$ be the number of transitively directed triangles of *height* h in $\mathcal{IG}(n)$, that is, oriented 3-cycles for which h is the maximum of the heights of the three directed edges. We call the sequence

$$0^{t(0)}, 1^{t(1)}, \cdots, h^{t(h)}, \cdots$$

the *triangle height sequence* for $\mathcal{IG}(n)$. Table 14 gives the triangle height sequence for each $\mathcal{IG}(n)$ with $4 \leq n \leq 8$.

Table 14: The triangle height sequence for $\mathcal{IG}(n)$.

n	Total	Triangle height sequence	First max
4	3	$0^{1}\ 1^{2}$	4,5,7
5	290	$0^{68}\ 1^{154}\ 2^{62}\ 3^{6}$	5,11,22
6	83970	$0^{7806}\ 1^{31192}\ 2^{27214}\ 3^{12368}\ 4^{3992}\ 5^{1160}\ 6^{196}\ 7^{42}$	9,16,132
7	47998164	$0^{2197562}\ 1^{11408828}\ 2^{15418546}\ 3^{11032360}$ $4^{5138998}\ 5^{1909632}\ 6^{632542}\ 7^{193478}\ 8^{52120}$ $9^{12510}\ 10^{1536}\ 11^{52}$	19,35,983
8	96775426396	$0^{3183817290}\ 1^{18122040894}\ 2^{30988057738}$ $3^{32798374390}\ 4^{23239287708}\ 5^{11552154228}$ $6^{4532438526}\ 7^{1581909774}\ 8^{525404152}\ 9^{172139476}$ $10^{55895834}\ 11^{17458378}\ 12^{4900842}\ 13^{1226188}$ $14^{275170}\ 15^{39816}\ 16^{4214}\ 17^{1778}$	20,42,12181

From Table 14 we deduce the following theorems:

Theorem 8 *For $n \geq 1$, the number of independent triples in $\mathcal{G}(n)$ is*

$$0, 0, 0, 3, 290, 83970, 47998164, 96775426396, \cdots$$

Theorem 9 *For $n \leq 8$, the triangle height sequence of $\mathcal{IG}(n)$ is index unimodal. The peak supports and index peaks are*

0:0, 0:0, 0:0, 1:2, 1:154, 1:31192, 2:15418546, 3:32798374390.

We conjecture that the triangle height sequence of every $\mathcal{IG}(n)$ is index unimodal.

This concludes our descriptive summary of the structure of the posets $\mathcal{G}(n)$ for $4 \leq n \leq 8$. We hope that it will motivate readers to seek proofs of the conjectures, and to conjecture and prove other general theorems suggested by the data. Once again, we remind readers that extensive data used to produce our summaries is available for reference or downloading at **[3]**.

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