Homework 2

Due: Thursday April 19, 2012

1. Find the size of the conjugacy class of the element $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ in $GL_2(\mathbb{F}_5)$. (The conjugacy class of $g \in G$ is the set of all elements of the form $hgh^{-1}$ for some $h \in G$.)

2. Let $G$ be a group. Prove that the map from $G$ to itself given by $g \mapsto g^2$ is a homomorphism if and only if $G$ is abelian.

3. Consider the set of necklaces with $p$ beads, with each bead coloured in one of $a$ different colours. The cyclic group of order $p$ acts on this set by rotating the necklace. In this setup, show how the formula

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g)$$

yields a proof of Fermat’s Little Theorem.

4. Let $X$ be a set with 4 elements. Let $Y$ be the set of unordered pairs of subsets $\{A, B\}$ of $X$ with $|A| = |B| = 2$ and $A \cap B = \emptyset$. Then $Y$ is a set with 3 elements. The symmetric group $S_4$ acts on $X$ in the usual manner and this induces an action of $S_4$ on $Y$. Hence we have obtained a homomorphism from $S_4$ to $S_3$. Prove that this homomorphism is surjective and its kernel is isomorphic to $C_2 \times C_2$ (this is called the Klein four group).

5. Let $G$ be an abelian group. Prove that the set $\{g \in G \mid g^n = 1 \text{ for some } n \geq 1\}$ is a subgroup of $G$ (it goes by the name of the torsion subgroup). Give an example of a nonabelian group $G$ where this statement fails.


7. Find all possible subgroups of the alternating group $A_4$ and determine which are normal.

8. Let $F$ be a field. Show that the determinant defines a surjective group homomorphism from $GL_n(F)$ to $F^\times$, with kernel $SL_n(F)$. 