Improved Fixed Point Methods for Loss Networks with Linear Structure

M.S. Bebbington* P.K. Pollett** and I. Ziedins***
*Department of Statistics Massey University
**Department of Mathematics University of Queensland
***Department of Statistics University of Auckland

ABSTRACT *: This paper is concerned with the performance evaluation of loss networks. For the simplest networks there are explicit analytical formulae for the important measures of performance, but for networks which involve some level of dynamic control, exact analytical methods have had only limited success. Under several regimes the Erlang Fixed Point (EFP) method provides a good approximation for the blocking probabilities, but when these regimes are not operative the method can perform badly. In many cases this is because the key assumption of independent blocking does not hold. We derive methods for estimating the blocking probabilities which specifically account for the dependencies between neighbouring links. For the network considered here, namely a ring network with two types of traffic, our method produced relative errors typically* 10^{-5} *of that found using the EFP approximation.*

1. INTRODUCTION

We shall be concerned with circuit-switched networks of the kind depicted in Figure 1. These consist of a set of links indexed by $j = 1, 2 \ldots, K$, with C_j circuits comprising each link j, and a collection of routes $\mathcal{R}$. Each route $r \in \mathcal{R}$ is a set of links. Calls using route r are offered at rate ν_r as a Poisson stream, and use $a_{jr} (\geq 0)$ circuits from link j. $\mathcal{R}$ indexes *independent* Poisson processes. Calls requesting route r are blocked and lost if, on *any* link j, there are fewer than a_{jr} free circuits. Otherwise, the call is connected and simultaneously holds a_{jr} circuits on each link j for the duration of the call. For simplicity, we shall take $a_{jr} \in \{0, 1\}$. Call durations are independent and identically distributed exponential random variables with unit mean, and are independent of the arrival processes.

Let $\boldsymbol{n} = (n_r,\ r \in \mathcal{R})$, where n_r is the number of calls in progress using route r, let $\boldsymbol{C} = (C_j,\ j = 1, \ldots, K)$, and let $\boldsymbol{A} = (a_{jr},\ r \in \mathcal{R},\ j = 1, \ldots, K)$. Then, the usual model for a circuit-switched network (see for example [5]) is a continuous-time Markov chain $(\boldsymbol{n}(t),\ t \geq 0)$ taking values in

$$S = S(\boldsymbol{C}) = \{\boldsymbol{n} \in \mathrm{Z}_+^{\mathrm{R}} \ :\ \boldsymbol{An} \leq \boldsymbol{C}\}$$

and its unique stationary distribution is given by

$$\pi(\boldsymbol{n}) = \Phi^{-1} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \quad \boldsymbol{n} \in S,$$

where

$$\Phi = \Phi(\boldsymbol{C}) = \sum_{\boldsymbol{n} \in S(\boldsymbol{C})} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}.$$

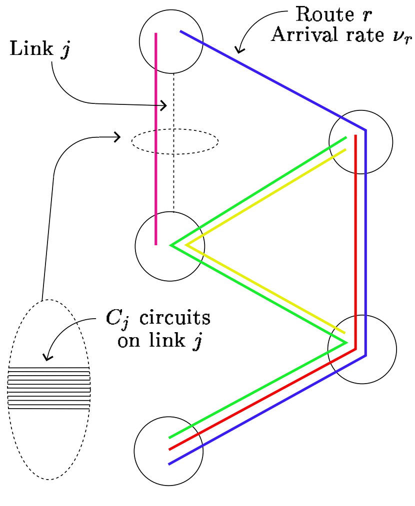

Fig 1. A typical circuit-switched network
(5 nodes, 6 links and 5 routes)

The stationary probability that a route-r call is blocked is then given by

$$L_r = 1 - \frac{\Phi(\boldsymbol{C} - \boldsymbol{A}\boldsymbol{e}_r)}{\Phi(\boldsymbol{C})},$$

where $\boldsymbol{e}_r$ is the unit vector from $S(\boldsymbol{C})$ describing just one call in progress on route r. However, although we have an explicit expression for L_r in terms of Φ, the latter can't (usually) be computed in polynomial time. To see this, we only need consider the trivial case of a fully-connected network with all possible one-link routes (that is, $|\mathcal{R}| = K$ and $\boldsymbol{A} = \boldsymbol{I}$) and with $C_j = C$; clearly $|S| = C^K$. Thus, for networks with even moderate capacity, one is forced to use alternative methods, and, arguably the most important of *these* is the EFP approximation.

2. THE EFP APPROXIMATION

Kelly [4] proved that there is a unique vector $(B_1, \ldots, B_K) \in [0,1]^K$ satisfying

$$B_j = E(\rho_j, C_j), \tag{1}$$

$$\rho_j = (1 - B_j)^{-1} \sum_r a_{jr} \nu_r (1 - L_r), \tag{2}$$

for $j = 1, \ldots, K$, and

$$L_r = 1 - \prod_i (1 - B_i)^{a_{ir}}, \ r \in \mathcal{R}, \tag{3}$$

where

$$E(\nu, C) = \frac{\nu^C}{C!} \left(\sum_{n=0}^{C} \frac{\nu^n}{n!} \right)^{-1} .$$

$E(\nu, C)$ is *Erlang's Formula* for the loss probability on a *single link* with C circuits and Poisson traffic offered at rate ν. The EFP approximation is obtained by using B_j to estimate the probability that link j is full, and L_r to estimate the route-r blocking probability.

The rationale for the EFP approximation is one of *independent blocking*. If links along route r were blocked independently (they are clearly not) and if B_j *were* the link-j blocking probability, then L_r would be the route-r blocking probability:

$$L_r = 1 - \prod_{i \in r} (1 - B_i) = 1 - \prod_i (1 - B_i)^{a_{ir}}.$$

Carrying this further, the traffic offered to link j would be Poisson (at rate ρ_j, say) and the *carried traffic* (that which is accepted) on link j would be

$$\sum_r a_{jr} \nu_r (1 - L_r) \ (= (1 - B_j)\rho_j \ ) \ .$$

The EFP approximation therefore stipulates that the link blocking probabilities $(B_1, \ldots, B_K)$ should be consistent with this level of carried traffic:

$$B_j = E(\rho_j, C_j), \quad j = 1, \ldots, K.$$

On combining (1), (2) and (3) we obtain a set of equations for $(B_1, \ldots, B_K)$:

$$B_j = E\left(\left(\sum_r a_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - B_i) \right), \, C_j \right) .$$

The existence of an Erlang Fixed Point, namely a fixed point of these equations is easy to prove using the *Brouwer fixed point theorem*; they define a continuous mapping from a compact convex set $[0,1]^K$ into itself. The uniqueness is considerably more difficult to prove [4]. We note that for more complex systems, there may be more than one fixed point (see for example [1]).

The EFP approximation performs particularly well under two limiting regimes. The first is one in which the topology of the network is held fixed, while capacities and arrival rates at the links become large [4]; this has become known as the *Kelly limiting regime*, or (somewhat misleadingly) as the *heavy traffic limit*. Under the second limiting regime, called *diverse routing*, the number of links, and the number of routes which use these links, become large, while the capacities are held fixed and the arrival rates on multi-link routes become small. Examples of this are star networks and fully-connected networks with alternate routing [2, 3, 6, 9, 11]. If neither regime is operative, the EFP method can perform badly: in particular, in highly linear networks and/or networks with low capacities. Further, adding controls to the network may cause the method to perform badly under otherwise favourable regimes. A particularly useful control is *trunk reservation*. Here, traffic streams are assigned priorities and calls are accepted only if the occupancies of links along their route are not above a given threshold, the size of which depends on the type of call. This widely used control mechanism is typically very robust to fluctuations in arrival rate and has the added advantage of eliminating pathological behaviour such as bistability [1]. With such a control operating in a network of reasonable size, the occupancy of neighbouring links may be highly dependent and the equilibrium distribution will no longer have a product form, as it does for the corresponding uncontrolled network. Modelling dependencies in *this* context is thus critical. For an excellent review of the theory of loss networks, and in particular EFP methods, see Kelly [5].

We shall focus attention on simple, highly linear networks, since here the EFP approximation is expected to perform poorly.

3. A SYMMETRIC RING NETWORK

To illustrate our methods, consider a loss network with K links forming a loop, and each link having the same capacity C. Such a network is depicted in Figures 2 and 3. There are two types of traffic: 1-link routes (type-1 traffic) and 2-link routes comprising pairs of adjacent links (type-2 traffic). Type-t traffic is offered at rate ν_t on each type-t route. If L_t is the common loss probability of

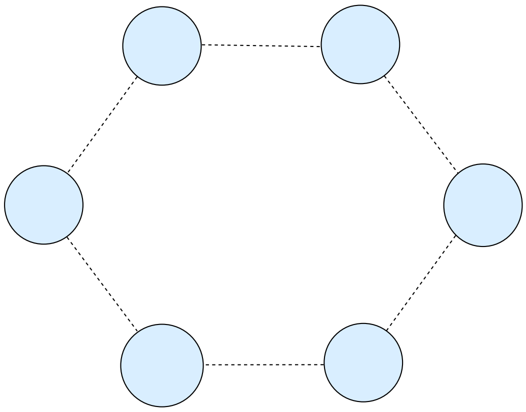

Fig 2. A Ring Network (6 nodes)

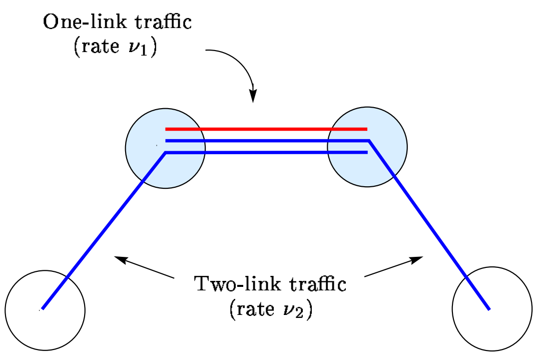

Fig 3. One- and two-link traffic using a given link

type-t calls, then it is easy to show that the EFP approximation is given by

$$L_1 \simeq B \quad \text{and} \quad L_2 \simeq 1 - (1-B)^2,$$

where the Erlang Fixed Point B is the unique solution to

$$B = E(\nu_1 + 2\nu_2(1-B), C),$$

where recall that $E(\nu, C)$ is Erlang's Formula. Figure 4 shows the EFP approximation for the blocking probability of type-1 calls in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$.

In order to assess the accuracy of the EFP approximation, as well as the improved methods described below, we shall need to evaluate the exact blocking probabilities. This will be done using an iterative technique based on the equilibrium distribution, one which has some interest in its own right.

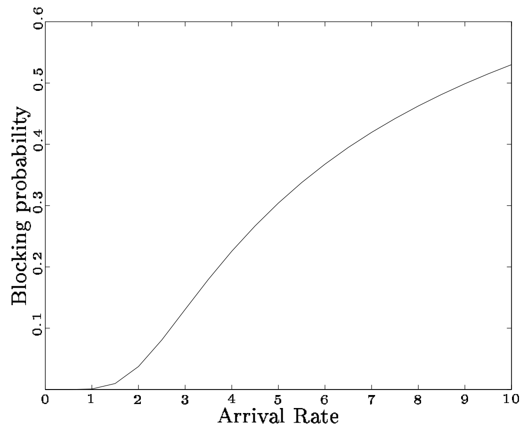

Fig 4. EFP approximation for the blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

The state space for the ring network is given by

$$S_K = \{\boldsymbol{n} : n_i + n_{i-1,i} + n_{i,i+1} \le C, i = 1, \dots, K\},$$

where, in a convenient notation, route $\{K, 1\}$ is denoted by $\{K, K+1\}$; since we shall be varying K, it will be necessary to make any dependence on K explicit in our notation. The equilibrium distribution is given by

$$\pi_K(\boldsymbol{n}) = \Phi_K^{-1} \frac{\nu_1^{\sum_i n_i} \nu_2^{\sum_i n_{i,i+1}}}{\prod_i n_i!\, n_{i,i+1}!}, \quad \boldsymbol{n} \in S_K,$$

where

$$\Phi_K = \sum_{\boldsymbol{n} \in S_K} \frac{\nu_1^{\sum_i n_i} \nu_2^{\sum_i n_{i,i+1}}}{\prod_i n_i!\, n_{i,i+1}!}$$

is the normalizing constant for the network with K links.

Now consider a *line network* consisting of a series of K links. This is obtained if the ring is disconnected at one node. In a similar fashion we define the normalizing constant Ψ_K for this network. We also define $\Psi_K^{(i,j)}$ to be the normalizing constant for the line network with $C_1 = i$, $C_k = C$, for $1 < k < K$, and $C_K = j$. Note that $\Psi_K^{(C,C)} = \Psi_K$. Note also that $\Psi_K^{(i,j)} = \Psi_K^{(j,i)}$. Then, the $\Psi_K^{(i,j)}$ satisfy the following recursion:

$$\Psi_K^{(i,j)} = \sum_{\beta=0}^{i} \left[\sum_{\alpha=0}^{i-\beta} \frac{\nu_1^\alpha}{\alpha!} \frac{\nu_2^\beta}{\beta!} \Psi_{K-1}^{(C-\beta,j)} \right]$$

with

$$\Psi_1^{(i,j)} = \sum_{\alpha=0}^{\min(i,j)} \frac{\nu_1^\alpha}{\alpha!},$$
$$\Psi_0^{(i,j)} = 1.$$

This recursion is obtained by considering the number of one-link and two-link calls on link 1. Consider the contribution to the normalizing constant made by some fixed configuration of calls. Suppose that this configuration has $n_{12} = \beta$, where β must lie between 0 and i inclusive. Then we must have $n_1 \leq \beta - i$. Thus, the contribution from these two routes for this particular configuration is exactly

$$\frac{\nu_1^{n_1}}{n_1!}\frac{\nu_2^\beta}{\beta!}.$$

We now consider the remaining routes; they use links $\{2, \ldots, K\}$ and form a $(K-1)$-link network. Since there are β calls on route $\{1,2\}$, link 2 only has $C - \beta$ free circuits, and so the contribution from the remaining routes is $\Psi_{K-1}^{(C-\beta,j)}$.

Let us return to the ring network. An expression for Φ_K in terms of the Ψ_K is obtained as follows. Consider links K and 1. By conditioning on n_{K1}, we can break the ring network into a line network, and write

$$\Phi_K = \sum_{n_{K1}=0}^{C} \frac{\nu_2^{n_{K1}}}{n_{K1}!} \Psi_K^{(C-n_{K1},C-n_{K1})}.$$

Note that links K and 1 have been chosen as the reference links here, but of course the recursion would be the same if any other pair of adjacent links had been chosen.

The blocking probabilities can now be written in terms of the normalizing constants. To do this, we introduce some further notation. Let $\Phi_K^{(i)}$ denote the normalizing constant for the ring network in which all the links have capacity C, except for one link, which has capacity i. Similarly, let $\Phi_K^{(i,j)}$ be the normalizing constant for the ring network in which all links except two, have capacity C; the exceptions have capacities i and j, and are adjacent. Then, the probability that a one-link call is accepted (which is also the probability that a link has free capacity) is given by

$$\frac{\Phi_K^{(C-1)}}{\Phi_K},$$

and the probability that a two-link call is accepted is given by

$$\frac{\Phi_K^{(C-1,C-1)}}{\Phi_K},$$

where, just as for Φ_K, we can write

$$\Phi_K^{(C-1)} = \sum_{n_{K1}=0}^{C-1} \frac{\nu_2^{n_{K1}}}{n_{K1}!} \Psi_K^{(C-n_{K1},C-1-n_{K1})}$$

and

$$\Phi_K^{(C-1,C-1)} = \sum_{n_{K1}=0}^{C-1} \frac{\nu_2^{n_{K1}}}{n_{K1}!} \Psi_K^{(C-1-n_{K1},C-1-n_{K1})}.$$

These recursions are easily implemented to obtain the exact blocking probabilities numerically.

Figure 5 shows the relative error in using the EFP approximation to estimate the blocking probability of type-1 calls in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$. Notice that the exact blocking probabilities are overestimated for small values of the arrival rate ν and underestimated for larger values, and, that the accuracy improves as the arrival rate becomes very large. Notice also that the approximation is most accurate around the point of "critical loading", namely when $\nu_1 + 2\nu_2 = C$; for the parameter values used, this is when $\nu = 10/3$, a point just before the graph crosses the x-axis. For type-2 calls the trend is no different.

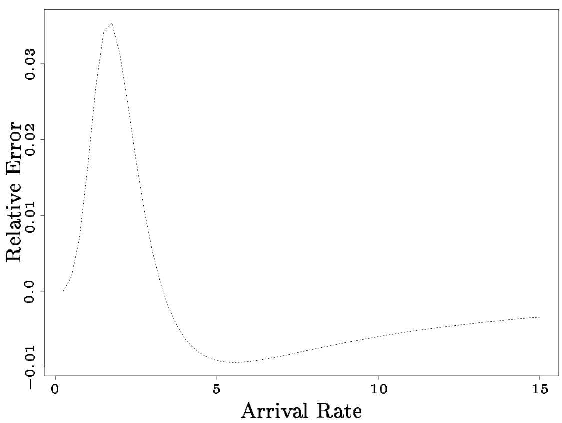

Fig 5. Relative error in the EFP approximation for the blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

To illustrate why we might expect the EFP approximation to perform badly in the present context, we shall assess the dependence between two adjacent links. Take links 1 and 2 as reference links and consider the subnetwork depicted in Figure 6. We identify three routes: $\{1\}$, $\{2\}$ and $\{1,2\}$. If m_r denotes the number of calls on route r, then m_1 is the number of calls occupying capacity on link 1 *but not* on link 2, that is $m_1 = n_1 + n_{K1}$, m_2 is the number occupying capacity on link 2 *but not* on link 1, that is

$m_2 = n_2 + n_{23}$, and, $m_{12} (= n_{12})$ is the number of calls occupying capacity on both links. Figure 7 shows the correlation between links 1 and 2 for the

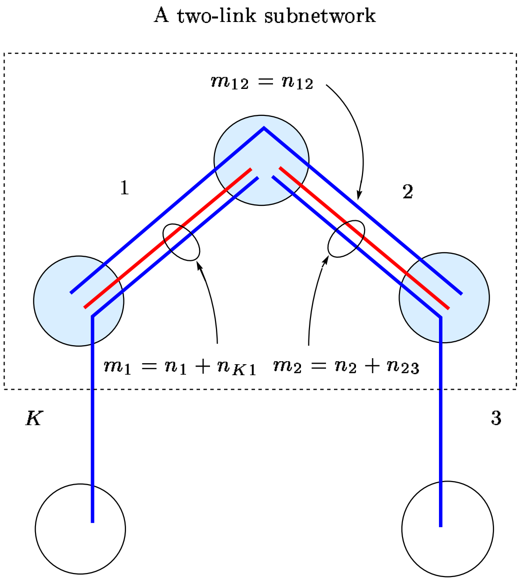

Fig 6. Definition of m_1, m_2 and m_{12} for the symmetric ring network

network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$; to be precise, we have plotted

$$\mathrm{Corr}\left(I_{\{m_1+m_{12}<C\}}, I_{\{m_2+m_{12}<C\}}\right)$$

against the arrival rate ν.

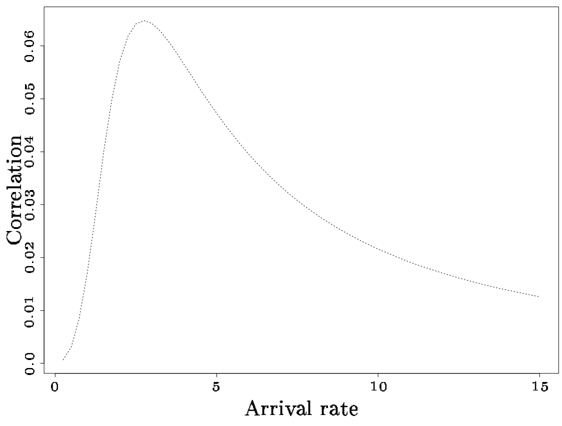

Fig 7. Correlation between adjacent links ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

We can estimate the blocking probabilities more accurately by specifically accounting for the dependencies between adjacent links. Our first approximation (Approximation I) is obtained by adapting the method of Pallant [7]. In Pallant's method, the network is decomposed into independent subnetworks and the stationary distribution is evaluated for each. For example, if we take our subnetwork to be the one depicted in Figure 6, then its state space will be

$$S = \{(m_1, m_2, m_{12}) : m_i + m_{12} \le C, i = 1, 2\}$$

and its stationary distribution will be

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{(\nu_1 + \nu_2(1-B))^{m_1+m_2} \nu_2^{m_{12}}}{m_1! \, m_2! \, m_{12}!},$$

where Φ is a normalizing constant. We then estimate B, the probability that a link adjacent to the two-link subnetwork is fully occupied, using the subnetwork itself; set

$$\begin{aligned} B &= \sum_{m \, : \, m_1+m_{12}=C} \pi(m_1, m_2, m_{12}) \\ &= \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}) \, . \end{aligned}$$

These expressions are used iteratively to determine a fixed point B, and we then set $L_1 = B$ and

$$L_2 = 2L_1 - \sum_{m_{12}=0}^{C} \pi(C - m_{12}, C - m_{12}, m_{12}).$$

Our second, and more accurate, approximation (Approximation II) uses additional knowledge of the state of a given link in estimating the probability that the adjacent link is full. We use *state-dependent* arrival rates, $\rho_n = \nu_1 + \nu_2(1 - b_n)$, $n \in \{0, 1, \ldots, C-1\}$, where b_n is the probability that link K is fully occupied, conditional on $m_1 = n$ (b_n is also the probability that link 3 is fully occupied, *conditional on* $m_2 = n$), so that

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{\nu_2^{m_{12}} \left(\prod_{n=0}^{m_1-1} \rho_n\right) \left(\prod_{n=0}^{m_2-1} \rho_n\right)}{m_1! \, m_2! \, m_{12}!} \, .$$

Once b_n is estimated and π determined, we set L_1 and L_2 as for Approximation I. An estimate of b_n is found by assuming that b_n does not depend on m_{12}. For $n = 0, \ldots, C-1$, we set

$$b_n = \frac{\sum_{m=0}^{n} p(n-m, C-m, m)}{\sum_{m=0}^{n} \sum_{r=0}^{C-m} p(n-m, r, m)},$$

where

$$p(n_1, m_K, n_{K1}) = \frac{\nu_1^{n_1} \nu_2^{n_{K1}} \left(\prod_{s=0}^{m_K-1} \rho_s\right)}{n_1! \, n_{K1}! \, m_K!} \, .$$

The dependence of b_n on m_{12} is due to the cyclic nature of the network, but is expected to be slight for large networks.

This approximation is exact for the infinite line network, as shown by Zachary[10] for an equivalent network with $\nu_1 = 0$ (that is, no one-link traffic). Our expression for b_n is the same as that obtained in his paper for the infinite line network, although written in a different form.

State-dependent arrival rates such as we have here are also discussed by Pallant and Taylor [8].

Figures 8 and 9 show the relative error in using each of the three approximations to estimate the blocking probability of type-1 and type-2 calls, respectively, in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$. Notice that, while Approximation I gives some improvement in accuracy over the EFP approximation, the improvement obtained using Approximation II is considerable. Indeed, the maximum error for Approximation II is of order 10^{-8} for both types of traffic.

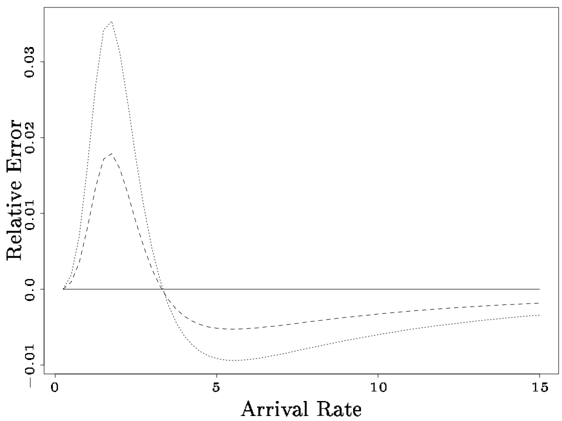

Fig 8. Relative error in the estimated blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)
············· EFP - - - - - Approx. I
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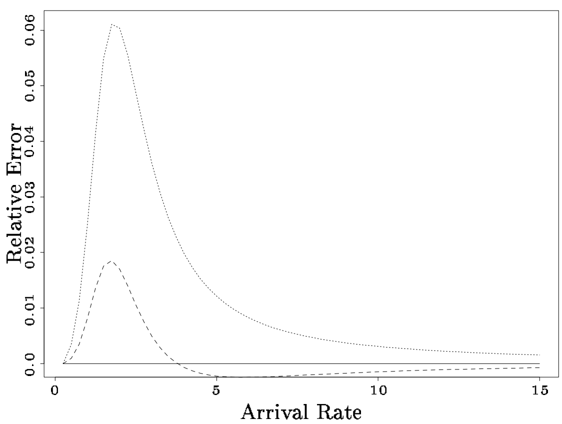

Fig 9. Relative error in the estimated blocking probability of type-2 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)
············· EFP - - - - - Approx. I
________________ Approx. II

We are presently working on extending our methods to deal with trunk reservation and networks with a more general topology.

4. REFERENCES

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