Solutions to 116 Homework 6

1. Integrating in the Riemann-Stieltjes sense, we have

\[ \pi(x) = \sum_{p \leq x} \frac{\log p}{\log p} = \lim_{\epsilon \downarrow 0} \int_{2-\epsilon}^{x+\epsilon} \frac{1}{\log t} \, d\theta(t) \]

\[ = \lim_{\epsilon \downarrow 0} \frac{\theta(t)}{\log t} \bigg|_{2-\epsilon}^{x+\epsilon} - \int_{2-\epsilon}^{x+\epsilon} \theta(t) \frac{1}{\log t} \, dt \]

\[ = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log t^2} \, dt \]

Or we can evaluate the identity explicitly. Note that

\[ \int_a^b \frac{1}{t \log t^2} \, dt = \frac{1}{\log a} - \frac{1}{\log b} \]

and let \( p_1 < p_2 < \ldots < p_n \) be the primes \( \leq x \), so that \( \pi(x) = n \). Then

\[ \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log t^2} \, dt \]

\[ = \frac{\theta(p_n)}{\log p_n} + \theta(p_n) \int_{p_n}^x \frac{\theta(t)}{t \log t^2} \, dt + \sum_{k=1}^{n-1} \theta(p_k) \int_{p_k}^{p_{k+1}} \frac{\theta(t)}{t \log t^2} \, dt \]

\[ = \frac{\theta(p_n)}{\log p_n} + \sum_{k=1}^{n-1} \theta(p_k) \left( \frac{1}{\log p_k} - \frac{1}{\log p_{k+1}} \right) \]

\[ = \sum_{k=1}^{n-1} \frac{\theta(p_{k+1}) - \theta(p_k)}{\log p_k} + \frac{\theta(p_1)}{\log p_1} \]

\[ = n \]

2. Since \( z \in \mathfrak{h} \) is never real, \( cz + d \neq 0 \) on \( \mathfrak{h} \), so \( \rho_A : \mathfrak{h} \rightarrow \mathbb{C} \) is well-defined. Take \( z \in \mathfrak{h} \), then we calculate

\[ \Im \rho_A(z) = \frac{\det A}{|cz + d|^2} \Im z > 0 \]

so \( \rho_A(\mathfrak{h}) \subset \mathfrak{h} \).

Let \( B \) be the matrix \( \begin{pmatrix} d & -b \\ c & a \end{pmatrix} \), and write \( \rho_B \) for the corresponding Mobius transform. Since \( \det B = \det A \), as above \( \rho_B \) is a well-defined map \( \mathfrak{h} \rightarrow \mathfrak{h} \). An easy calculation shows that \( \rho_A^{-1} = \rho_B \).
3. Let $|f| > \epsilon > 0$ on $C = \partial D$. By assumption, $f_n \to f$ uniformly on the closed disc $\overline{D}$, so we can find an $N$ such that $|f_n - f| < \epsilon/2$ on $C$ whenever $n > N$. In particular, this means that $|f| > |f_n - f|$. So by Rouche’s theorem, $f$ and $f + (f_n - f) = f_n$ have the same number of zeros when $n > N$.

4. Take $\Re s > 1$. For any integer $N$,

$$
\sum_{n=1}^{N} \frac{1}{n^s} = \lim_{\epsilon \downarrow 0} \int_{1-\epsilon}^{N+\epsilon} \frac{1}{x^s} d\lfloor x \rfloor
$$

$$
= \frac{N}{N^s} - \int_{1}^{N} \lfloor x \rfloor \left( -\frac{s}{x^{s+1}} dx \right)
$$

$$
= N^{1-s} + s \int_{1}^{N} \frac{x}{x^{s+1}} dx - s \int_{1}^{N} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx
$$

$$
= N^{1-s} + s \frac{1}{s-1} (1 - N^{1-s}) - s \int_{1}^{N} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx
$$

(again we use Riemann-Stieltjes integral)

Since $|N^{1-s}| = N^{1-\Re s}$, take $N \to \infty$ and the result follows.

The last integral converges absolutely iff $\Re s > 0$, since $|x - \lfloor x \rfloor| \leq 1$ but $|x - \lfloor x \rfloor| \geq 1/2$ for $x \in \mathbb{N} + [1/2,1]$. So the right hand side defines a meromorphic extension of $\zeta$ to $\Re s > 0$. 
