Q1: First we need to check that if \( z \in \mathbb{D} \), then \( f(z) \in \mathbb{D} \).

With \( z = x + iy \). For \( z \in \mathbb{D} \), \( y > 0 \).

Then \( |z - i|^2 = x^2 + (y - 1)^2 = x^2 + y^2 + 1 - 2y \).

\( |z + i|^2 = x^2 + (y + 1)^2 = x^2 + y^2 + 1 + 2y \).

For \( y > 0 \), \( |z - i|^2 < |z + i|^2 \Rightarrow |z - i| < |z + i| \)

\( \Rightarrow \left\| \frac{z - i}{z + i} \right\| < 1 \) as required.

Injectivity of \( f \):

If \( f(z_1) = f(z_2) \):

\[
\frac{z_1 - i}{z_1 + i} = \frac{z_2 - i}{z_2 + i}
\]

\( (z_1 - i)(z_2 + i) = (z_2 - i)(z_1 + i) \)

\( z_1 z_2 - i z_1 \bar{z}_2 + i + 1 = z_1 z_2 - i z_2 \bar{z}_1 + 1 \)

\( i(\bar{z}_1 - z_2) = -i(\bar{z}_2 - z_1) \)

\( \bar{z}_1 - z_2 = 0 \Rightarrow z_1 = z_2 \).

Surjectivity of \( f \):

Let \( w \in \mathbb{D} \). Let \( z = -i \frac{w + 1}{w - 1} = i \frac{1 + w}{1 - w} \).

Then \( f(z) = i \frac{1 + w}{1 - w} - i \frac{1 + w}{1 - w} = i \frac{1 + w}{1 - w} + i \frac{1 + w}{1 - w} = \frac{1 + w - (1 - w)}{1 + w + (1 - w)} = \frac{2w}{2} = w \).

So provided \( z \in \mathbb{D} \) we've shown \( f \) is surjective.

To show \( z \in \mathbb{D} \) we need to show \( \text{Re} \left( \frac{1 + w}{1 - w} \right) > 0 \), i.e. \( -\frac{\pi}{2} < \text{arg} \left( \frac{1 + w}{1 - w} \right) < \frac{\pi}{2} \).

The line segment between \( 1 + w \) and \( 1 - w \) has centre 1 and lies entirely within the circle centred at 1 with radius 1 as \( |w| < 1 \).

Thus the angle \( \triangle ABC \) is acute so it is smaller than the \( 90 \)° angle subtended by the diameter.

\( \Rightarrow -\frac{\pi}{2} < \text{arg} \left( \frac{1 + w}{1 - w} \right) < \frac{\pi}{2} \) as required.
The previous computations show that the inverse of $f$ is given by

$$f^{-1}(w) = i \cdot \frac{1+w}{1-w}.$$ 

\[ i \cdot \frac{1+w}{1-w} \] 

is a rational function with a pole at $\frac{1}{2}$, which is not in $\mathbb{D}$.

ii. $f^{-1}$ is a holomorphic function on $\mathbb{D}$.
Q2 (a): Let $r$ be a real number greater than 1 and \\
\[ \sum_{i=0}^{n-1} |a_i| \]
\[ \frac{1}{|a_0|} \]

Then for $|z| = r$, we have
\[ |q(z)| \leq \sum_{i=0}^{n-1} |a_i| r^i \] (by the triangle inequality)
\[ \leq r^{n-1} \sum_{i=0}^{n-1} |a_i| \] as $r > 1$
\[ \leq |a_n| r^n \] as $r > \frac{\sum_{i=0}^{n-1} |a_i|}{|a_0|}$
\[ = |a_n z^n| \] as required.

Q2 (b): In part (a) we showed that the functions $q(z)$ and $a_n z^n$ satisfy $|q(z)| \leq |a_n z^n|$ on the circle with $|z| = r$.

Both the functions $q(z)$ and $a_n z^n$ are polynomials, hence are entire.

Therefore, by Rouche's theorem, the functions $q(z) + a_n z^n$ and $a_n z^n$ have the same number of zeros inside the disc $\{ z \in \mathbb{C} : |z| < r \}$, counted with multiplicity.

The function $a_n z^n$ has a zero at $z = 0$.

Therefore, the function $q(z) + a_n z^n$ must have a zero.

i.e. $q(z)$ has a zero, proving the fundamental theorem of algebra.
Q3: The set \( X = \{ z \in \mathbb{C} \mid 0 \leq \text{Re}(z) \leq 1, 0 \leq \text{Im}(z) \leq |\text{Im}(z)|^2 \} \)

is a closed and bounded subset of \( \mathbb{C} \), hence is compact.

As \( f \) is continuous, \( |f(z)| \) is bounded on \( X \).

i.e. \( \exists M \in \mathbb{R} \) such that \( |f(z)| < M \) for all \( z \in X \).

Now let \( z \in \mathbb{C} \) be arbitrary. As \( \text{Im}(z) \neq 0 \), there exists an integer \( n \) such that \( n \) \( z' = z - \frac{2\pi}{n} \mathbb{Z} \) satisfies

\[ 0 \leq \text{Im}(z') \leq |\text{Im}(z)|. \]

There exists an integer \( m \) such that \( z'' = z' - m \) satisfies

\[ 0 \leq \text{Re}(z'') \leq 1. \]

As \( m \in \mathbb{R} \), \( 0 \leq \text{Im}(z'') \leq |\text{Im}(z)| \), so \( z'' \in X \).

We have

\[ f(z) = f(z') = f(z'') \]

(iterating \( f(z + z) = f(z) \))

\[ = f(z'') \]

(iterating \( f(z + z) = f(z) \))

\[ |f(z)| = |f(z'')| < M \quad \text{as} \quad z'' \in X. \]

Thus \( f \) is an entire, bounded function. By Liouville's Theorem, \( f \) is constant.
Q4: Note that \( \cos(x) = \text{Re} \left( e^{ix} \right) \).

\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} \, dx = \text{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \, dx = \text{Re} \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{1+x^2} \, dx.
\]

Consider the function \( f(z) = \frac{e^{iz}}{1+z^2} \). It has two poles at \( i \) and \(-i\). The residue at \( z = i \) is

\[
\text{Res}_{z=i} f(z) = \lim_{z \to i} \frac{e^{iz}}{1+z^2} (z-i) = \lim_{z \to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}.
\]

Let \( C_R \) be the semicircular arc \( \{ z \in \mathbb{C} \mid \, |z| = R, \, \text{Im}(z) \geq 0 \} \).

If \( z \in C_R \) then \( \text{Im}(z) > 0 \Rightarrow \text{Re}(iz) \leq 0 \Rightarrow |e^{iz}| \leq 1 \).

Also \( |z^2| \geq |z^2| - 1 = R^2 - 1 \) \( (\text{triangle inequality}) \).

\[
\left| \frac{e^{iz}}{1+z^2} \right| \leq \frac{1}{R^2 - 1}.
\]

\[
\left| \int_{C_R} \frac{e^{iz}}{1+z^2} \, dz \right| \leq \pi R \cdot \frac{1}{R^2 - 1} \quad \text{as the length of } C_R \text{ is } \pi R.
\]

As \( \lim_{R \to \infty} \frac{1}{R^2 - 1} = 0 \) this implies \( \lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{1+z^2} \, dz = 0 \).

By the residue formula, for \( R > 1 \):

\[
\int_{C_R} \frac{e^{iz}}{1+z^2} \, dz + \int_{-R}^{R} \frac{e^{ix}}{1+x^2} \, dx = 2\pi i \text{Res}_{z=i} f(z) = \frac{\pi e^{-1}}{2}.
\]

Taking the limit as \( R \to \infty \) yields \( \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{1+x^2} \, dx = \frac{\pi}{2} \).

And hence \( \int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} \, dx = \frac{\pi}{2} \).
Q5(a) Let \( K \) be a compact subset of \( D \).

Let \( r = \sup_{z \in K} |z| \).

As \( K \) is compact and the absolute value function is continuous, there exists \( w \in K \) with \( |w| = r \). \( \therefore r < 1 \) (as we \( D \)).

Choose \( R \) such that \( r < R < 1 \).

By Cauchy's inequality
\[
|f_n^{(k)}(0)| \leq \frac{\pi k! B}{R^k}.
\]

Let
\[
a_k = \lim_{n \to \infty} f_n^{(k)}(0).
\]

Taking the limit \( \infty \) in the above inequality implies
\[
|a_k| \leq \frac{k! B}{R^k}.
\]

Let \( f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \). We will show that \( \{f_n\} \) converges uniformly to \( f \) on \( K \).

We know
\[
f_n(z) = \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0)}{k!} z^k.
\]

For \( z \in K \):
\[
|f_n(z) - f(z)| = \left| \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0)}{k!} z^k - \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \right|
\leq \sum_{k=0}^{N-1} \frac{|f_n^{(k)}(0) - a_k|}{k!} |z|^k + \sum_{k=N}^{\infty} \frac{|f_n^{(k)}(0)| + |a_k|}{k!} \frac{r^k}{k!}
\leq \sum_{k=0}^{N-1} \frac{|f_n^{(k)}(0) - a_k|}{k!} \frac{r^k}{k!} + \sum_{k=N}^{\infty} \frac{2^k}{k!} \frac{B}{R^k} \frac{r^k}{k!}
\leq \sum_{k=0}^{N-1} \frac{|f_n^{(k)}(0) - a_k|}{k!} \frac{r^k}{k!} + 2B \left( \frac{r^N}{R} \right) \frac{1}{1 - \frac{r}{R}}.
\]
Pick $\varepsilon > 0$. Then as $n \frac{f_n^k}{R^n} < 1$, there exists $N_0 \in \mathbb{N}$ such that

$$2B \left( \frac{R}{r} \right)^N \frac{1}{1 - \frac{r}{R}} < \frac{\varepsilon}{2}.$$ Fix this choice of $N_0$.

For each $k$ with $0 \leq k < N_0$, let $\hat{\phi}_n^{(k)}(0)$ converges to $a_k$.

For each $k$ with $0 \leq k < N_0$, there exists $N_k$ such that for $n > N_k$,

$$\left| f_n^{(k)}(0) - a_k \right| < \frac{\varepsilon}{2N^k}. 

Then for $n > \max \{ N_0, N_1, \ldots, N_{N-1} \}$ we have, from the inequality on the previous page:

$$\left| f_n^{(k)}(z) - \hat{\phi}_n^{(k)}(z) \right| \leq \left( \sum_{k=1}^{N_k} \varepsilon \frac{k!}{2N^k} \frac{r^k}{k!} \right) + \frac{\varepsilon}{2} = \varepsilon.$$

This is true for all $z \in K$, so we have shown the desired uniform convergence.

(b) Let $\hat{f}_n(z) = 3^n z^n$.

Then $\hat{f}_n^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n \\ 3^n n! & \text{if } k = n. \end{cases}$

For each $k$, $\{ f_n^{(k)}(0) \}_{n=0}^{\infty}$ is convergent, as it is eventually constant.

The sequence $\{ f_n \}$ does not converge as

$$f_n \left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^n$$

which tends to $\infty$ as $n \to \infty$ (which also shows condition (i) does not hold).