A Note on the Existence and Uniqueness of a Bounded Mean-Reverting Process

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February 9, 2008

Abstract

We study a stochastic differential equation (SDE) describing a class of mean-reverting diffusions on a bounded interval. The drift coefficient is not continuous near the boundaries. Nor does it satisfy either of the usual Lipschitz or linear growth conditions. We characterize the boundary behaviour, identifying two possibilities: entrance boundary and regular boundary. In the case of an entrance boundary we establish existence and uniqueness of the solution to the SDE.

1 Introduction

We study the behaviour of a mean-reverting process described by the SDE

\[ dS(t) = -\mu \frac{S(t)}{1 - S(t)^2} dt + \sigma dB(t), \quad S(0) = S_0, \]  

where \( B(t) \) is one-dimensional Brownian motion, and \( \mu \) and \( \sigma \) are positive constants. The process behaves like the ubiquitous Ornstein-Uhlenbeck (OU) process when \( S(t) \) is close to zero, but as \( S(t) \) gets close to the boundaries at \( \pm 1 \), the influence of the drift increases. Consequently (1) will be useful for modelling phenomena that have inherent maxima and minima, such as the difference in two proportions (see for example [6], where the difference in proportions of voters for two major political parties is modelled using (1)). We identify two cases: (1) \( \mu < \sigma^2 \), where the boundaries are regular, and (2) \( \mu \geq \sigma^2 \), where both are entrance boundaries and the process is driven away from the boundaries. In the practically important case (2) the solution \( S(t) \) remains within \((-1, 1)\) whenever it starts there; we note that for the data reported in [7] concerning Australian Federal elections (1993-2002), the maximum likelihood estimates of \( \mu \) and \( \sigma \) were 3.98 and 0.28, giving an estimate of 50.765 for the ratio \( \mu/\sigma^2 \).

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The drift coefficient of our SDE is not continuous near the boundaries, and it does not satisfy either of the (standard) linear growth condition or the Lipschitz condition, which guarantee existence and then uniqueness of a solution (see Sections 5 and 6 of [5]). Despite this, we prove that there exists uniquely a solution to (1) when $\mu \geq \sigma^2$. We then extend this result to the case where numerator of the drift coefficient is a polynomial function of $S(t)$.

We begin by examining the corresponding deterministic model, and prove that the exact solution and an approximating solution remain in ($-1, 1$) whenever they start there.

2 The deterministic model

The deterministic model corresponding to (1) is

$$\frac{dS}{dt} = -\mu \frac{S}{1 - S^2}, \quad S(0) = S_0. \tag{2}$$

Its solution satisfies

$$Se^{-\frac{1}{2}(s^2-s_0^2)} = S_0e^{-\mu t}. \tag{3}$$

Since the right-hand side of (2) can be written as a Taylor series

$$-\mu \frac{S}{1 - S^2} = -\mu(S + S^3 + S^5 + \cdots), \quad -1 < S < 1,$$

we consider the related initial value problem

$$\frac{dS}{dt} = -\mu(S + S^3 + \cdots + S^{2N+1}), \quad S(0) = S_0, \tag{4}$$

and ask if the solution $S_N$ of (4) tends to the solution $S_a$ of (2), as $N \to \infty$. We begin by showing that both solutions remain in ($-1, 1$).

**Lemma 2.1** Provided $|S_0| < 1$ and $\mu > 0$, the solution $S(t)$ to (2) or (4) satisfies

$$|S(t)| \leq |S_0|, \quad \forall t. \tag{5}$$

**Proof:** We use the Lyapunov function $V(S) = S^2$, which is positive definite in ($-1, 1$). The orbital derivative of $V$ is given by

$$\dot{V}(t) = \nabla V(S) \dot{S}(t) = 2S \left( -\mu \frac{S}{1 - S^2} \right) = -2\mu \frac{S^2}{1 - S^2} (\leq 0).$$

Since this derivative is negative semidefinite, 0 is stable. That means, starting with $|S_0| < 1$, the solution to (2) or (4) remains in ($-S_0, S_0$). □

**Theorem 2.2** Given any $T > 0$, $-1 < S_0 < 1$ and small $\delta > 0$, there exists an $N \in \mathbb{N}$ such that $\|S_a - S_N\| < \delta$ over $[0, T]$, where $S_N$ solves (4) and $S_a$ solves (2).
The last inequality holds when 

Using the solution to (2), we can simplify the error term and deduce that 

The first term in the numerator of the right-hand side is obtained using the fact that 

Denoting the right-hand side of (2) and (4) by 

On substituting the right-hand side of (4) we get the approximation 

Proof: The solution of (2) is given by the function implicitly defined in (3). Differentiating both sides of (3) gives 

that is, 

On substituting the right-hand side of (4) we get the approximation 

its error being 

The first term in the numerator of the right-hand side is obtained using the fact that 

Using the solution to (2), we can simplify the error term and deduce that 

as \( N \to \infty \) (since \(-1 < S < 1\)). Now define 

We have 

The last inequality holds when \( N \) is large, since the solution is always between \(-1\) and \(1\).

Denoting the right-hand side of (2) and (4) by \( f(S) \) and \( g_n(S) \), respectively (with the same initial condition), we have two initial value problems, whose solutions are known to be asymptotically stable. We can write (2) as \(|S'_N - f(S_N)| = 0\) and (4) as \(|S'_N - g_n(S_N) - (g_n(S_N) - f(S_N))| = 0\). Hence, 

\[
|S'_N - f(S_N)| = |g_n(S_N) - f(S_N)| \leq \mu \sum_{i=N}^{\infty} S^{2i+3}_N = \mu S^{2N+3}_N (1 - S^2_N),
\]
which can be made arbitrarily small because \(|S_N| \leq S_0\). Indeed, for \(N\) sufficiently large, \(|S'_N - f(S_N)| < \epsilon\), for all \(t > 0\). It then follows from the fundamental inequality ([3], p. 171) that

\[
|S_a(t) - S_N(t)| \leq \frac{\epsilon}{K} (e^{K|t-t_0|} - 1),
\]

where \(\epsilon\) is small whenever \(N\) is large, and \(K\) is the Lipschitz constant

\[
K := \sup_{-S_0 \leq S^* \leq S_0} \left| \frac{d}{dS} \left( -\mu S^* + \frac{S}{1 - S^2} \right) \right|_{S=S^*}.
\]

By taking \(N\) large, \(\epsilon\) can be made arbitrarily small, and hence \(\|S_a - S_N\| < \delta\) over any bounded time interval.

\[\square\]

### 3 Boundary behaviour and stationary density

We will classify the boundary points, \(-1\) and \(1\), according to the classification laid out in Section 15.6 of [4]. It will be clear that \(-1\) and \(1\) share the same classification, and so we will focus our attention on the upper boundary point. We will see that the boundary behaviour is characterized simply in terms of the ratio \(\mu/\sigma^2\).

For regular diffusions on an interval \(D = (l, u)\) satisfying an SDE of the form

\[
dS(t) = b(S(t)) \, dt + \sigma(S(t)) \, dB(t), \quad S(0) = S_0 \in D,
\]

the classification of the boundary points \(l\) and \(u\) is given in terms of the scale function and scale measure, and the speed density and speed measure (see [4], pp. 226–227). The scale function \(H\) is given by

\[
h(s) = \exp \left\{ -\int_{x_0}^{s} \frac{2b(\eta)}{\sigma^2(\eta)} \, d\eta \right\}, \quad s \in D,
\]

and \(x_0\) and \(\xi_0\) are arbitrary values in \(D\). We write \(H[I] = H[c, d] = H(d) - H(c)\) for scale measure of the interval \(I = [c, d] \subset D\), and then \(H[c, u]\) for \(\lim_{d \uparrow u} H[c, d]\). The interpretation is as follows. If \(H[c, u] < \infty\) for some (and then all) \(c \in D\), then there is a positive probability that the diffusion reaches \(u\) before \(c\) starting from \(x \in (c, u)\); the boundary is said to be attracting. If \(H[c, u] = \infty\) for some (and then all) \(c \in D\), then the probability of reaching \(u\) before \(c\) is 0, and the boundary is non-attracting.

For the SDE (1), we have the following result.

**Lemma 3.1** Both boundaries are attracting, or both are non-attracting according as \(\mu < \sigma^2\) or \(\mu \geq \sigma^2\).

**Proof:** Taking \(\xi_0 = 0\), we have, for all \(s \in (-1, 1)\),

\[
H(s, 1) = \lim_{b \downarrow 1} \int_s^b \exp \left\{ \int_s^\xi \frac{2\mu}{\sigma^2} \left( \frac{\eta}{1 - \eta^2} \right) \, d\eta \right\} \, d\xi = \lim_{b \downarrow 1} \int_s^b \exp \left\{ -\frac{\mu}{\sigma^2} \left[ \log(1 - \eta^2) \right]_0^\xi \right\} \, d\xi
\]

\[
= \lim_{b \downarrow 1} \int_s^b (1 - \xi^2)^{-\mu/\sigma^2} \, d\xi = \lim_{b \downarrow 1} \int_s^b (1 + \xi)^{-\mu/\sigma^2} (1 - \xi)^{-\mu/\sigma^2} \, d\xi.
\]
Thus $H[s, 1]$ is bounded below by

$$\lim_{b \uparrow 1} (1 + b)^{-\mu/\sigma^2} \int_s^b (1 - \xi)^{-\mu/\sigma^2} d\xi,$$

which is infinite if $\mu \geq \sigma^2$, and bounded above by

$$(1 + s)^{-\mu/\sigma^2} \lim_{b \uparrow 1} \int_s^b (1 - \xi)^{-\mu/\sigma^2} d\xi,$$

which is finite if $\mu < \sigma^2$.

Next, for a regular diffusion satisfying (6), the \textit{speed density} $m$ and \textit{speed measure} $M$ are given by

$$m(s) = \frac{1}{\sigma^2(s)h(s)}, \quad s \in D,$$

and

$$M[I] = M[c, d] = \int_c^d m(s) \, ds,$$

where $I = [c, d] \subset D$ and then $M[c, u] = \lim_{d \uparrow u} M[c, d]$. ($M[c, u]$ is either finite for all $c \in D$ or infinite for all $c \in D$). Also define

$$\Sigma(u) = \int_s^u M[s, \eta]dH(\eta) = \int_s^u H(\eta, u)dM(\eta)$$

and

$$N(u) = \int_s^u H[s, \eta]dM(\eta) = \int_s^u M[\eta, u)dH(\eta),$$

for arbitrary $s$ in $D$ (whether each is finite or infinite does not depend on $s$). The interpretation is as follows. If $u$ is attracting, then $\Sigma(u) < \infty$ if and only if the expected time to leave the interval $(s, u)$ starting from $x \in (s, u)$ is finite; we say that $u$ is \textit{attainable} (otherwise \textit{unattainable}). Also, $N(u) < \infty$ if and only if the expected time taken to reach an interior point $s$ starting from the boundary point $u$ is finite. In particular, an attainable (and hence attracting) boundary is called \textit{regular} if $N(u) < \infty$ (otherwise \textit{absorbing}), while a non-attracting (and hence unattainable) boundary is called an \textit{entrance} boundary\footnote{Feller nomenclature.} if $N(u) < \infty$.

For the SDE (1), we have the following result.

\textbf{Lemma 3.2} If $\mu < \sigma^2$ then both boundaries are regular, while if $\mu \geq \sigma^2$ then both are entrance boundaries.

\textbf{Proof:} Suppose $\mu < \sigma^2$. We have already seen that $H[s, 1] < \infty$. Therefore, because $\Sigma(1) + N(1) = H[s, 1]M[s, 1]$, it suffices to prove that $M[s, 1] < \infty$. Again putting $\xi_0 = 0$, we have

$$h(s) = \exp \left\{ - \int_0^s \frac{2b(\eta)}{\sigma^2(\eta)} \, d\eta \right\} = \exp \left\{ \int_0^s \frac{2\mu\eta}{(1 - \eta^2)\sigma^2} \, d\eta \right\} = (1 - s^2)^{-\mu/\sigma^2},$$

$$m(s) = \frac{1}{\sigma^2(s)h(s)} = \frac{1}{\sigma^2} (1 - s^2)^{\mu/\sigma^2},$$

$\Box$
and hence for any \( s \in (-1, 1) \),
\[
M(s, 1) = \lim_{b \uparrow 1} \frac{1}{\sigma^2} \int_s^b (1 - \xi^2)^{\mu/\sigma^2} \, d\xi = \frac{1}{\sigma^2} \lim_{b \uparrow 1} \int_s^b (1 - \xi)^{\mu/\sigma^2} (1 + \xi)^{\mu/\sigma^2} \, d\xi \\
\leq \frac{1}{\sigma^2} \lim_{b \uparrow 1} (1 + b)^{\mu/\sigma^2} \int_s^b (1 - \xi)^{\mu/\sigma^2} \, d\xi = \frac{2^{\mu/\sigma^2}}{\mu + \sigma^2} (1 - s)^{1+\mu/\sigma^2} < \infty.
\]
(Notice that \( M(s, 1) \) is finite whether or not \( \mu < \sigma^2 \).

Now suppose that \( \mu \leq \sigma^2 \), so that \( H(s, 1) = \infty \). We need to prove that \( N(1) < \infty \).

Using the bound immediately above,
\[
N(1) = \int_s^1 M(\xi, 1) h(\xi) d\xi \leq \frac{2^{\mu/\sigma^2}}{\mu + \sigma^2} \int_s^1 (1 - \xi)^{1+\mu/\sigma^2} (1 - \xi^2)^{-\mu/\sigma^2} \, d\xi \\
= \frac{2^{\mu/\sigma^2}}{\mu + \sigma^2} \int_s^1 (1 - \xi) (1 + \xi)^{-\mu/\sigma^2} \, d\xi \\
\leq \frac{2^{\mu/\sigma^2}}{\mu + \sigma^2} \int_s^1 (1 - \xi) \, d\xi = \frac{2^{\mu/\sigma^2-1}}{\mu + \sigma^2} (1 - s)^2 < \infty.
\]
This completes the proof. 

The following is a simple consequence of Lemma 3.2.

**Corollary 3.3** If \( \mu \geq \sigma^2 \), then the process \( S(t) \) satisfying (1) remains in \((-1, 1)\) if it starts there.

Finally, for a diffusion satisfying (1) the unique stationary density is given by the normalized speed density, which for our model is
\[
\pi(s) = \frac{(1 - s^2)^{\mu/\sigma^2}}{\int_{-1}^1 (1 - u^2)^{\mu/\sigma^2} \, du} = \frac{\Gamma(1 + \frac{\mu}{\sigma^2} + \frac{1}{2})}{\Gamma(1 + \frac{\mu}{\sigma^2}) \Gamma(\frac{1}{2})} (1 - s^2)^{\mu/\sigma^2}, \quad -1 < s < 1,
\]
being the symmetric Beta density on \((-1, 1)\) with parameter \( \beta = 1 + \mu/\sigma^2 \). The interpretation is obvious for the diffusion process of Corollary 3.3, but \( \pi \) also admits a stationary interpretation when the boundary is regular: \( \pi \) is the stationary density of the process obtained through instantaneous reflection at the boundaries. And, not unexpectedly, these two cases correspond to the light and heavy tail behaviour, respectively, exhibited by (7).

### 4 Uniqueness of the Solution

Next we tackle the question of the uniqueness of the solution to (1) in the case when \( \mu \geq \sigma^2 \). Györögy [1] considered the general SDE on a given stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}(\mathcal{F}_t)_{t \geq 0})\)
\[
dS(t) = b(t, S(t)) \, dt + \sigma(t, S(t)) \, dB(t), \quad S(0) = S_0,
\]
in a domain \( D \) of \( \mathbb{R}^d \), where \((B(t, \mathcal{F}_t))\) is a \( d_1 \)-dimensional Wiener process, \( S_0 \) is an \( \mathcal{F}_0 \)-measurable random vector in \( D \); \( b \) and \( \sigma \) are Borel functions on \( \mathbb{R}_+ \times D \) taking values
in $\mathbb{R}^d$ and in $\mathbb{R}^{d \times d_1}$. Under Gyöngy’s assumptions (given below), the solution of (8) will never leave $D$. Therefore, the values of $\sigma$ and $b$ outside $D$ are irrelevant, and thus can be defined as $\sigma(t, s) = 0, b(t, s) = 0$ for $s \notin D, t \geq 0$.

The three assumptions used by Gyöngy are as follows:

(a) There exists an increasing sequence $\{D_k\}_{k=1}^{\infty}$ of bounded domains with $\cup_{k=1}^{\infty} D_k = D$ and, for every $k$ and $t \in [0, k],$

$$\sup_{s \in D_k} |b(t, s)| \leq M_k \quad \text{and} \quad \sup_{s \in D_k} |\sigma(t, s)|^2 \leq M_k, \quad (9)$$

where $M_k$ is a constant.

(b) There exists a non-negative function $V \in C^{1,2}(\mathbb{R}_+ \times D)$ such that

$$LV(t, s) \leq MV(t, s), \quad \forall t \in [0, T], s \in D,$$

$$V_k(T) := \inf_{s \in \partial D_k, t \leq T} V(t, s) \to \infty,$$

as $k \to \infty$ for every finite $T$, where $M = M(T)$ is a constant, $\partial D_k$ denotes the boundary of $D_k$, and $L$ is a differential operator

$$L := \frac{\partial}{\partial t} + \sum_i b_i(t, s) \frac{\partial}{\partial s^i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(t, s) \frac{\partial^2}{\partial s^i \partial s^j}.$$

(c) $P(S_0 \in D) = 1$.

We approximate the solution to (8) using the Euler ‘polygonal’ approximation, defined to be the diffusion process $(S_n(t))$ satisfying the SDE

$$dS_n(t) = b(t, S_n(\kappa_n(t))) \, dt + \sigma(t, S_n(\kappa_n(t))) \, dB(t), \quad S_n(0) = S_0,$$

with $\kappa_n(t) := t_n^i := i/n$ for $t \in [t^n_0, t^n_{i+1})$. In order to establish convergence of the approximation, Gyöngy also assumed that $b$ should satisfy the monotonicity condition. A function $f : \mathbb{R}_+ \times D \to \mathbb{R}$ satisfies the monotonicity condition in $D \subseteq \mathbb{R}$ if, for every positive real number $T$ and for every integer $k \geq 1$, there is a function $f_k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $f_k(t, s) = f(t, s)$ for $t \in [0, T], s \in D_k$ and $(s - u)(f_k(t, s) - f_k(t, u)) \leq C_k |s - u|^2$, for all $t \geq 0, s, u \in \mathbb{R}$ where $C_k$ is a constant.

Assuming further that $b$ is continuous in $D$ and $\sigma$ is locally Lipschitz in $D$, Gyöngy proved that $S_n(t)$ converges to a process $S(t)$ almost surely, uniformly in $t$ on bounded intervals, and that $S(t)$ is the unique solution of (8). Higham, et al. [2] showed that an implicit variant of Euler-Maruyama approximation also converges if the diffusion coefficient is globally Lipschitz but the drift coefficient only satisfies one-sided Lipschitz condition.

**Lemma 4.1** Let $D = (-1, 1)$. If $P(S_0 \in D) = 1$ and $\mu \geq \sigma^2$, then there exists uniquely a solution to (1).
Proof: In our case we have $D = (-1, 1)$, $b(t, s) = -\mu s/(1 - s^2)$ and $\sigma(t, s) = \sigma$ for positive constants $\mu$ and $\sigma$, and hence

$$\mathcal{L} = \frac{\partial}{\partial t} - \mu \left( \frac{s}{1 - s^2} \right) \frac{\partial}{\partial s} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial s^2}.$$  

We will first confirm that (1) satisfies Conditions (a) and (b) above (Condition (c) is an assumption).

Let $D_k$ be the interval $(-1 + 2^{-k}, 1 - 2^{-k})$. Then, $\{D_k\}_{k=1}^\infty$ is an increasing sequence of bounded domains such that $\cup_{k=1}^\infty D_k = D$. Since $s \in (-1, 1)$ and $\sigma$ is constant, we can always find constants $M_k$ appropriate for (9) to hold for every $k$ and $t \in [0, k]$. Thus, Condition (a) holds. Next, taking the non-negative function $V$ to be $V(t, s) := \ln \left( \frac{2 + s^2}{1 - s^2} \right)$, the first part of Condition (b) stipulates that

$$\frac{3\sigma^2(3s^4 + s^2 + 2) - 6\mu s^2(2 + s^2)}{(2 + s^2)^2(1 - s^2)^2} \leq M \ln \left( \frac{2 + s^2}{1 - s^2} \right).$$

Rearranging gives $M(2 + s^2)^2(1 - s^2)^2 \ln \left( \frac{2 + s^2}{1 - s^2} \right) + 6\mu s^2(2 + s^2) - 3\sigma^2(3s^4 + s^2 + 2) \geq 0$.

The condition $\mu \geq \sigma^2$ appears by taking $s \to 1$ in the last inequality. When $s = 0$, we should have $M \geq \frac{3\sigma^2}{2\ln 2}$. For any other values of $s \in (-1, 1)$, we should have

$$M \geq \frac{3\sigma^2(3s^4 + s^2 + 2) - 6\mu s^2(2 + s^2)}{(2 + s^2)^2(1 - s^2)^2 \ln \left( \frac{2 + s^2}{1 - s^2} \right)}.$$

The second part of (b) holds because, for any $T > 0$,

$$V_k(T) = \min \left\{ \ln \left( \frac{2 + (2^{-k} - 1)^2}{1 - (2^{-k} - 1)^2} \right), \ln \left( \frac{2 + (1 - 2^{-k})^2}{1 - (1 - 2^{-k})^2} \right) \right\} = \ln \left( \frac{3 + 2 - 2^{-k} - 2^{-k+1}}{1 - (1/2^{k+1})^2} \right) \to \infty,$$

as $k \to \infty$.

Since our $b$ clearly satisfies the monotonicity condition in $D = (-1, 1)$, and is continuous in $D$, and $\sigma(t, s)$ being constant in $D$ is locally Lipschitz in $D$, we may employ Theorem 2.3 of [1] to conclude that there is a unique solution to (1).  

From Corollary 3.3 and Lemma 4.1 we have the following.

Corollary 4.2 If $P(S_0 \in D) = 1$, the moments of $S(t)$ are bounded for all $t$.

5 An extension

We extend the results of the previous section to an SDE obtained by replacing the numerator $S$ of the drift coefficient in (1) by a polynomial in $S$:

$$dS(t) = -\mu \frac{f(S(t))}{1 - S(t)^2} \, dt + \sigma dB(t), \quad S(0) = S_0,$$

where $f$ is a polynomial of degree $n$:

$$f(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n.$$  

We assume that $P(S_0 \in D) = 1$, where $D = (-1, 1)$, and $\mu \geq C\sigma^2$ for a positive constant $C$. Our aim is to prove existence and uniqueness, again using Theorem 2.3 of [1].
Theorem 5.1 If the $a_i$ of (11) satisfy $\sum_{i=0}^{\infty} |a_i| < \infty$, then there exists uniquely a solution to (10).

Proof: Since $-1 < s < 1$, we can find an upper bound for $f(s)$ and also for $b(t, s)$, so that the monotonicity condition is always satisfied. Also, $b(t, s)$ is continuous in $D$ and $\sigma$ is locally Lipschitz in $D$. As before, we define $D_k := (-1 + 2^{-k}, 1 - 2^{-k})$ to ensure that Győngy’s Condition (a) holds. To verify Condition (b) we take $V(t, s) := 1/(1 - s^2)$ and show that there is a constant $M$ for which
\[
MV(t, s) - LV(t, s) \geq 0, \quad \forall t \in [0, T], \ s \in D,
\]
where now
\[
L = \frac{\partial}{\partial t} - \mu f(s) - \frac{\partial}{(1 - s^2)} \sigma \frac{\partial^2}{\partial s^2}.
\]
Using (11) the inequality becomes
\[
M(1 - s^2)^2 + 2\mu s(a_0 + a_1 s + \cdots + a_n s^n) - \sigma^2(1 + 3s^2) \geq 0.
\] (12)
Since $-1 < s < 1$, the first term of (12) is always positive but the third term is always negative. The second term is bounded above by $2\mu s \sum_{i=1}^{n} |a_i|$. So, since $\sum_{i=1}^{\infty} |a_i|$ is finite, we can always find a positive number $M$ such that (12) holds.

Further extension are possible. For example, if $f$ is a continuous function and has a Taylor series expansion at $s = s_a$ where $|s_a| < 1$, then an upper bound for $f(s)$ can be found and we can show that there exists a unique solution to (10) with $f(s)$ is given by (13). If the $n^{th}$ derivative of $f$ at $s = s_a$ is finite, then $f(s)$ can be written as
\[
f(s) = f(s_a) + f'(s_a)(s - s_a) + f''(s_a)\frac{(s - s_a)^2}{2!} + \cdots + f^{(n-1)}(s_a)\frac{(s - s_a)^{n-1}}{(n-1)!} + f^{(n)}(t_a)\frac{(s - s_a)^n}{n!},
\]
for some $t_a \in (s_a, s)$.

Following the same approach as the proof of Theorem 5.1, and taking the same non-negative function $V(t, s)$, we will end up with the following inequality:
\[
M(1 - s^2)^2 + 2\mu s \left( f(s_a) + f'(s_a)(s - s_a) + \cdots + f^{(n-1)}(s_a)\frac{(s - s_a)^{n-1}}{(n-1)!} + f^{(n)}(t_a)\frac{(s - s_a)^n}{n!} \right) - \sigma^2(1 + 3s^2) \geq 0.
\] (14)
Since the absolute values of both $s$ and $s_a$ are less than one and the $n^{th}$ derivative of $f(s)$ at $s = s_a$ is finite for $n = 0, 1, 2, \ldots$, then the second term of (14) is bounded above by
\[
2\mu s \max_{n} f^{(n)}(s_a) \sum_{n=0}^{\infty} \frac{2^n}{n!},
\]
which is clearly finite. Therefore, we can always find a positive number $M$ such that (14) holds.
6 Conclusions

We have classified the boundary behaviour of a mean-reverting SDE whose drift coefficient does not satisfy neither the Lipschitz nor linear growth conditions. The boundary behaviour is either entrance or regular boundary depending on the values of $\mu$ and $\sigma^2$ (whether $\mu \geq \sigma^2$ or $\mu < \sigma^2$). Existence and uniqueness of the solution of the SDE was established when the boundary is an entrance boundary ($\mu \geq \sigma^2$).

Extensions were also considered by first replacing the numerator $S$ of the drift coefficient of the SDE by a polynomial $f(S)$ of degree $n$ and then establishing existence and uniqueness of the solution. A further extension was to consider $f(S)$ as a continuous function and has a Taylor series expansion at $s = s_a$ with $|s_a| < 1$. In both extensions, uniqueness of the solution can be proved under some conditions on the coefficients of the polynomial.

Acknowledgements We thank Thomas Taimre for helpful comments. Dharma Lesmono acknowledges the support of the Department of Education, Science and Training (DEST) Australia, which awarded him an Endeavour-Indonesia Research Fellowship to work at the University of Queensland. The work of Phil Pollett is supported by the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems. Kevin Burrage would like to thank the Australian Research Council for its funding of a Federation Fellowship.

References


