EXTINCTION TIMES FOR A GENERAL BIRTH, DEATH AND CATASTROPHE PROCESS

BEN CAIRNS * ** AND
P. K. POLLETT,* *** University of Queensland

Abstract
The birth, death and catastrophe process is an extension of the birth–death process that incorporates the possibility of reductions in population of arbitrary size. We will consider a general form of this model in which the transition rates are allowed to depend on the current population size in an arbitrary manner. The linear case, where the transition rates are proportional to current population size, has been studied extensively. In particular, extinction probabilities, the expected time to extinction, and the distribution of the population size conditional on nonextinction (the quasi-stationary distribution) have all been evaluated explicitly. However, whilst these characteristics are of interest in the modelling and management of populations, processes with linear rate coefficients represent only a very limited class of models. We address this limitation by allowing for a wider range of catastrophic events. Despite this generalisation, explicit expressions can still be found for the expected extinction times.

Keywords: Catastrophe process; persistence time; hitting time

2000 Mathematics Subject Classification: Primary 60J27
Secondary 92B05; 60J80

1. Introduction
Accounting for catastrophic events has become an important part of stochastic population modelling, particularly in ecology, but also in an array of other fields, including economics, chemistry and telecommunications. In the context of population processes, catastrophes are sudden declines in the population and, according to Shaffer [13] and others, they are one of the primary sources of variation in the abundance of species. The use of catastrophe processes, such as those described here, is not limited to studying the numbers of individuals in a population. Mangel and Tier [7], for example, have used a catastrophe process to model the number of occupied habitat patches in a metapopulation. For a review of the significance of catastrophes in ecological modelling, see [12].

Of primary importance is the effect of catastrophes on the persistence of a population, and in particular their effect on the expected time to extinction. Recent work, beginning with Brockwell et al. [3], examines extinction probabilities, conditions for eventual extinction, and expected extinction times in a variety of different cases. Brockwell [2] laid out a programme for evaluating the probability of extinction and the Laplace transform of the distribution of the time to extinction for a general birth, death and catastrophe process. As Anderson pointed out (see Section 9.2 of [1]), Brockwell’s argument can be used to evaluate expected extinction times, and

Received 3 October 2003; revision received 2 March 2004.
* Postal address: Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia.
** Email address: bjc@maths.uq.edu.au
*** Email address: pkp@maths.uq.edu.au
indeed his entire programme extends, at least in principle, to any upwardly skip-free Markov chain. However, explicit results can only be obtained in particular cases. Here we examine one such case, in which the transition rates are allowed to depend on the current population size. Our main result gives an explicit expression for the expected extinction times. We illustrate our result with several examples.

2. The model

Let \( X(t) \) be the number in the population at time \( t \), and suppose that \((X(t), t \geq 0)\) is a continuous-time Markov chain taking values in \( S = \{0, 1, \ldots\} \). Let \( f_i \) \((i > 0)\) be the rate at which the population size changes when there are \( i \) individuals present, and suppose that, when a change occurs, it is a birth with probability \( a \) \((> 0)\) or a catastrophe of size \( k \) \((the removal of \( k \) individuals) with probability \( d_k \), \(k \geq 1\). (Simple death events are to be interpreted as catastrophes of size 1.) Assume that \( d_k > 0 \) for at least one \( k \geq 1 \) and that \( a + \sum_{k=1}^{\infty} d_k = 1 \). Thus, the process has transition rates \( Q = (q_{ij}) \) given by

\[
q_{ij} = \begin{cases} 
  f_i \sum_{k \geq j} d_k, & j = 0, \, i \geq 1, \\
  f_i d_{i-j}, & j = 1, 2, \ldots, i - 1, \, i \geq 2, \\
  -f_i, & j = i, \, i \geq 1, \\
  f_i a, & j = i + 1, \, i \geq 1, \\
  0 & \text{otherwise}.
\end{cases}
\]

Notice, in particular, that \( q_{0j} = 0 \) for \( j \geq 0 \) and that \( q_{ij} > 0 \) for at least one \( i \geq 1 \). Thus, the sole absorbing state 0, corresponding to population extinction, is accessible from \( \{1, 2, \ldots\} \) (an irreducible class). The special case \( f_i = \rho i \), where \( \rho \) \((> 0)\) is a per-capita transition rate, was studied by Brockwell [2], Pakes [8], [9] and Pollett [10]. Our purpose here is to evaluate the expected time to extinction for the present general model, thus extending Brockwell’s results for the linear case.

Whilst this model is quite general, it does have limitations. Firstly, it is frequently useful to separate death and catastrophe events, and to assign different rate functions to births, deaths and catastrophes, as in [7]. Another drawback is that the catastrophe size distribution does not depend on the number of individuals present. For example, it rules out two important cases: where the size of a catastrophe has (i) a binomial \( B(i, p) \) distribution (each of the \( i \) individuals present is removed with probability \( p \)) and (ii) a uniform distribution on the set \( \{1, 2, \ldots, i\} \); see for example [3].

3. Extinction probabilities

The probability of extinction does not depend on the event rates \((f_i, \, i \geq 1)\), because the jump chain is the same in all cases. It was shown by Pakes [8] that the probability of extinction \( \alpha_i \), starting with \( i \) individuals, is 1 for all \( i \geq 1 \) if and only if the drift \( D \) (drift away from 0), given by

\[
D = a - \sum_{i=1}^{\infty} i d_i = 1 - \sum_{i=1}^{\infty} (i + 1) d_i,
\]

is less than or equal to 0. Note that the process is said to be subcritical, critical, or supercritical according to whether \( D < 0 \), \( D = 0 \), or \( D > 0 \). In the latter case, extinction is of course still
Extinction times possible, and the extinction probabilities can be expressed in terms of the probability generating function

\[ d(s) = a + \sum_{i=1}^{\infty} d_is^{i+1}, \quad |s| < 1. \tag{2} \]

(Note that \( D = 1 - d'(1-) \) and that this satisfies \(-\infty \leq D \leq 1 \).) It follows from Theorem 4 of [5] (see also [8]) that, when \( D > 0 \),

\[ \sum_{i=1}^{\infty} (1 - \alpha_i) s^i = Ds/(d(s) - s). \]

Thus, writing \( b(s) = d(s) - s \), we see that

\[ \sum_{i=1}^{\infty} \alpha_is^{i-1} = \frac{1}{1-s} - \frac{D}{b(s)}. \tag{3} \]

It is interesting to note that \( \alpha_i \) tends to 0 as \( i \) tends to \( \infty \); roughly speaking, the larger the initial population, the less likely the population is to become extinct (in the supercritical case). However, as Pakes [8] noted, the convergence of \( \alpha_i \) to 0 can be very slow. For example, it is easy to show, letting \( s \uparrow 1 \) in (3) and using L'Hôpital's rule twice, that

\[ \sum_{i=1}^{\infty} \alpha_i = d''(1-) / 2D, \]

and so this is finite if and only if the variance of the catastrophe size distribution is finite.

4. Explosions

One interesting aspect of the present model is that the process may explode (that is, the population size may reach infinity in a finite time). Of course, this can only occur in the supercritical case, for, as we have already noted, the process hits 0 with probability 1 in the subcritical and critical cases.

It is easy to exhibit explosive behaviour: imagine that there is no catastrophe component in the model; we then obtain the pure-birth process with birth rates \( q_{i,i+1} = f_i a \), and this is well known to be explosive if and only if \( \sum_{i=1}^{\infty} g_i < \infty \), where \( g_i = 1/f_i \). If death transitions are included, that is, \( q_{i,i-1} = f_i b \), where \( a + b = 1 \), then the resulting birth–death process is explosive if and only if \( a > b \) (supercritical) and \( \sum_{i=1}^{\infty} g_i < \infty \) (apply Theorem 3.2.2 of [1]). It might therefore be conjectured that the general birth, death and catastrophe process is explosive if and only if it is supercritical and \( \sum_{i=1}^{\infty} g_i < \infty \). Our first result establishes that this is indeed the case.

**Theorem 1.** The birth, death and catastrophe process with transition rates (1) is nonexplosive if and only if \( \sum_{i=1}^{\infty} 1/f_i = \infty \) or \( \sum_{i=1}^{\infty} id_i \geq a \).

**Proof.** Recall that \( D = a - \sum_{i=1}^{\infty} id_i \). We will use results contained in [4] to prove that \( Q \) is always regular (the process is nonexplosive) if \( D \leq 0 \), while if \( D > 0 \) then \( Q \) is regular if and only if \( \sum_{i=1}^{\infty} 1/f_i = \infty \).

Theorem 1 of [4] applies to any conservative q-matrix \( Q = (q_{ij}) \), defined on the nonnegative integers, that is upwardly skip free in that \( q_{i,i+1} > 0 \) for all \( i \geq 1 \) and \( q_{ij} = 0 \) if \( j > i + 1 \). It states that \( Q \) is regular if and only if \( \sum_{n=1}^{\infty} R_n = \infty \), where \( R_0 = 1 \) and

\[ R_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{m=1}^{n-1} \sum_{k=0}^{m-1} q_{nk} R_{m-k} \right), \quad n \geq 1. \tag{4} \]
This result was obtained using Reuter’s condition (see [11] and Theorem 2.2.7(2) of [1]) to first establish that $Q$ is regular if and only if
\[ q_{n,n+1}(u_{n+1} - u_n) = u_n + \sum_{m=1}^{n} \sum_{k=0}^{m-1} q_{nk}(u_m - u_{m-1}), \quad 0 \leq u_n \leq 1, \quad n \geq 0, \]
has only the trivial solution, and then proving that this is equivalent to $\sum_{n=1}^{\infty} R_n = \infty$ using a generalization of Reuter’s lemma (Lemma 3.2.1 of [1]).

Set $g_i = 1/f_i$, so that (4) can be written as
\[ R_n = \frac{1}{q_{n,n+1}} + \sum_{m=1}^{n} R_{m-1} v_n^{(m-1)}, \quad n \geq 1, \]
where, for $m < n + 1$,
\[ v_n^{(m-1)} = \frac{1}{q_{n,n+1}} \sum_{k=0}^{m-1} q_{nk} = \frac{g_n}{a} \left( \sum_{k=n}^{\infty} \frac{d_k}{g_n} + \sum_{k=1}^{m-1} \frac{d_{n-k}}{g_n} \right) = \frac{1}{a} \sum_{k=n-m+1}^{\infty} d_k. \]

Since $v_n^{(m)}$ depends on $m$ and $n$ only through the difference $n - m$, let us write $v_n^{(m)} = v_{n-m}$. Then (5) becomes
\[ R_n = \frac{g_n}{a} + \sum_{m=0}^{n-1} R_{n-1} v_{n-m}, \quad n \geq 1, \]
remembering that $R_0 = 1$. On introducing generating functions $R(s) = \sum_{n=0}^{\infty} R_n s^n$, $v(s) = \sum_{i=1}^{\infty} v_i s^i$ and $g(s) = \sum_{i=1}^{\infty} g_i s^i$, we find that $R(s)(1 - v(s)) = 1 + g(s)/a$. It is then easy to prove that $b(s) = a(1 - s)(1 - v(s))$, which implies that
\[ R(s) \frac{b(s)}{1 - s} = a + g(s) \]
and, hence, if $b(s) \neq 0$,
\[ R(s) \frac{a + g(s)}{b(s)} = \frac{a + g(s)}{b(s)}. \]

We know, from Section V.12 of [6], that $1/b(s)$ has a power series expansion with positive coefficients and with radius of convergence $\sigma$, where $\sigma$ is the smallest zero of $b$ on $(0, 1]$. Furthermore, $\sigma = 1$ or $\sigma < 1$ according to whether $D \geq 0$ or $D < 0$, and $b(s) > 0$ for all $s \in [0, \sigma)$. Thus, if $D < 0$, then letting $s \uparrow \sigma$ in (7) gives $R(\sigma) = \infty$, implying that $\sum_{n=0}^{\infty} R_n = \infty$. If $D \geq 0$, then because $D = -b'(1-)$, letting $s \uparrow 1$ in (6) gives $D \sum_{n=0}^{\infty} R_n = a + \sum_{n=0}^{\infty} g_n (> 0)$. Therefore, if $D = 0$, then $\sum_{n=0}^{\infty} R_n = \infty$, while if $D > 0$, then $\sum_{n=0}^{\infty} R_n = \infty$ if and only if $\sum_{i=1}^{\infty} g_i = \infty$. This completes the proof.

5. Expected extinction times

We shall restrict our attention to the subcritical case ($D < 0$), where extinction occurs with probability 1, and make some remarks about the other cases at the end of this section.

For a general Markov chain with transition rates $Q = (q_{ij}, \ i, j \in S)$, whose state space consists of an irreducible class $\{1, 2, \ldots\}$ and a single absorbing state 0 that is reached with
probability 1, the expected absorption time $\tau_i$, starting in state $i$, is the minimal nonnegative solution to the system of equations

$$\sum_{j \geq 0} q_{ij} z_j + 1 = 0, \quad i \geq 1,$$

(8)

with $z_0 = 0$. In the present case, (8) becomes

$$f_i a z_{i+1} - f_i z_i + f_i \sum_{j=1}^{i-1} d_{i-j} z_j + 1 = 0, \quad i \geq 1,$$

with the empty sum being interpreted as 0 when $i = 1$. This can be written as

$$az_{i+1} - z_i \sum_{j=1}^{i-1} d_{i-j} z_j + g_i = 0, \quad i \geq 1.$$  

(9)

On multiplying by $s^i$ and then summing over $i$, we find that the generating function $h(s) = \sum_{i=1}^{\infty} z_i s^i$ of any solution $(z_i, i \geq 1)$ to (9) must satisfy $b(s) h(s) - az_1 + g(s) = 0$. (We delay addressing the question of whether the solution is nonnegative.)

We have already noted that $1/b(s)$ has a power series expansion with positive coefficients and with radius of convergence $\sigma$, and that $b(s) > 0$ for all $s \in [0, \sigma)$. (In the present subcritical case, $\sigma < 1$.) Let us write $e(s) = 1/b(s) = \sum_{j=0}^{\infty} e_j s^j$ when $|s| < \sigma$, where $e_j > 0$, noting that $a = b(0) = 1/e_0$. Letting $\kappa = az_1$, we obtain that

$$h(s) = z_1 + \sum_{i=1}^{\infty} \left( \kappa e_i - \sum_{j=1}^{i} g_j e_{i-j} \right) s^i$$

and, hence, for $i \geq 2$,

$$z_i = \kappa e_{i-1} - \sum_{j=1}^{i-1} g_j e_{i-1-j}.$$  

(10)

Now, $(e_i)$ is an increasing sequence. To see this, observe that if we had $g_i = 0$ for all $i$, then we would have $z_i = \kappa e_{i-1}$ and then, from (9),

$$ae_i - e_{i-1} + \sum_{j=1}^{i-1} d_{i-j} e_{i-1-j} = 0, \quad i \geq 1,$$

(11)

with the empty sum being interpreted as 0 when $i = 1$. Hence, $(e_i)$ satisfies

$$a(e_i - e_{i-1}) = \sum_{j=1}^{i-1} d_{i-j} (e_{i-1} - e_{j-1}) + e_{i-1} \sum_{j=i}^{\infty} d_j, \quad i \geq 1.$$  

Since $e_0 = 1/a > 0$ and $d_j > 0$ for at least one $j \geq 1$, a straightforward inductive argument shows that $(e_i)$ is increasing.
Theorem 2. For the subcritical birth, death and catastrophe process with transition rates (1), we have therefore proved the following result.

Thus, in order to ensure that the expected extinction time is finite, we require that \( \kappa \geq \sup_{i \geq 1} h_i \), where

\[
h_i = \frac{1}{\epsilon_i} \sum_{j=1}^{i} g_{j} e_{i-j} = \sum_{j=1}^{i} \frac{g_{j}}{\epsilon_{i}} e_{i-j}.
\]

The minimal solution is then obtained on setting \( \kappa = \sup_{i \geq 1} h_i \).

Since \( \{e_i\} \) is increasing, \( 0 < e_{i-j}/e_i \leq 1 \) for all \( i \geq j \geq 0 \). Moreover, because \( \sigma \) is the radius of convergence of \( \sum_{j=0}^{\infty} e_j s^j \), \( \sigma = \lim_{i \to \infty} e_i \) if and only if this limit exists, implying that \( e_{i-j}/e_i \to \sigma^j \) for each \( j \). Hence, formally, \( h_i \to g(\sigma) \). Once we prove that this limit exists and equals \( \sup_{i \geq 1} h_i \), we may set \( \kappa = g(\sigma) \) to obtain the minimal nonnegative solution to (9).

To achieve this, we will draw further on branching process theory. Since \( e(s) \) has a power series expansion with positive coefficients, so does \( \pi(s) = a\pi_1 \int_{0}^{s} \frac{du}{b(u)} \), \( |s| < \sigma \), where \( \pi_1 = \pi'(0) \). Indeed, writing \( \pi(s) = \sum_{i=1}^{\infty} \pi_i s^i \), it is easy to see that \( \pi_i/\pi_1 = ae_{i-1}/i \) for \( i \geq 1 \). However, the coefficients \( \{\pi_i\} \) form a stationary measure on \( \{1, 2, \ldots\} \) for the Markov branching process with offspring distribution \( (q_i, i \geq 0) \), where \( q_0 = a, q_1 = 0 \), and \( q_i = d_{i-1} \) for \( i \geq 2 \) (refer to the corollary to Theorem V.12.2 of [6]). Theorem 1(e) of [15] then gives \( i\pi_i \sigma^i \uparrow a\pi_1/(1 - d'(\sigma)) \) as \( i \to \infty \) whenever \( D \neq 0 \) (note that \( d'(\sigma) < 1 \), since \( b'(\sigma) < 0 \) when \( D \neq 0 \)). Consequently, \( e_{i-1}/e_i \to \sigma^j \) as \( i \to \infty \). Applying the dominated convergence theorem to (12) shows that if \( g(\sigma) < \infty \) then \( h_i \to g(\sigma) \) and, hence, \( \sup_{i \geq 1} h_i = g(\sigma) \) because \( h_i \leq g(\sigma) \). On the other hand, Fatou's lemma always implies that \( \lim \inf_{i \to \infty} h_i \geq g(\sigma) \), so that if \( g(\sigma) = \infty \) then \( \sup_{i \geq 1} h_i = \infty \).

We have therefore proved the following result.

**Theorem 2.** For the subcritical birth, death and catastrophe process with transition rates (1), let \( \{e_i, i \geq 0\} \) be the coefficients of the power series expansion of \( e(s) = 1/(d(s) - s) \) when \( |s| < \sigma \), where \( d(s) \) is given by (2) and \( \sigma (< 1) \) is the smallest solution of \( d(s) = s \) in \( (0, 1) \). Then the expected extinction time \( \tau_i \), starting in state \( i \), is finite if and only if \( \kappa := \sum_{i=1}^{\infty} \sigma^j/f_i < \infty \), in which case \( \tau_0 = 0 \) and

\[
\tau_i = \kappa e_{i-1} - \sum_{j=1}^{i-1} \frac{e_{i-1-j}}{f_j}, \quad i \geq 1.
\]

We conclude this section with some remarks on the critical and supercritical cases, both of which have \( \sigma = 1 \). Theorem 2 is certainly valid in the critical case provided that \( d''(1-)<\infty \) (which corresponds to the variance of the catastrophe size distribution being finite): the expected extinction times are all finite if and only if \( \kappa := \sum_{i=1}^{\infty} 1/f_i < \infty \), in which case (13) holds. This is true because, as \( i \to \infty \), \( e_{i-1}/e_i \to 2/d''(1) > 0 \) (Theorem 1(c) of [15]) and hence \( e_{i-1}/e_i \to 1 \). (The infinite variance case is mathematically delicate, and we will not pursue it further here.) In the supercritical case, where there is a probability \( \alpha_i \) of extinction, starting in \( i \), which is strictly less than 1 (and therefore the expected extinction times are infinite), it is possible to evaluate expected extinction times conditional on extinction occurring. This can be done by interpreting the result of Theorem 2 for the (subcritical) birth, death and catastrophe process with transitions rates \( \bar{q}_{ij} = q_{ij} \alpha_j/\alpha_i \), and following the programme laid out in [14].
6. Examples

First let us examine the linear case studied by Brockwell [2]. This has $f_i = \rho i$, where $\rho > 0$. So, $g(s) = -\log(1-s)/\rho$ when $|s| < 1$, implying that $g(\sigma)$ is finite whenever $\sigma < 1$. Setting $\kappa = -\log(1-\sigma)/\rho$, it follows from (13) that

$$\tau_i = \frac{1}{\rho} \left( e_{i-1} \log \left( \frac{1}{1-\sigma} \right) - \sum_{j=0}^{i-2} \frac{e_j}{i-j-1} \right), \quad i \geq 1.$$  

This is equivalent to

$$\sum_{i=1}^{\infty} \tau_i s^{i-1} = \frac{1}{\rho b(s)} \log \left( \frac{1-s}{1-\sigma} \right), \quad |s| < \sigma,$$

which is Equation (3.1) of [2].

Further examples of the subcritical birth, death and catastrophe process for which the expected extinction times are finite include the case where events occur at a constant rate $f_i = \rho > 0$, independent of the population size, giving $\kappa \rho = \sigma/(1-\sigma)$, and the case where $f_i = \rho i(i+1)$, giving $\kappa \rho = 1 - \frac{1-\sigma}{\sigma} \log \left( \frac{1}{1-\sigma} \right)$.

Explicit results can be obtained in the case where the catastrophe size follows a geometric law. Suppose that $d_i = b(1-q)q^{i-1}$ for $i \geq 1$, where $b > 0$ satisfies $a+b = 1$ and $0 \leq q < 1$. The simple birth–death process with linear birth and linear death rates is recovered on setting $q = 0$. It is easy to see that $D = a - b/(1-q)$, and so $D < 0$ or $D \geq 0$ according to whether $c > 1$ or $c \leq 1$, where $c = q + b/a$. We also have

$$b(s) = \frac{(b+qa)s^2 - (1+qa)s + a}{1-qs} = \frac{a(1-s)(1-cs)}{1-qs}$$

and, hence, if $D < 0$ then $\sigma = 1/c (< 1)$. The coefficients of the power series $1/b(s) = \sum_{j=0}^{\infty} e_j s^j$ are easily evaluated using partial fractions. If $D < 0$ (or indeed if $D > 0$) then

$$e_j = \frac{1-q - (c-q)c^j}{a(1-c)}, \quad j \geq 0.$$  

We may evaluate $\tau_i$ by substituting these expressions into (13). If, for example, $f_i = \rho \beta^{i-1}$, where $\rho, \beta > 0$, then, if $\beta = 1$,

$$\tau_i = \frac{1 + (1-q)(i-1)}{\rho(b - a(1-q))}, \quad i \geq 1,$$

while if $\beta \neq 1$, the expected extinction times are finite only if $\beta > a/(b + qa)$, in which case

$$\tau_i = \frac{1 - q - (\gamma - q)\gamma^{i-1}}{\rho(b - a(\gamma - q))(1-\gamma)}, \quad i \geq 1,$$

where $\gamma = 1/\beta$. 
Acknowledgements

The support of the Australian Research Council (grant A00104575) is gratefully acknowledged. The work of the first author was supported by a PhD scholarship from the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.

References