Path Integrals for Continuous-Time Markov Chains

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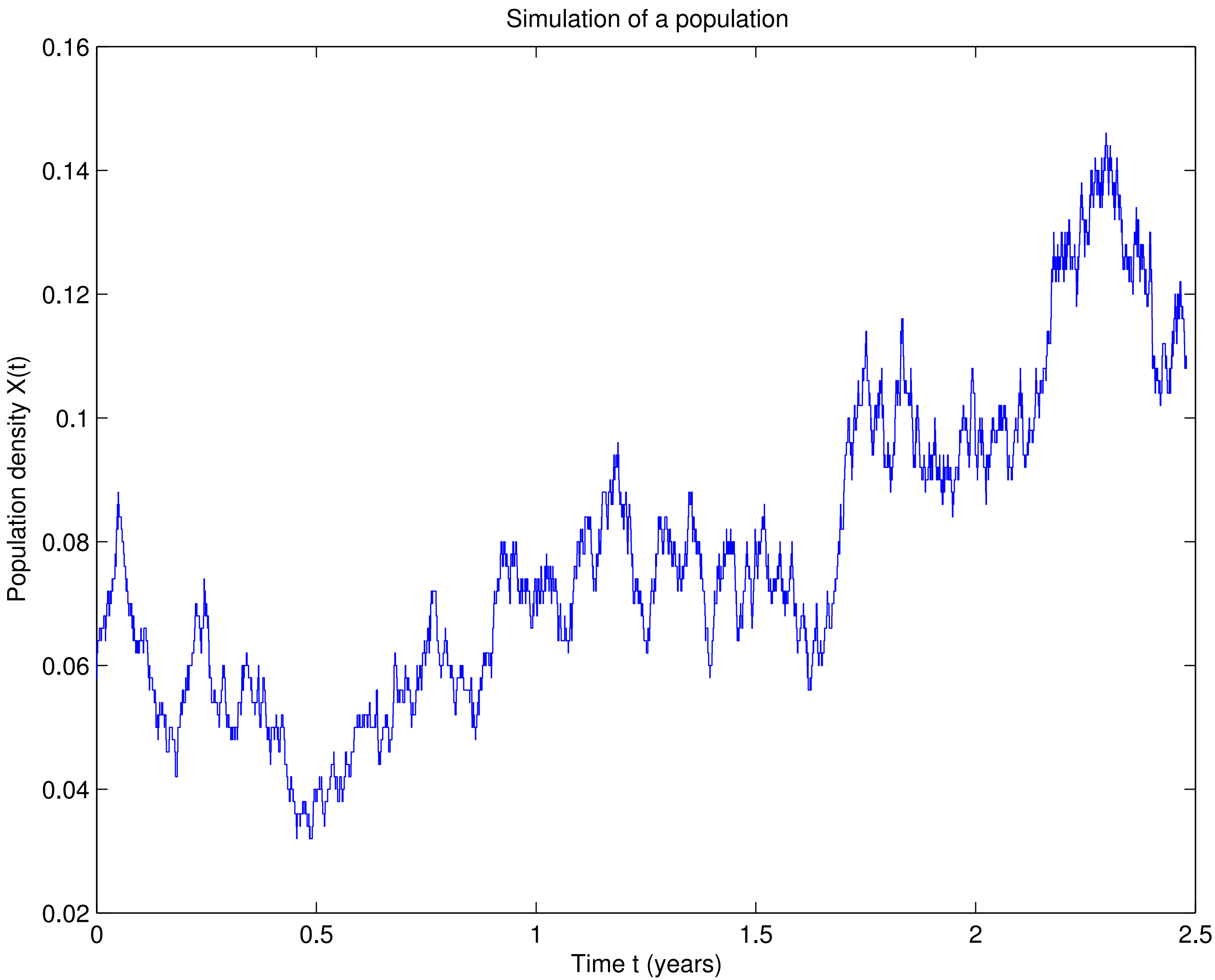
AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

A population process



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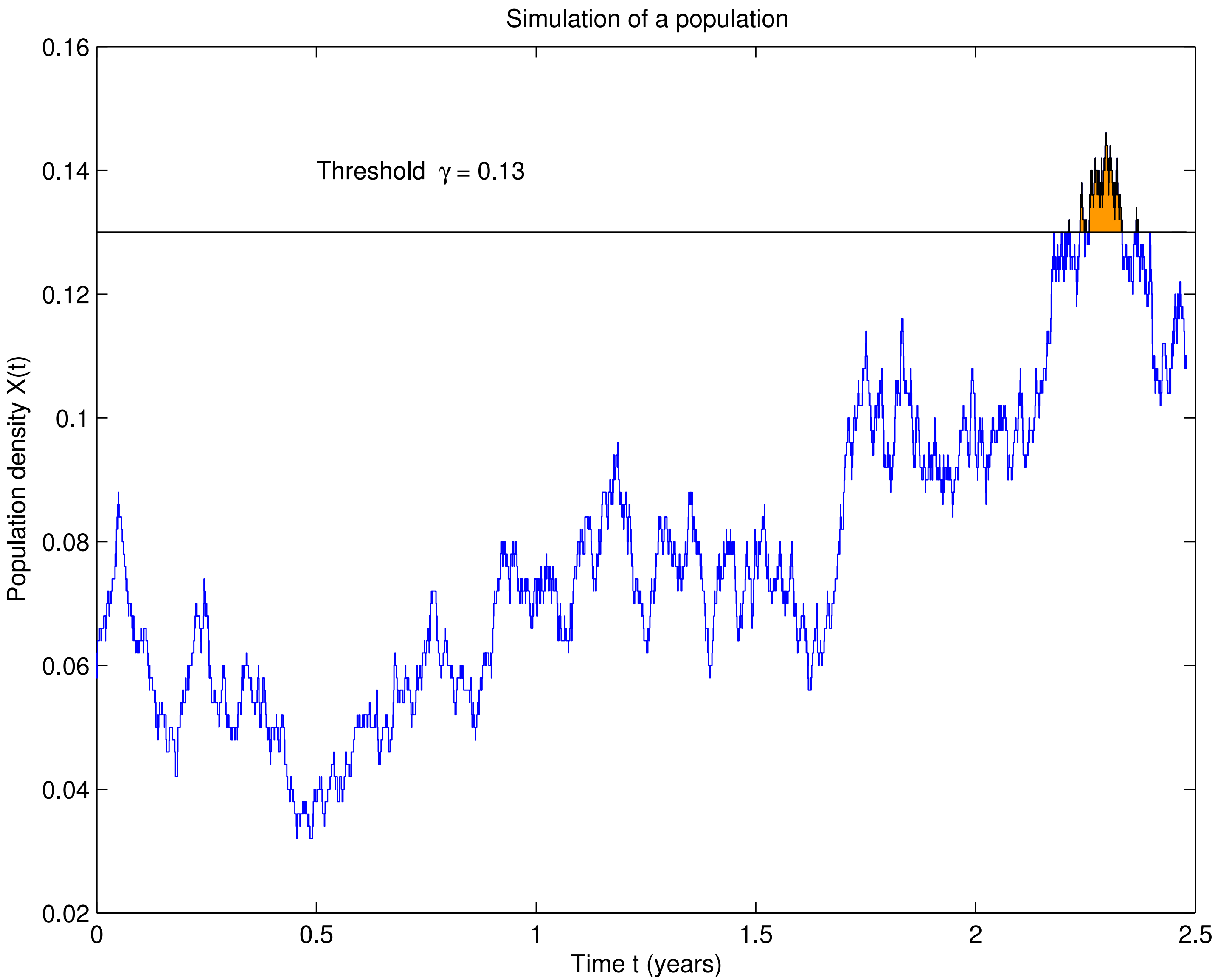


Total cost

Let $X(t)$ be the population density at time t.

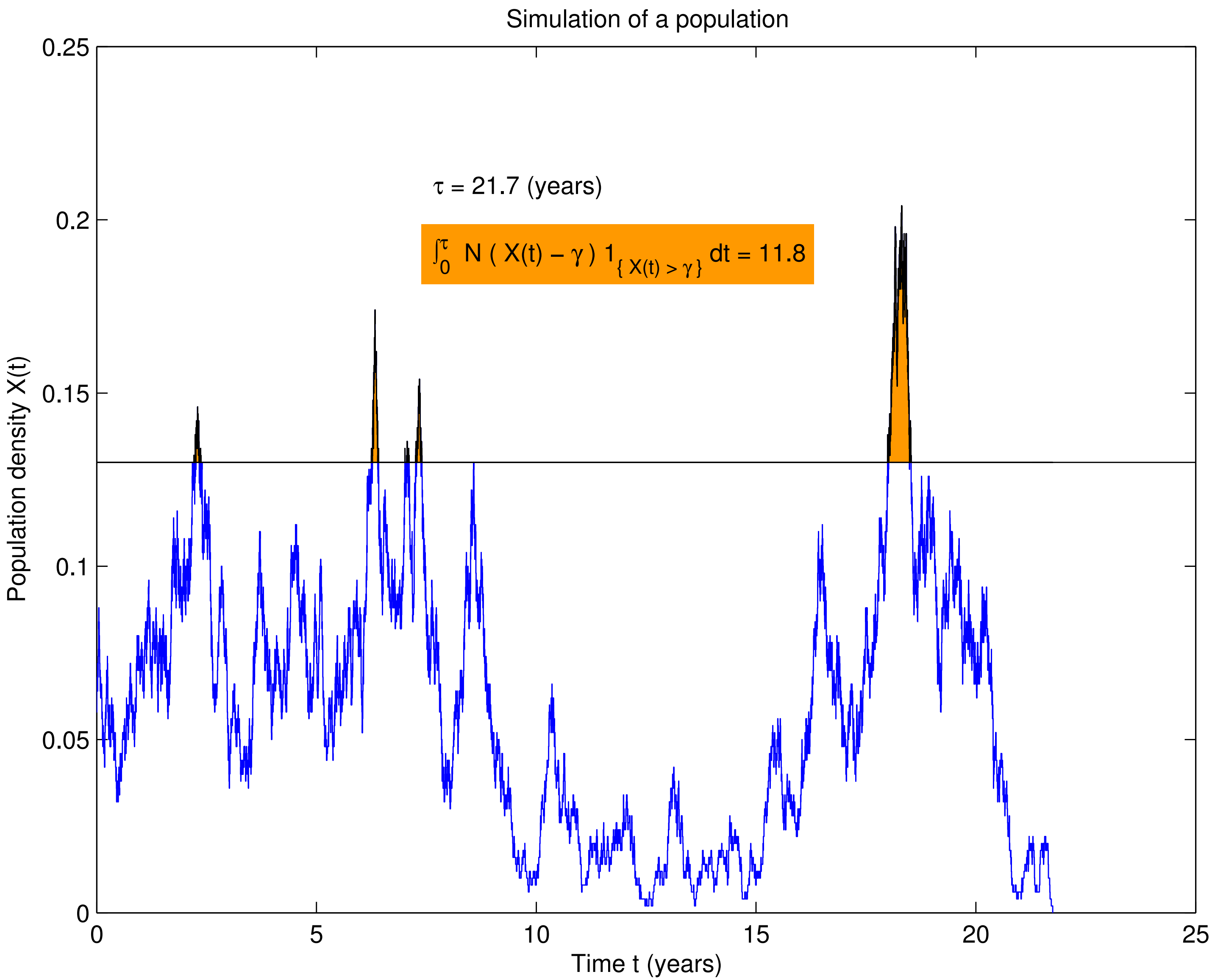
Let $c(x)$ be the cost per unit time of maintaining the population when its density is x units above a threshold γ.

Then, if τ is the time to extinction,

$$\int_0^{\tau} c(X(t) - \gamma) 1_{\{X(t) > \gamma\}} \, dt$$

is the total cost over the life of the population.

A population process



Ingredients

- A random process $(X(t), t \geq 0)$ in continuous time

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- A set of states A
- The (random) time τ to first exit from A
- The cost (per unit time) f_x of being in state x
- The "path integral"

$$\Gamma = \int_0^{\tau} f_{X(t)} \, dt,$$

the total cost incurred before leaving A (also random)

Other examples

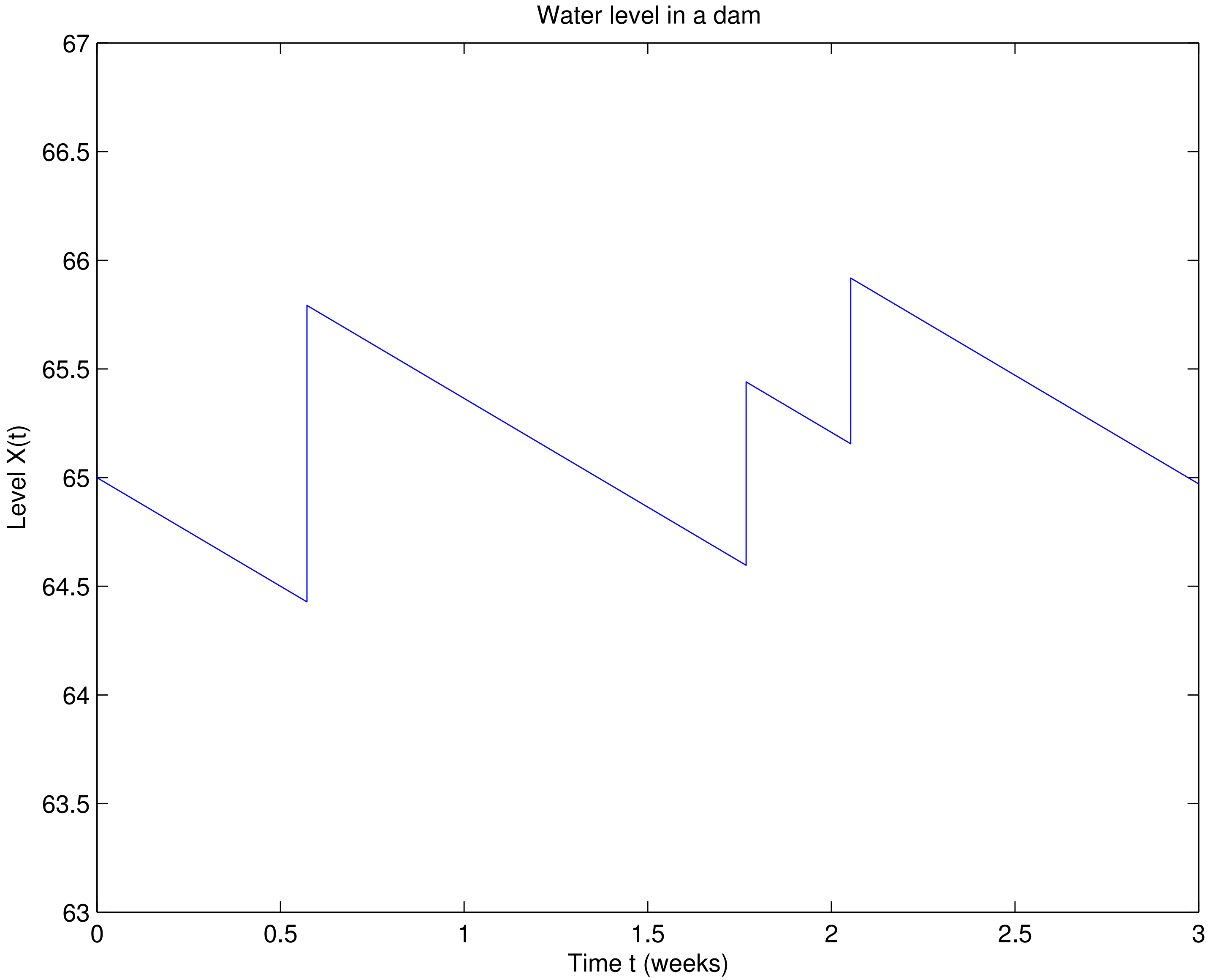
- Consider a dam with finite capacity V, and let $X(t)$ be the water level at time t.

  We might wish to estimate the total time for which the level was below a given value γ,

  $$\Gamma = \int_0^{\tau} 1_{\{X(t)<\gamma\}}\, dt,$$

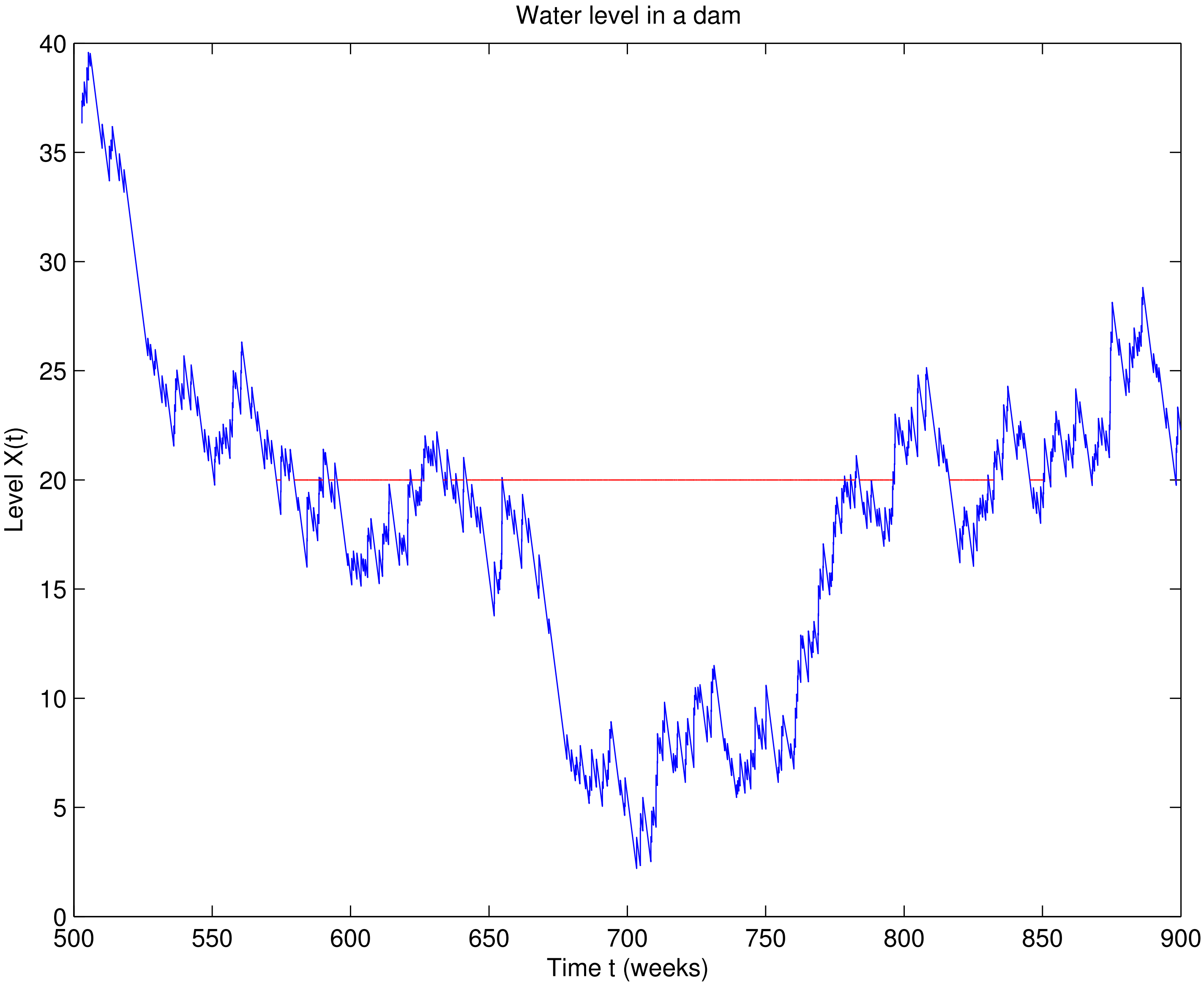
  where τ is (say) the time to reach capacity or to empty (whichever occurs first).

Dam



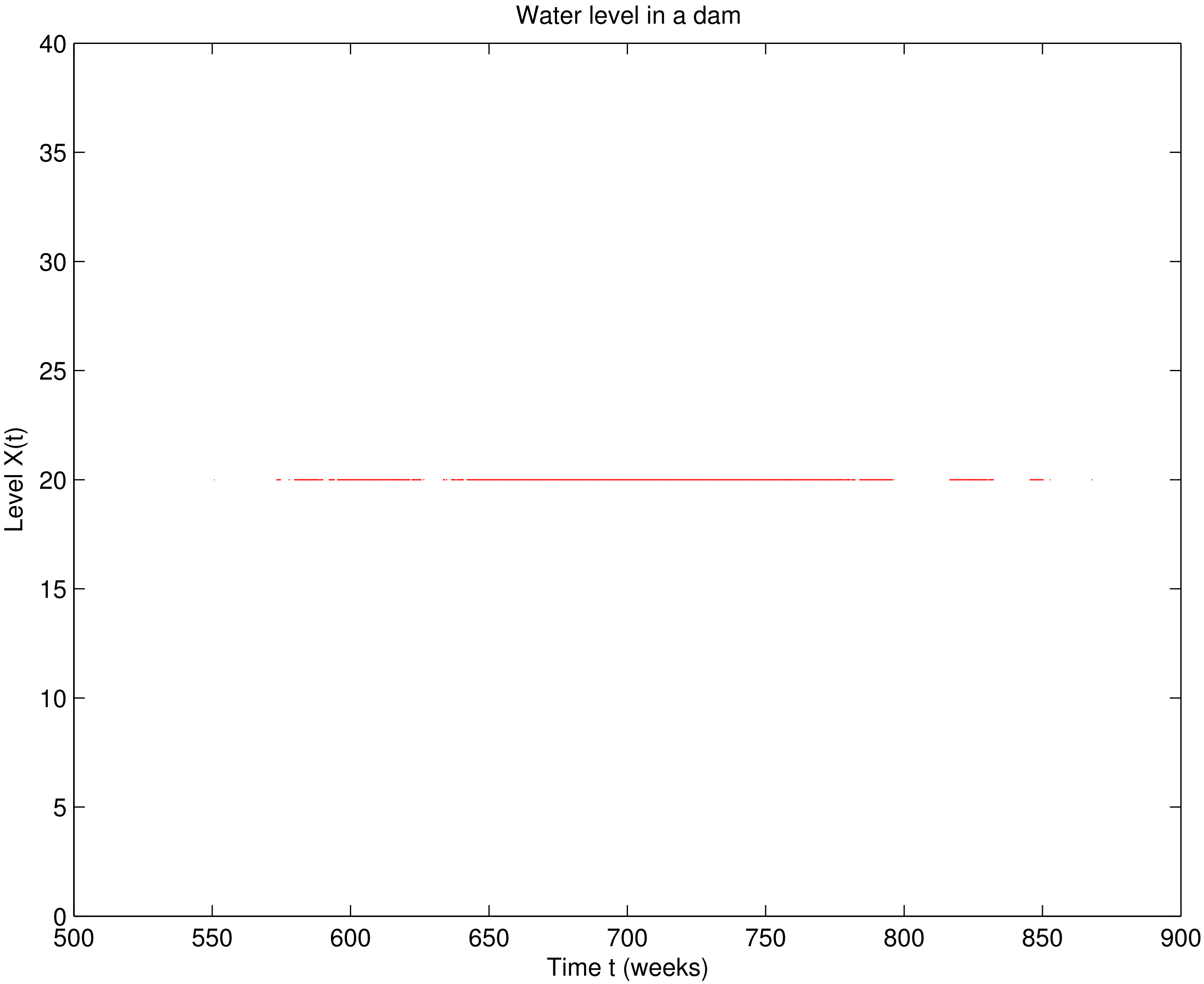
Water level in a dam
Level X(t)
Time t (weeks)
67
66.5
66
65.5
65
64.5
64
63.5
63
0
0.5
1
1.5
2
2.5
3

Dam

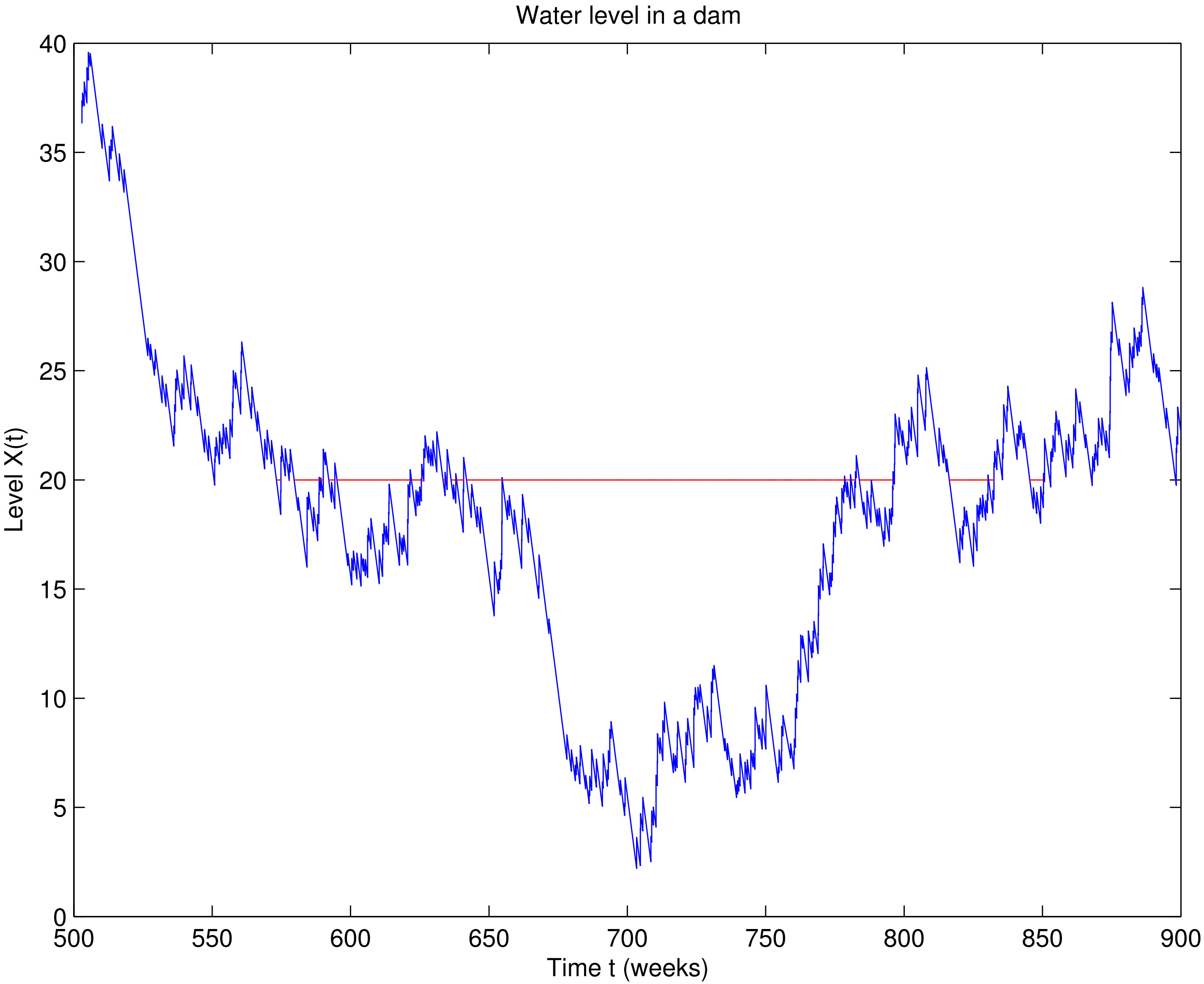


Water level in a dam
Level X(t)
Time t (weeks)
40
35
30
25
20
15
10
5
0
500
550
600
650
700
750
800
850
900

Dam


Water level in a dam
Level X(t)
Time t (weeks)
0
5
10
15
20
25
30
35
40
500
550
600
650
700
750
800
850
900

Dam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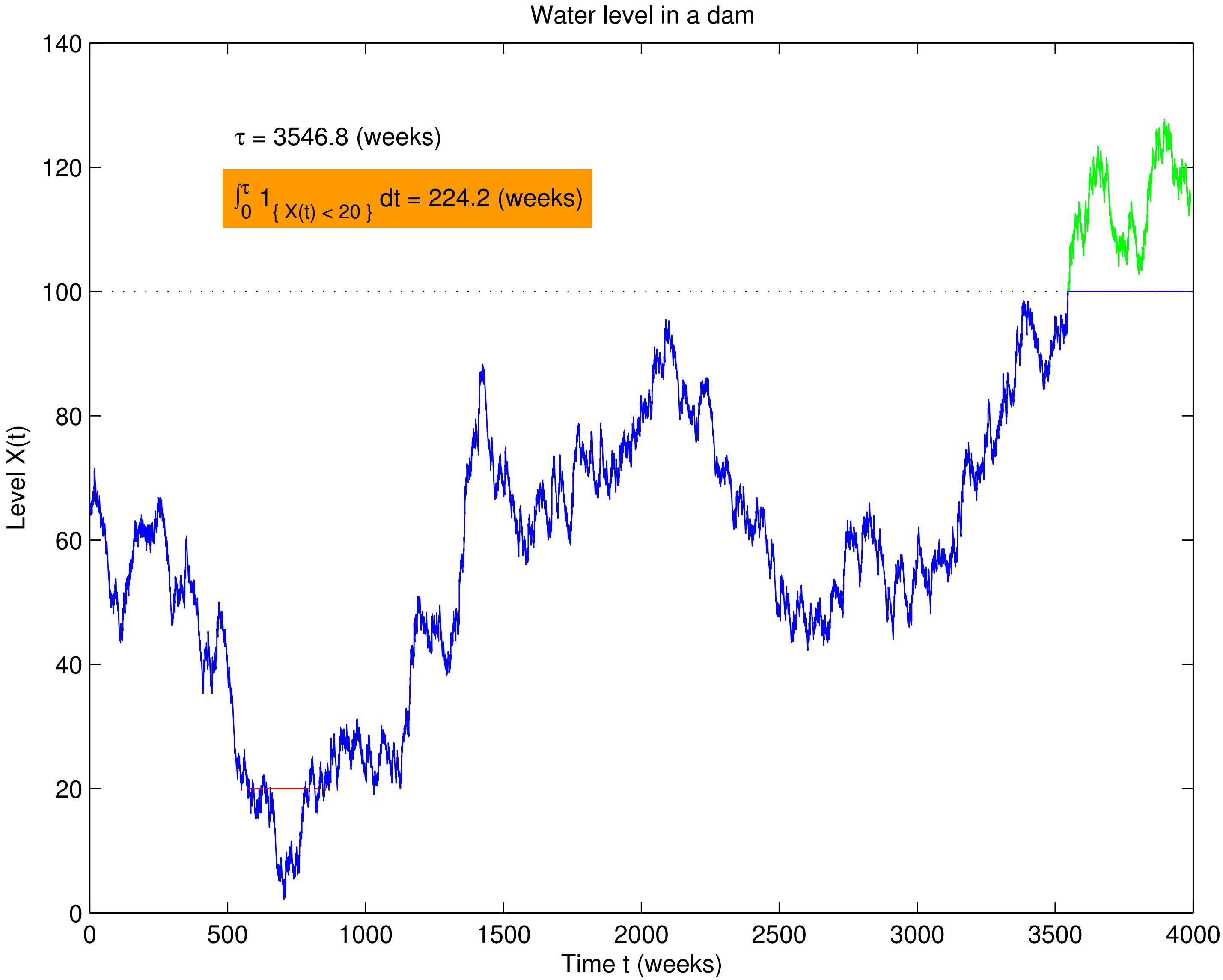
Dam


Water level in a dam
τ = 3546.8 (weeks)
$\int_0^\tau 1_{\{X(t) < 20\}} dt = 224.2$ (weeks)
Level X(t)
Time t (weeks)
0
20
40
60
80
100
120
140
0
500
1000
1500
2000
2500
3000
3500
4000

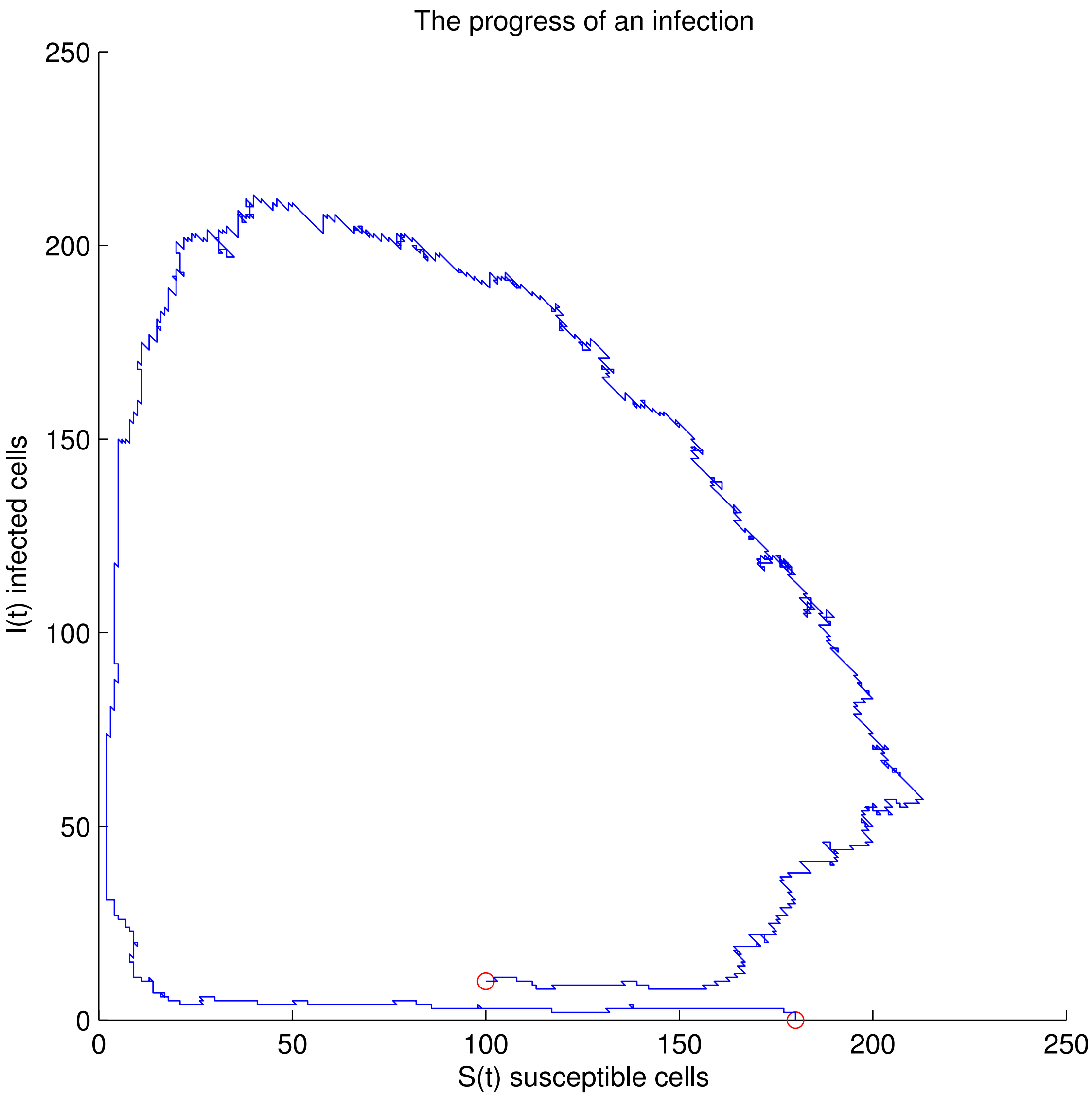
Other examples

- Let $(S(t), I(t))$ be the number of susceptibles and infectives in an epidemic at time t.

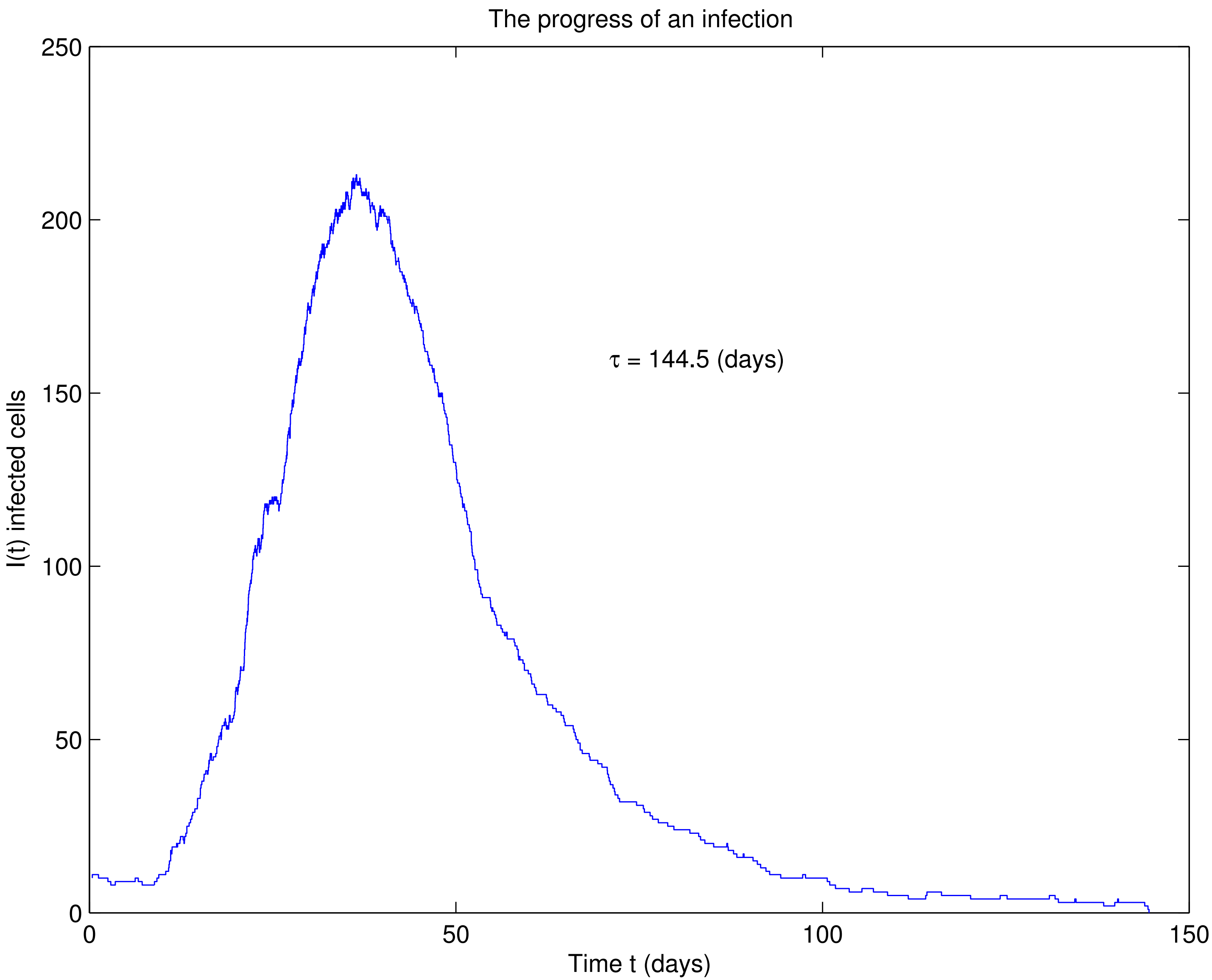
  If τ is the period of infection and $f_{(s,i)} = i$, then Γ is the total amount of infection:

$$\Gamma = \int_0^{\tau} I(t)\, dt.$$

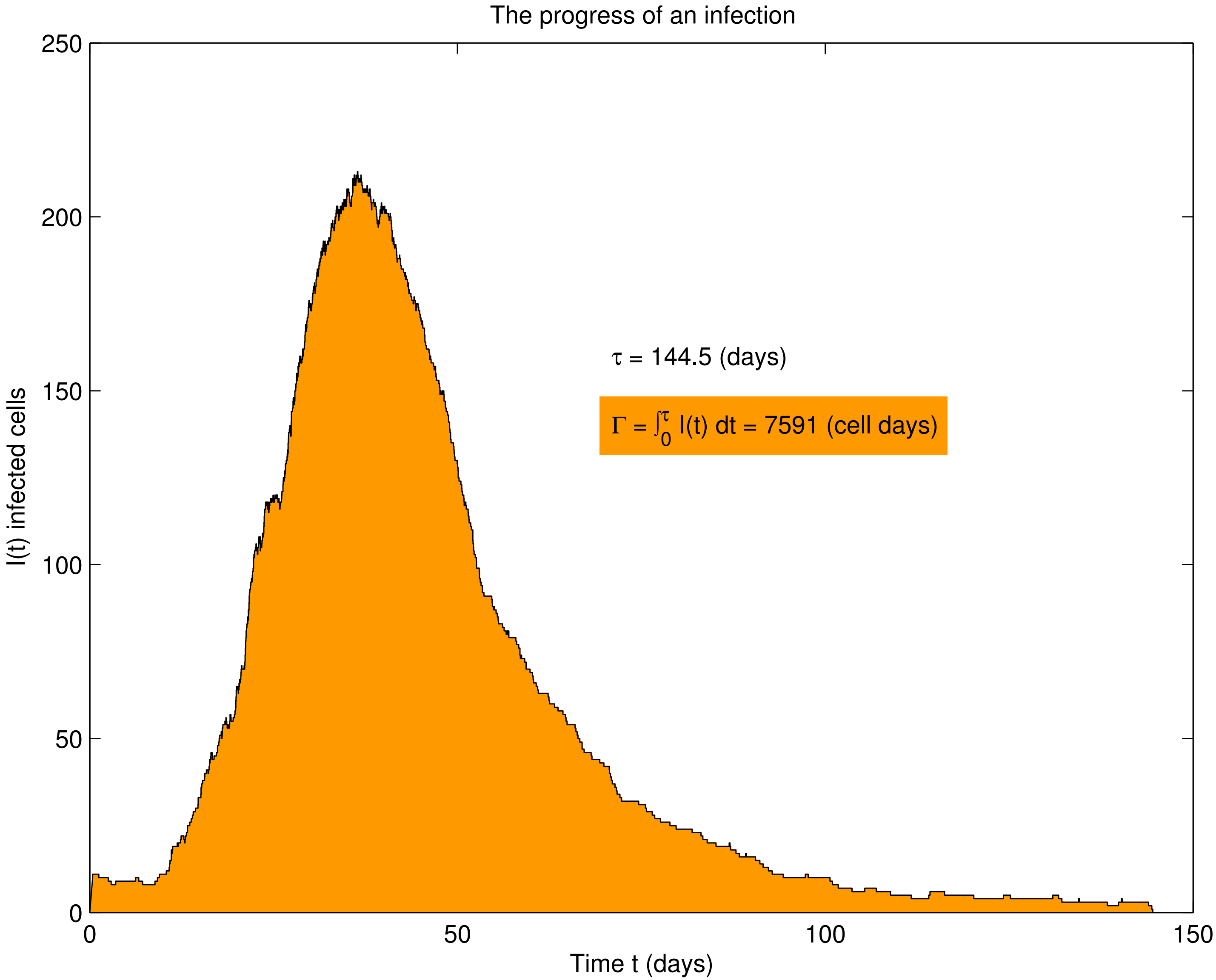
Epidemic


The progress of an infection
I(t) infected cells
S(t) susceptible cells
0
50
100
150
200
250

Epidemic


The progress of an infection
τ = 144.5 (days)
I(t) infected cells
Time t (days)
0
50
100
150
200
250
0
50
100
150

Epidemic


The progress of an infection
250
200
150
100
50
0
0
50
100
150
I(t) infected cells
Time t (days)
τ = 144.5 (days)
Γ = ∫₀^τ I(t) dt = 7591 (cell days)

The problem

Our problem is to determine the *expected value*, and the *distribution* of the total cost

$$\Gamma = \int_0^{\tau} f_{X(t)}\, dt,$$

where recall that τ is the time to first exit from a set A and f_x is cost per unit time of being in state x.

For simplicity, suppose that $X(t)$ takes values in $S = \{0, 1, \dots\}$.

For example, $X(t)$ might be the number in a population at time t, and $A = \{1, 2, \dots\}$, so that τ is the time to extinction.

A first attempt at evaluating $E(\Gamma)$

Let T_j be the total time that the process spends in state j during the period up to time τ and let N_j be the number of visits to j during that period. Then,

$$\Gamma = \sum_{j \in A} f_j T_j$$

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$$\Gamma = \sum_{j \in A} f_j T_j \quad \text{and} \quad T_j = \sum_{n=1}^{N_j} X_{jn},$$

where X_{jn}, $n = 1, 2, \ldots$, are the successive occupancy times for state j.

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where X_{jn}, $n = 1, 2, \ldots$, are the successive occupancy times for state j. Then, under mild conditions, $E(\Gamma) = \sum_{j \in A} f_j E(N_j) \mu_j$, where μ_j is the mean occupancy time for state j.

Markovian models

We will assume that $(X(t),\, t \geq 0)$ is a *Markov chain* with *transition rates*

$$Q = (q_{ij},\, i, j \in S),$$

so that q_{ij} represents the rate of transition from state i to state j, for $j \neq i$, and $q_{ii} = -q_i$, where

$$q_i := \sum_{j \neq i} q_{ij} \;\; (< \infty)$$

represents the total rate out of state i.

Markovian models

An example is the *birth-death process*, which has

$$q_{i,i+1} = \lambda_i \qquad \text{(birth rates)}$$
$$q_{i,i-1} = \mu_i \qquad \text{(death rates)},$$

with $\mu_0 = 0$ and otherwise 0 ($q_i = \lambda_i + \mu_i$):

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Example

The *Stochastic Logistic Model* (simulated earlier) is a birth-death process on $S = \{0, 1, \ldots, N\}$, with

$$\lambda_i = \frac{\lambda}{N} i(N - i) \quad \text{and} \quad \mu_i = \mu i,$$

where $\lambda, \mu > 0$.

Interlude

These birth and death rates can be written

$$\frac{\lambda_i}{N} = \lambda \left(\frac{i}{N}\right)\left(1 - \frac{i}{N}\right) \quad \text{and} \quad \frac{\mu_i}{N} = \mu \left(\frac{i}{N}\right)$$

Intuition: for large N the population *density* $X(t)/N$ becomes more deterministic (non-random):

$$\frac{dx}{dt} = \lambda(x) - \mu(x),$$

where

$$\lambda(x) = \lambda x\,(1 - x) \quad \text{and} \quad \mu(x) = \mu x.$$

Interlude

Soit p la population : représentons par dp l'accroissement infiniment petit qu'elle reçoit pendant un temps infiniment court dt. Si la population croissait en progression géométrique, nous aurions l'équation $\frac{dp}{dt} = mp$. Mais comme la vitesse d'accroissement de la population est retardée par l'augmentation même du nombre des habitans, nous devrons retrancher de mp une fonction inconnue de p; de manière que la formule à intégrer deviendra

$$\frac{dp}{dt} = mp - \varphi(p).$$

L'hypothèse la plus simple que l'on puisse faire sur la forme de la fonction φ, est de supposer $\varphi(p) = np^2$. On trouve alors pour intégrale de l'équation ci-dessus

$$t = \frac{1}{m}\left[\log.\, p - \log.(m - np)\right] + \text{constante},$$

et il suffira de trois observations pour déterminer lés deux coefficiens constans m et n et la constante arbitraire.

Interlude

En résolvant la dernière équation par rapport à p, il vient

$$p = \frac{mp' e^{mt}}{np' e^{mt} + m - np'} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

en désignant par p' la population qui répond à $t = o$, et par e la base des logarithmes népériens. Si l'on fait $t = \infty$, on voit que la valeur de p correspondante est $P = \frac{m}{n}$. Telle est donc *la limite supérieure de la population.*

Au lieu de supposer $\varphi p = np^2$, on peut prendre $\varphi p = np^\alpha$, α étant quelconque, ou $\varphi p = n \log. p$. Toutes ces hypothèses satisfont également bien aux faits observés ; mais elles donnent des valeurs très-différentes pour la limite supérieure de la population.

J'ai supposé successivement

$$\varphi p = np^2,\ \varphi p = np^3,\ \varphi p = np^4,\ \varphi p = n \log. p;$$

et les différences entre les populations calculées et celles que fournit l'observation ont été sensiblement les mêmes.

Interlude

This is from ...

> P.F. Verhulst, Notice sur la loi que la population suit dans son accroisement, *Corr. Math. et Phys*. X (1838), 113–121.

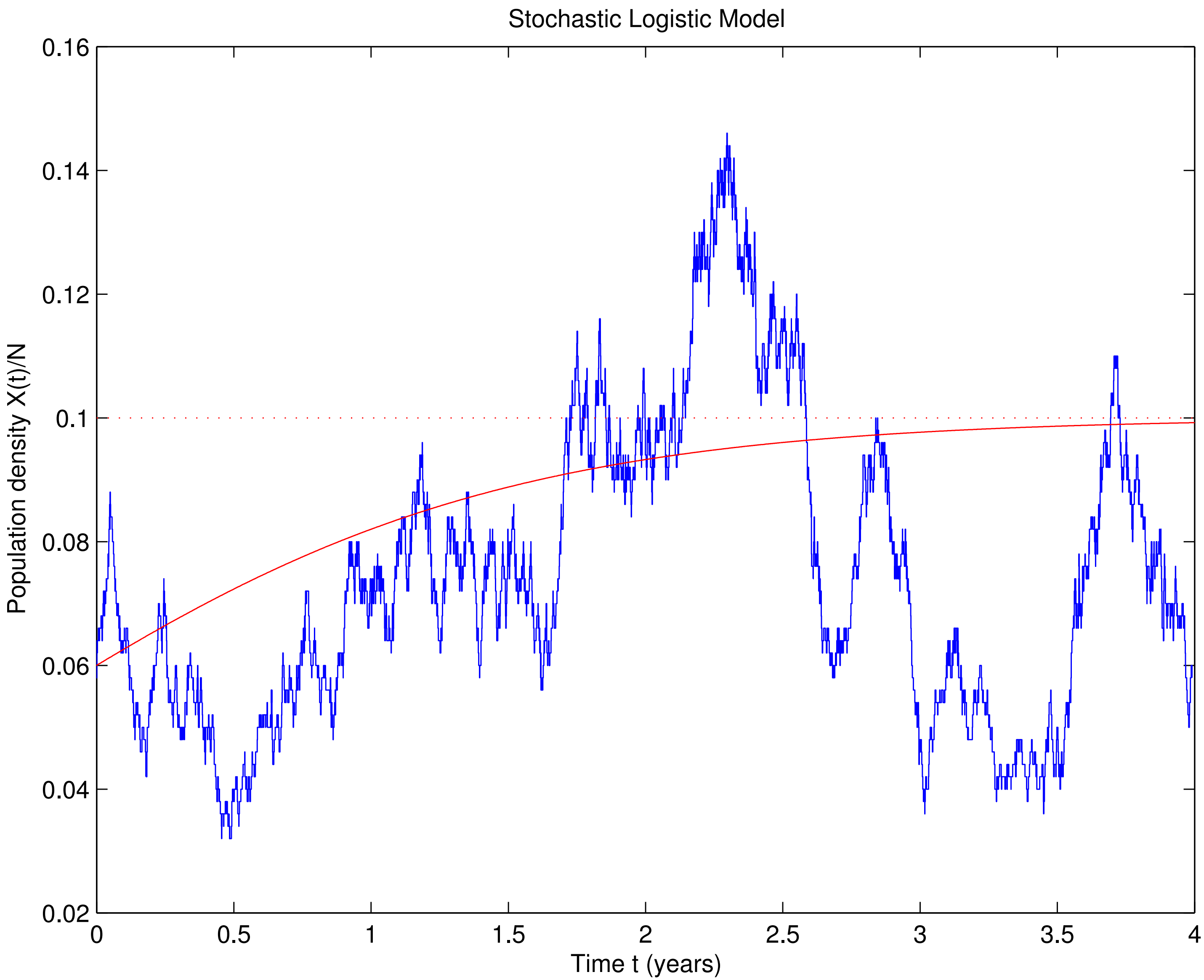
We learn that

$$p(t) = \frac{mp_0}{np_0 + (m - np_0)e^{-mt}}, \qquad t \geq 0.$$

For us,

$$\frac{X(t)}{N} \sim \frac{(1-\rho)x_0}{x_0 + (1 - \rho - x_0)e^{-\lambda(1-\rho)t}}, \quad \text{where } \rho = \frac{\mu}{\lambda}.$$

A population process



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$$\lambda_i = \frac{\lambda}{N} i(N - i) \quad \text{and} \quad \mu_i = \mu i,$$

where $\lambda, \mu > 0$.

The *epidemic model* mentioned earlier is a two-dimensional Markov chain with transition rates

$$q_{(s\ i),(s+1\ i)} = \alpha s, \qquad q_{(s\ i),(s\ i-1)} = \gamma i,$$

$$q_{(s\ i),(s-1\ i+1)} = \beta s i,$$

where $\alpha, \gamma, \beta > 0$ are the *splitting*, *removal* and *infection* rates.

The expected value of Γ

Returning to our general Markov chain, let $e_i = E_i(\Gamma) := E(\Gamma | X(0) = i)$, and condition on the time of the first jump and the state visited at that time, to get

$$E_i(\Gamma) = \int_0^\infty \sum_{k \neq i} \left(\frac{f_i}{q_i} + E_k(\Gamma) \right) \frac{q_{ik}}{q_i} q_i e^{-q_i u} \, du,$$

which leads to

$$q_i e_i = f_i + \sum_{k \neq i} q_{ik} e_k,$$

so that

$$\sum_k q_{ik} e_k + f_i = 0.$$

The expected value of Γ

We can do better:

Theorem 1 $\boldsymbol{e} = (e_i,\, i \in A)$, where $e_i = E_i(\Gamma)$, is the *minimal* non-negative solution to

$$\sum_{k \in A} q_{ik} z_k + f_i = 0, \quad i \in A,$$

in the sense that $\boldsymbol{e}$ satisfies these equations, and, if $\boldsymbol{z} = (z_i,\, i \in A)$ is any non-negative solution, then $e_i \leq z_i$ for all $i \in A$.

The expected value of Γ

So, we solve a system of linear equations to obtain the vector of expected total costs starting in the various states:

$$Q\boldsymbol{z} = -\boldsymbol{f}$$

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Transition rates restricted to A (the model)

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Transition rates restricted to A (the model)

Unit costs

The expected value of Γ

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$$Q\boldsymbol{z} = -\boldsymbol{f}$$

Transition rates restricted to A (the model)

Expected total cost (minimal solution)

Unit costs

Birth-death processes

Let's apply this to *birth-death processes*:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Assume that the birth rates $(\lambda_i,\ i \geq 1)$ and the death rates $(\mu_i,\ i \geq 0)$ are all strictly positive, except that $\lambda_0 = 0$. So, all states in $A = \{1, 2, \ldots\}$ intercommunicate, and 0 is an absorbing state (corresponding to population extinction).

Birth-death processes

Define $(\pi_i,\, i \geq 1)$ by $\pi_1 = 1$ and

$$\pi_i = \prod_{j=2}^{i} \frac{\lambda_{j-1}}{\mu_j}, \qquad i \geq 2,$$

and assume that

$$\sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} = \infty,$$

a condition that corresponds to extinction being certain.

Birth-death processes

On applying Theorem 1 we get:

Proposition The expected cost up to the time of extinction, starting in state i (≥ 1), is given by

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{\infty} f_k \pi_k,$$

this being finite if and only if $\sum_{k=1}^{\infty} f_k \pi_k < \infty$.

Birth-death processes

In the finite state-space case ($S = \{0, 1, \ldots, N\}$), we get

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{N} f_k \pi_k, \qquad i = 1, 2, \ldots, N.$$

For the Stochastic Logistic Model,

$$E_i(\Gamma) = \frac{1}{\mu} \sum_{j=1}^{i} \sum_{k=0}^{N-j} \left(\frac{1}{N\rho}\right)^k \frac{f_{j+k}}{j+k} \frac{(N-j)!}{(N-j-k)!},$$

where $\rho = \mu/\lambda$. If $\rho < 1$ (the interesting case),

$$E_i(\Gamma) \sim \frac{\rho}{\mu(1-\rho)} \left(\frac{e^{-(1-\rho)}}{\rho}\right)^N \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{i} f_j \rho^j \quad \text{as } N \to \infty.$$

The distribution of Γ

Can we evaluate the *distribution* of Γ, that is,

$$\Pr(\Gamma \le x | X(0) = i)\,?$$

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$$\Pr(\Gamma \leq x | X(0) = i)\,?$$

I will explain how to evaluate $y_i(\theta) = E_i(e^{-\theta\Gamma})$, the Laplace-Steiltjes Transform (LST) of the distribution:

$$y_i(\theta) = \int_0^\infty e^{-\theta x}\, d\Pr(\Gamma \leq x | X(0) = i).$$

The distribution of Γ

An argument similar to that used to evaluate $E_i(\Gamma)$ leads to:

Theorem 2 For each $\theta > 0$, $\boldsymbol{y}(\theta) = (y_i(\theta), i \in S)$ is the *maximal* solution to

$$\sum_{k \in S} q_{ik} z_k = \theta f_i z_i, \quad i \in A,$$

with $0 \le z_i \le 1$ for $i \in A$ and $z_i = 1$ for $i \notin A$.

A catastrophe process

Assume that the transition rates have the form

$$q_{ij} = \begin{cases} i\rho a, & i \geq 0,\ j = i+1, \\ -i\rho, & i \geq 0,\ j = i, \\ i\rho d_{i-j}, & i \geq 2,\ 1 \leq j < i, \\ i\rho \sum_{k \geq i} d_k, & i \geq 1,\ j = 0, \end{cases}$$

with all other transition rates equal to 0. Here ρ and a are positive, d_i is positive for at least one i in $A = \{1, 2, \ldots\}$ and $a + \sum_{i=1}^{\infty} d_i = 1$.

Clearly 0 is an absorbing state for the process and A is a communicating class.

A catastrophe process

We will consider only the *subcritical case*, where the drift D, given by $D = a - \sum_{i=1}^{\infty} i d_i$, is strictly negative and extinction is certain.

Let $b(s) = d(s) - s$, where d is the probability generating function $d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}$, $|s| < 1$.

There is a unique solution, σ, to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

We can evaluate $E_i(e^{-\theta\Gamma})$ for specific choices of f.

For example, take $f_i = i$.

We seek the maximal solution to

$$\sum_{j=0}^{\infty} q_{ij} z_j = \theta i z_i, \qquad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

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We seek the maximal solution to

$$\rho a z_{i+1} - \rho z_i + \rho \sum_{j=1}^{i-1} d_{i-j} z_j + \rho z_0 \sum_{j=i}^{\infty} d_j = \theta z_i, \quad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

Multiplying by s^{i-1} and summing over i gives

$$\sum_{i=1}^{\infty} E_i(e^{-\theta\Gamma})s^{i-1} = \frac{1}{1-s} - \frac{\theta(\gamma_\theta - s)}{(1-\gamma_\theta)(1-s)(\rho b(s) - \theta s)},$$

where γ_θ is the unique solution to $\rho b(s) = \theta s$ on the interval $0 < s < \sigma$, where σ itself is the unique solution to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

In the case of "geometric catastrophes" ($d_i = d(1-q)q^{i-1}$, $i \geq 1$, where $d > 0$ satisfies $a + d = 1$, and $0 \leq q < 1$), we get

$$E_i(e^{-\theta\Gamma}) = \frac{\beta(\theta) - q}{1-q} \left(\beta(\theta)\right)^{i-1}, \quad i \geq 1,$$

where $\beta(\theta)$ is the smaller of the two zeros of $a\rho s^2 - (\rho(1+qa)+\theta)s + \rho(d+qa) + q\theta$.

Workshop

ARC Centre of Excellence for Mathematics and Statistics of Complex Systems

Workshop on Metapopulations

The University of Queensland
Thursday 2nd September 2004

Invited speakers: Andrew Barbour (University of Zürich) Ben Cairns, Phil Pollett, Hugh Possingham, Tracey Regan, Joshua Ross, Severine Vuilleumier and Chris Wilcox (University of Queensland).

URL: http://www.maths.uq.edu.au/~pkp/MetaPop04.html