On the existence of uni-instantaneous $Q$-processes with a given finite $\mu$-invariant measure

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Abstract

Let $S$ be a countable set and let $Q = (q_{ij}, i, j \in S)$ be a conservative $q$-matrix over $S$ with a single instantaneous state $b$. Suppose that we are given a real number $\mu \geq 0$ and a strictly positive probability measure $m = (m_j, j \in S)$ such that $\sum_{i \in S} m_i q_{ij} = -\mu m_j$, $j \neq b$. We prove that there exists a $Q$-process $P(t) = (p_{ij}(t), i, j \in S)$ for which $m$ is a $\mu$-invariant measure, that is, $\sum_{i \in S} m_i p_{ij}(t) = e^{-\mu t} m_j$, $j \in S$. We illustrate our results with reference to the Kolmogorov K1 chain, and a birth-death process with catastrophes and instantaneous resurrection.

Keywords: Markov chain; $q$-matrix; birth-death process; construction theory

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1 Introduction

We begin with a conservative $q$-matrix over a countable set $S$, that is, a collection $Q = (q_{ij}, i, j \in S)$ of real numbers that satisfy

\[
0 \leq q_{ij} < \infty, \quad j \neq i, \; i, j \in S, \\
q_i := -q_{ii} \leq \infty, \quad i \in S, \\
\sum_{j \neq i} q_{ij} = q_i, \quad i \in S.
\]

We shall assume that $Q$ has a single instantaneous state, that is, a state $b \in S$ such that $q_b = \infty$ and $q_i < \infty$ for $i \neq b$. A set of real-valued functions $P(t) = (p_{ij}(t), i, j \in S)$ defined on $(0, \infty)$ is called a standard transition function or process if

\[
p_{ij}(t) \geq 0, \quad i, j \in S, \; t > 0, \quad (1) \\
\sum_{j \in S} p_{ij}(t) \leq 1, \quad i \in S, \; t > 0, \quad (2) \\
p_{ij}(s + t) = \sum_{k \in S} p_{ik}(s)p_{kj}(t), \quad i, j \in S, \; s, t > 0. \quad (3)
\]
$$\lim_{t \to 0} p_{ij}(t) = \delta_{ij}, \quad i, j \in S. \quad (4)$$

$P$ is then honest if equality holds in (2) for some (and then all) $t > 0$, and it is called a $Q$-transition function (or $Q$-process) if $p'_{ij}(0+) = q_{ij}$ for each $i, j \in S$.

If $\mu$ is some fixed non-negative real number, a collection of strictly positive numbers $m = (m_j, j \in S)$ is called a $\mu$-subinvariant measure (on $S$) for $Q$ if $\sum_{i \in S} m_i q_{ij} \leq -\mu m_j$, $j \in S$, and $\mu$-invariant if

$$\sum_{i \in S} m_i q_{ij} = -\mu m_j, \quad j \in S. \quad (5)$$

Here we shall suppose that $m$ is a finite measure ($\sum_{i \in S} m_i < \infty$) which is almost $\mu$-invariant for $Q$, that is,

$$\sum_{i \in S} m_i q_{ij} = -\mu m_j, \quad j \neq b, \quad (6)$$

and we will show that there always exists a $Q$-process $P$ such that $m$ is a $\mu$-invariant measure (on $S$) for $P$, that is,

$$\sum_{i \in S} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in S, \quad t > 0. \quad (7)$$

(When $\mu = 0$, all of the above reduce to the more common notions of invariance and subinvariance.) Note that if we were given a $\mu$-invariant measure $m$ for a particular $Q$-process $P$, then, since (7) may be rewritten as

$$\sum_{i \neq j} m_i p_{ij}(t) + (1 - e^{-\mu t}) m_j = (1 - p_{jj}(t)) m_j,$$

Fatou’s Lemma would give

$$\sum_{i \neq j} m_i q_{ij} + \mu m_j \leq q_j m_j,$$

for all $j \in S$, so that $m$ would be $\mu$-subinvariant for $Q$. But, under what conditions is $m$ be $\mu$-invariant for $Q$? In Section 2 we provide necessary and sufficient conditions for $m$ to be almost invariant for $Q$ and delay addressing the interesting question of whether or not $\sum_{i \neq b} m_i q_{ib} = \infty$, which would be the remaining requirement for (5); this will be considered in Section 6.

We are assuming here that $Q$ is uni-instantaneous. When $Q$ is totally stable, that is, $q_i < \infty$ for all $i \in S$, the relationship between (5) and (7) is well understood, and has been divined completely for the minimal $Q$-process $F$. It was shown by Tweedie [14] that if $m$ is a $\mu$-invariant measure for $F$, then it is $\mu$-invariant for $Q$. Conversely (Pollett [8, 9]), if $m$ is $\mu$-invariant for $Q$, then it is $\mu$-subinvariant for $F$, and $\mu$-invariant for $F$ if and only if the equations

$$\sum_{i \in S} y_i q_{ij} = -\nu y_j, \quad 0 \leq y_j \leq m_j, \quad j \in S,$$

have only the trivial solution for some (and then all) $\nu < \mu$. This result holds whether or not $S$ is irreducible and does not require $m$ to be finite. If, as we are assuming here, $m$ is
finite, then for \( \mu \) to be strictly positive, it is necessary that \( F \) be dishonest. Furthermore, if \( F \) is the unique \( Q \)-process satisfying the forward equations, then \( m \) is \( \mu \)-invariant for \( F \).

Recently, Zhang, Lin and Hou solved the existence problem for the case \( \mu = 0 \) in, respectively, the totally stable case [17], and the uni-instantaneous case [18]. They proved that if \( m \) is a strictly positive (almost) invariant probability measure for \( Q \), then there exists a \( Q \)-process \( P \) for which \( m \) is an invariant measure (and hence a stationary distribution) for \( P \). We will extend their results to the case when \( \mu > 0 \).

The structure of the paper is as follows. We begin, in Section 2, by examining the relationship between (6) and (7). Next, we recall the Resolvent Decomposition Theorem of Chen and Renshaw [2], which is the major tool for constructing uni-instantaneous \( Q \)-processes. This, and some other preliminary results, are presented in Section 3. Our main result on the existence of a \( Q \)-process with a given finite almost \( \mu \)-invariant measure for \( Q \) is proved in Section 4. Section 5 explores two examples which illustrate our results, and finally, in Section 6, we provide some necessary conditions for \( \mu \)-invariance. The terminology and notation used will follow that established by Anderson [1] and Yang [16].

## 2 Almost \( \mu \)-invariance

Our aim here is to provide necessary and sufficient conditions for a measure \( m \) (not necessarily finite) satisfying (7) to be almost \( \mu \)-invariant for \( Q \). We will need to recall the notion of an almost \( B \)-type and an almost \( F \)-type \( Q \)-process.

**Definition 1** (Chen and Renshaw [3]) A uni-instantaneous \( Q \)-process \( P \) with instantaneous state \( b \) is called almost \( B \)-type if it satisfies the Kolmogorov backward equations over the non-instantaneous states:

\[
p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \quad i \neq b, \ j \in S.
\]

It is called almost \( F \)-type if it satisfies the Kolmogorov forward equations over the non-instantaneous states:

\[
p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \quad i \neq b, \ j \in S.
\]

By adapting the proof of Theorem 1 of [11], we can establish the following.

**Theorem 1** If \( m \) is a \( \mu \)-invariant measure for \( P \), then \( m \) is almost \( \mu \)-invariant for \( Q \) if and only if \( P \) is almost \( F \)-type.

**Proof.** Since (7) holds, we may define an honest standard transition function \( P^*(t) = (p^*_{ij}(t), i, j \in S) \) over \( S \) by

\[
p^*_{ij}(t) = e^{\mu t} m_j p_{ji}(t) / m_i, \quad i, j \in S, \ t > 0.
\]

Indeed, \( P^* \) is a \( Q^* \)-transition function, where \( Q^* = (q^*_{ij}, i, j \in S) \) is the \( q \)-matrix with entries

\[
q^*_{ij} = m_j q_{ji} / m_i + \mu \delta_{ij}, \quad i, j \in S.
\]
(\(P^*\) is called the \(\mu\)-reverse of \(P\) with respect to \(m\) and \(Q^*\) the \(\mu\)-reverse of \(Q\) with respect to \(m\); see [9].) It is easy to see that \(Q^*\) is uni-instantaneous with instantaneous state \(b\), and, for \(i \neq b\),
\[
m_i \sum_{j \in S} q^*_ij = \sum_{j \neq i} m_j q_{ji} + \mu m_i - m_i q_i \leq 0.
\]
Moreover, all states \(i \neq b\) are conservative states for \(Q^*\) if and only if (6) holds. It is easy to verify that \(P^*\) is almost \(B\)-type if and only if \(P\) is almost \(F\)-type. So, if (6) holds, then \(Q^*\) is conservative for states \(i \neq b\). Hence, the backward equations (8) hold for \(P^*\) over states \(i \neq b\), implying that \(P\) is almost \(F\)-type. Conversely, if \(P\) is almost \(F\)-type, then if \(P^*\) is almost \(B\)-type. But \(P^*\) is honest, implying that states \(i \neq b\) are conservative states for \(Q^*\).

Hence, (6) holds.

3 The Resolvent Decomposition Theorem

Henceforth we will find it convenient to specify transition functions through their Laplace transforms. If \(P\) is a specified transition function, then the function \(\Psi(\alpha) = (\psi_{ij}(\alpha), i, j \in S)\), given by
\[
\psi_{ij}(\alpha) = \int_0^\infty e^{-\alpha t} p_{ij}(t) \, dt, \quad i, j \in S, \ \alpha > 0,
\]
is called the resolvent of \(P\). Indeed, if \(i, j \in C\), where \(C\) is any irreducible class, then the integral in (9) converges for all \(\alpha > -\lambda_P(C)\), where \(\lambda_P(C)\) is the decay parameter of \(C\) (for \(P\)); see Kingman [6]. Analogous to properties (1)–(4) of \(P\), the resolvent satisfies
\[
\psi_{ij}(\alpha) \geq 0, \quad i, j \in S, \ \alpha > 0, \quad (10)
\]
\[
\sum_{j \in S} \alpha \psi_{ij}(\alpha) \leq 1, \quad i \in S, \ \alpha > 0, \quad (11)
\]
\[
\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in S} \psi_{ik}(\alpha) \psi_{kj}(\beta) = 0, \quad i, j \in S, \ \alpha, \beta > 0, \quad (12)
\]
\[
\lim_{\alpha \to \infty} \alpha \psi_{ij}(\alpha) = \delta_{ij}, \quad i, j \in S. \quad (13)
\]
(Note that (12) is called the resolvent equation.) Indeed, any \(\Psi\) that satisfies (10)–(13) is the resolvent of a standard transition function \(P\) (Lemma 1.1 of Reuter [12]). Furthermore, (11) is satisfied with equality if and only if \(P\) is honest, in which case the resolvent is said to be honest. Also, the \(q\)-matrix of \(P\) can be recovered from \(\Psi\) using the following identity:
\[
q_{ij} = \lim_{\alpha \to \infty} \alpha (\alpha \psi_{ij}(\alpha) - \delta_{ij}). \quad (14)
\]
Finally, a resolvent \(\Psi\) that satisfies (14) is called a \(Q\)-resolvent.

We can identify \(\mu\)-invariant measures using resolvents. If \(P\) is a \(Q\)-process with resolvent \(\Psi\) and \(m = (m_j, j \in S)\) is a \(\mu\)-invariant measure for \(P\), then \(\mu \leq \lambda_P(S)\), where \(\lambda_P(S) = \inf_C \lambda_P(C)\) (the infimum taken over all the irreducible classes comprising \(S\)); see
Lemma 4.1 of Vere-Jones [15]. Furthermore, since the integral in (9) converges for all $\alpha > -\lambda_P(S)$, we have
\[
\sum_{i \in S} m_i \alpha \psi_{ij}(\alpha - \mu) = m_j,
\] for all $j \in S$ and $\alpha > 0$. We refer to $m$ as being $\mu$-invariant for $\Psi$ if (15) is satisfied. Finally, a simple extension of Lemma 1 of Pollett [10] establishes that $m$ is $\mu$-invariant for $\Psi$ if it is $\mu$-invariant for $P$, and, if $\mu \leq \lambda_P(S)$, then $m$ is $\mu$-invariant for $P$ if it is $\mu$-invariant for $\Psi$.

We are assuming that $Q$ is a uni-instantaneous $q$-matrix with instantaneous state $b$, so let us write $N = S \setminus \{b\}$ and $Q_N = (q_{ij}, i, j \in N)$ for the restriction of $Q$ to $N$, and, if $m = (m_i, i \in S)$ is a measure on $S$, then $m_N = (m_i, i \in N)$ will be the restriction of $m$ to $N$.

The following important result combines Theorems 7.7 and 7.8 of Chen and Renshaw [2]. It characterizes $Q$-processes with a single instantaneous state. In preparation, define families $H_\Psi$ and $K_\Psi$, for a given $Q_N$-resolvent $\Psi(\alpha) = (\psi_{ij}(\alpha), i, j \in N)$, as follows: $H_\Psi$ is the set of all non-negative row vectors $\eta(\alpha) = (\eta_i(\alpha), i \in N)$, $\alpha > 0$, satisfying $\sum_{j \in N} \eta_j(\alpha) < \infty$ and
\[
\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in N} \eta_k(\alpha) \psi_{kj}(\beta) = 0, \quad j \in N,
\]
and $K_\Psi$ is the set of all column vectors $\xi(\alpha) = (\xi_i(\alpha), i \in N)$, $\alpha > 0$, satisfying $0 \leq \xi_i(\alpha) \leq 1$, $i \in N$, and
\[
\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k \in N} \psi_{ik}(\alpha) \xi_k(\beta) = 0, \quad i \in N.
\]

**Theorem 2** (Resolvent Decomposition Theorem) For the uni-instantaneous $q$-matrix $Q$, every $Q$-resolvent $R(\alpha) = (r_{ij}(\alpha), i, j \in S)$ can be decomposed uniquely as
\[
R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \psi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha) \eta(\alpha) \end{pmatrix},
\]
where $\Psi(\alpha) = (\psi_{ij}(\alpha), i, j \in N)$ is a $Q_N$-resolvent, and $\eta(\alpha) = (\eta_i(\alpha), i \in N)$ and $\xi(\alpha) = (\xi_i(\alpha), i \in N)$ satisfy the following conditions:

1. $\eta(\alpha) \in H_\Psi$ and $\xi(\alpha) \in K_\Psi$,
2. $\xi_i(\alpha) \leq 1 - \sum_{j \in N} \alpha \psi_{ij}(\alpha), \ i \in N$,
3. $\lim_{\alpha \to -\infty} \alpha \eta_j(\alpha) = q_{bj}, \ j \in N$,
4. $\lim_{\alpha \to -\infty} \alpha \xi_i(\alpha) = q_{ib}, \ i \in N$, and,
5. $r_{bb}(\alpha) = (C + \alpha + \alpha \sum_{j \in N} \eta_j(\alpha) \xi_j)^{-1}$, with $\xi_j := \lim_{\alpha \to -\infty} \xi_j(\alpha)$ and $C (< \infty)$ satisfying
\[
C \geq \lim_{\alpha \to -\infty} \alpha \sum_{j \in N} \eta_j(\alpha)(1 - \xi_j),
\]
and $\lim_{\alpha \to -\infty} \alpha \sum_{j \in N} \eta_j(\alpha) \xi_j = \infty$ (equivalently, $\lim_{\alpha \to -\infty} \alpha \sum_{j \in N} \eta_j(\alpha) = \infty$).
Conversely, if there exists a $Q_N$-resolvent $Ψ$, and $η(α)$ and $ξ(α)$ satisfying the above conditions, then $R$ defined by (17) is a $Q$-resolvent.

Our main result rests on the following three lemmas.

**Lemma 1** Suppose that the uni-instantaneous $q$-matrix $Q$ admits an almost $µ$-invariant measure $m = (m_i, i \in S)$. Then, $d_i(α) = (d_i(α), i \in N)$, defined by

$$d_i(α) = m_i - (α + µ) \sum_{k \in N} m_k φ_{ki}(α), \quad i \in N, \ α > 0,$$

(19)

where $Φ_N(α) = (φ_{ij}(α), i, j \in N)$ is the minimal $Q_N$-resolvent, satisfies

$$\lim_{α \to ∞} αd_i(α) = m_i q_{ii}, \quad i \in N.$$

*Proof.* Since $m$ is almost $µ$-invariant for $Q$, it is easy to see that the restriction $m_N = (m_i, i \in N)$ is a $µ$-subinvariant measure for $Q_N$. Therefore, because $m_N$ is then $µ$-subinvariant for $Φ_N$, we have that $d_i(α) ≥ 0, i \in N, α > 0$. Also, since $Φ_N$ is the minimal $Q_N$-resolvent, it satisfies the resolvent equation

$$φ_{ij}(α) - φ_{ij}(β) + (α - β) \sum_{k \in N} φ_{ik}(α)φ_{kj}(β) = 0, \quad i, j \in N, \ α, β > 0,$$

and therefore

$$\tilde{d}_i(α) - \tilde{d}_i(β) + (α - β) \sum_{k \in N} \tilde{d}_k(α)φ_{kj}(β) = 0, \quad i \in N, \ α, β > 0,$$

(20)

where

$$\tilde{d}_i(α) = m_i - α \sum_{k \in N} m_k φ_{ki}(α), \quad i \in N, \ α > 0.$$

Since $d_i(α) ≥ 0, i \in N, α > 0$, we have $\tilde{d}_i(α) ≥ 0, i \in N, α > 0$. Using (20) we see that, for each $i \in N$, $d_i(α)$ is non-increasing in $α$ and hence $α \sum_{k \in N} m_k φ_{ki}(α)$ is non-decreasing in $α$. Therefore, $\lim_{α \to ∞} α \sum_{k \in N} m_k φ_{ki}(α)$ exists. But, by Fatou’s lemma, $\lim_{α \to ∞} α \sum_{k \in N} m_k φ_{ki}(α) ≥ m_i$, and hence $\lim_{α \to ∞} α \sum_{k \in N} m_k φ_{ki}(α) = m_i$ because $\tilde{d}_i(α) ≥ 0$. Since $Φ_N$ satisfies the forward equation

$$αφ_{ij}(α) = δ_{ij} + \sum_{k \in N} φ_{ik}(α)q_{kj}, \quad i, j \in N, \ α > 0,$$

and (19) can be rewritten as

$$d_i(α) = \sum_{k \in N} m_k (δ_{ki} - (α + µ)φ_{ki}(α)), \quad i \in N, \ α > 0,$$

we may deduce that

$$αd_i(α) = -α \sum_{k \in N} m_k \sum_{j \in N} φ_{kj}(α)q_{ji} - αµ \sum_{k \in N} m_k φ_{ki}(α),$$

$$= -\sum_{j \in N} q_{ji}α \sum_{k \in N} m_k φ_{kj}(α) - µα \sum_{k \in N} m_k φ_{ki}(α),$$

(21)
which leads to
\[
\lim_{\alpha \to \infty} \alpha d_i(\alpha) = -\sum_{j \in N} m_j q_{ji} - \mu m_i = m_b q_{bi}, \quad i \in N.
\]

Lemma 2 Let \( \Psi(\alpha) = (\psi_{ij}(\alpha), i, j \in N) \) be a \( Q_N \)-resolvent and let \( \xi_i = \lim_{\alpha \to 0} \xi_i(\alpha) \), where \( \xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha) \), \( i \in N \). If \( \eta(\alpha) \in H_\Psi \), then \( \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i) \) is finite and does not depend on \( \alpha \).

Proof. By the Dominated Convergence Theorem,
\[
\lim_{\beta \to 0} \alpha \beta \sum_{i \in N} \sum_{j \in N} \eta_i(\alpha)\psi_{ij}(\beta) = \alpha \sum_{i \in N} \eta_i(\alpha) \lim_{\beta \to 0} \beta \sum_{j \in N} \psi_{ij}(\beta) = \alpha \sum_{i \in N} \eta_i(\alpha) \lim_{\beta \to 0} (1 - \xi_i(\beta))
\]
\[
= \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i).
\]

On the other hand, using (16) we get
\[
\lim_{\beta \to 0} \alpha \beta \sum_{i \in N} \sum_{j \in N} \eta_i(\alpha)\psi_{ij}(\beta) = \lim_{\beta \to 0} \alpha \beta \sum_{j \in N} \sum_{i \in N} \eta_i(\alpha)\psi_{ij}(\beta) = \lim_{\beta \to 0} \beta \left( \frac{\alpha \beta}{\beta - \alpha} \sum_{j \in N} (\eta_j(\alpha) - \eta_j(\beta)) \right)
\]
\[
= \lim_{\beta \to 0} \frac{\alpha \beta}{\beta - \alpha} \sum_{j \in N} \eta_j(\alpha) + \lim_{\beta \to 0} \frac{\alpha \beta}{\beta - \alpha} \sum_{j \in N} \eta_j(\beta).
\]

The first term is 0 because \( \sum_{j \in N} \eta_j(\alpha) < \infty \). The second term is \( \lim_{\beta \to 0} \beta \sum_{j \in N} \eta_j(\beta) \), which exists, because it is easy to deduce from (16) that \( \beta \sum_{j \in N} \eta_j(\beta) \) is non-decreasing in \( \beta \). Since this limit does not depend on \( \alpha \), the proof is complete.

Lemma 3 Suppose that \( m = (m_i, i \in S) \) is a strictly positive probability measure. If \( m \) is \( \mu \)-invariant for the \( Q \)-resolvent \( R \) defined in (17), then
(i) \( m_N = (m_i, i \in N) \) is a \( \mu \)-subinvariant measure for \( \Psi \), and
(ii) \( \eta_i(\alpha) = d_i(\alpha)/m_b \), where \( d_i(\alpha) = m_i - (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) \), \( i \in N, \alpha > 0 \).

Conversely, if (i) and (ii) hold, then on setting \( \xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha) \), \( i \in N \), and \( C = \mu/m_b + \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i) \), where \( \xi_i = \lim_{\alpha \to 0} \xi_i(\alpha) \), (17) determines a \( Q \)-resolvent \( R \) for which \( m \) is a \( \mu \)-invariant measure.

Proof. If \( m \) is \( \mu \)-invariant for \( R \), that is,
\[
(\alpha + \mu) \sum_{i \in S} m_i r_{ij}(\alpha) = m_j, \quad j \in S, \alpha > 0,
\]
then \( (\alpha + \mu) \sum_{i \in N} m_i \psi_{ij}(\alpha) \leq m_j, j \in N \), since, from (17), we have \( \psi_{ij}(\alpha) \leq r_{ij}(\alpha), i, j \in N \). This proves (i). Next, from (17) and (22), we have
\[
(\alpha + \mu) r_{ib}(\alpha)m_b + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha)r_{ib}(\alpha) = m_b,
\]
and, for all $i \in N$ and $\alpha > 0$,

$$(\alpha + \mu)\eta_i(\alpha) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha) r_{bb}(\alpha) \eta_i(\alpha) = m_i. \quad (24)$$

These equations combine to give $m_b \eta_i(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) = m_i, \ i \in N$, and hence (ii) holds.

To prove the converse, set $\xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha)$ in (17) and take $\eta(\alpha)$ satisfying (16). Then, by Lemma 2, $\alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i)$ is finite and independent of $\alpha$, and so the given $C$ satisfies (18). It follows that

$$r_{bb}(\alpha) = \left(\frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha)\right)^{-1}.$$ 

Since (i) and (ii) hold, and $\sum_{i \in S} m_i = 1$, we have

$$(\alpha + \mu) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{i \in N} m_i \psi_i(\alpha) r_{bb}(\alpha)$$

$$= r_{bb}(\alpha) \left((\alpha + \mu) m_b + (\alpha + \mu)(1 - m_b) - \alpha \sum_{j \in N} (\alpha + \mu) \sum_{i \in N} m_i \psi_{ij}(\alpha)\right)$$

$$= r_{bb}(\alpha) \left(\mu + \alpha m_b + \alpha m_b \sum_{j \in N} \eta_j(\alpha)\right) = m_b$$

and, for $i \in N$,

$$(\alpha + \mu) \eta_i(\alpha) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha) r_{bb}(\alpha) \eta_i(\alpha)$$

$$= (\alpha + \mu) r_{bb}(\alpha) d_i(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + (\alpha + \mu) r_{bb}(\alpha) \frac{d_i(\alpha)}{m_b} \sum_{k \in N} m_k \xi_k(\alpha)$$

$$= (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + \frac{d_i(\alpha)}{m_b} \left((\alpha + \mu) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{i \in N} m_i \xi_i(\alpha) r_{bb}(\alpha)\right)$$

$$= (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + d_i(\alpha) = m_i.$$ 

Thus, (23) and (24) are satisfied. These in turn imply that (22) holds, that is, $m$ is a $\mu$-invariant measure for $R$.

4 Existence

We are now ready to state our main result.

**Theorem 3** Let $\mu \geq 0$ and suppose that the uni-instantaneous $q$-matrix $Q$ admits a finite almost $\mu$-invariant measure $m = (m_i, i \in S)$. Then, there exists a $Q$-process for which $m$ is a $\mu$-invariant measure.
Proof. Without loss of generality we may assume that \( \sum_{i \in S} m_i = 1 \). Let \( \Phi(\alpha) = (\phi_{ij}(\alpha), i, j \in N) \) be the minimal \( Q_N \)-resolvent. Since \( m \) is almost \( \mu \)-invariant for \( Q \), the restriction \( m_N = (m_i, i \in N) \) is a \( \mu \)-subinvariant measure for \( Q_N \), and hence \( \mu \)-subinvariant for \( \Phi \). Set

\[
\begin{align*}
d_i(\alpha) &= m_i - (\alpha + \mu) \sum_{k \in N} m_k \phi_{ki}(\alpha), \quad i \in N, \ \alpha > 0, \\
\eta_i(\alpha) &= d_i(\alpha)/m_b, \quad i \in N, \ \alpha > 0, \\
\xi_i(\alpha) &= 1 - \alpha \sum_{j \in N} \phi_{ij}(\alpha), \quad i \in N, \ \alpha > 0,
\end{align*}
\]

and

\[
\eta_i(\alpha) - \eta_i(\beta) + (\alpha - \beta) \sum_{k \in N} \eta_k(\alpha) \phi_{ki}(\beta) = 0, \quad i \in N,
\]

and

\[
\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k \in N} \phi_{ik}(\alpha) \xi_k(\beta) = 0, \quad i \in N.
\]

Using Lemma 1, we see that

\[
\lim_{\alpha \to \infty} \alpha \eta_j(\alpha) = \lim_{\alpha \to \infty} \alpha \frac{d_j(\alpha)}{m_b} = q_j, \quad j \in N,
\]

and

\[
\lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha) = \lim_{\alpha \to \infty} \frac{1}{m_b} \sum_{j \in N} \alpha d_j(\alpha) = \sum_{j \in N} q_{bj} = \infty.
\]

Also,

\[
\lim_{\alpha \to \infty} \alpha \xi_i(\alpha) = \lim_{\alpha \to \infty} \sum_{k \in N} \alpha (\delta_{ik} - \alpha \phi_{ik}(\alpha)) = -\sum_{k \in N} q_{ik} = q_{ib}, \quad i \in N.
\]

Therefore, using (27), (29) and Lemma 2, we deduce that \( \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i) \) is finite and independent of \( \alpha \). Now set

\[
C = \frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i),
\]

where \( \xi = \lim_{\alpha \to 0} \xi(\alpha) \) and observe that \( C \) satisfies (18) of Theorem 2. Hence, in view of Theorem 2, we may use (25)-(28) to construct a \( Q \)-resolvent \( R \) by setting

\[
R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{ib}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha) \eta(\alpha) \end{pmatrix},
\]

and then use the second part of Lemma 3 to deduce that \( m \) is a \( \mu \)-invariant measure for \( R \). This completes the proof.

Remark When \( \mu = 0 \), Theorem 3 reduces to the result of Zhang, Lin and Hou [18].
5 Examples

Example 1 We will begin with an example, generally known as “K1”, described by Kolmogorov [7] and analysed by Kendall and Reuter [5] and Reuter [13] (see also the discussion in Chung [4] and Anderson [1]). It has a $q$-matrix over the non-negative integers, given by

$$Q = \begin{pmatrix} -\infty & 1 & 1 & 1 & \cdots \\ q_1 & -q_1 & 0 & 0 & \cdots \\ q_2 & 0 & -q_2 & 0 & \cdots \\ q_3 & 0 & 0 & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (30)$$

where $q_i > 0$, $i \geq 1$. If a $\mu$-subinvariant measure exists for $Q$ then $\mu \leq \inf_i q_i$ (Corollary 1 of Kingman [6]). We will assume that $\mu < q_i$, for all $i \geq 1$. Then, for any such $\mu$, $Q$ admits a $\mu$-invariant measure $m = (m_i, i \geq 0)$ given by $m_i = m_0/(q_i - \mu)$, $i \geq 1$, with $m_0$ arbitrary. This is finite if and only if

$$\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty, \quad (31)$$

in which case $Q$ has the unique $\mu$-invariant probability measure

$$m_0 = 1/A, \quad m_i = \frac{m_0}{q_i - \mu}, \quad i \geq 1, \quad (32)$$

where $A = 1 + \sum_{i=1}^{\infty} 1/(q_i - \mu)$. Therefore, an immediate consequence of Theorems 2 and 3 and Lemma 3, is the following simple result:

**Proposition 1** If $Q$ defined in (30) satisfies (31), then there exists a $Q$-process for which $m$, defined by (32), is a $\mu$-invariant probability measure. The resolvent of one such process is given by

$$R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha)\eta(\alpha) \end{pmatrix},$$

where

$$\phi_{ij}(\alpha) = \frac{\delta_{ij}}{\alpha + q_i}, \quad i, j \geq 1, \quad \alpha > 0,$$

$$\xi_i(\alpha) = \frac{q_i}{\alpha + q_i}, \quad i \geq 1, \quad \alpha > 0,$$

$$\eta_j(\alpha) = \frac{1}{\alpha + q_j}, \quad j \geq 1, \quad \alpha > 0,$$

and

$$r_{bb}(\alpha) = \left( \frac{\mu}{m_0} + \alpha + \alpha \sum_{i=1}^{\infty} \eta_i(\alpha) \right)^{-1}.$$
Example 2 Next we consider the following $q$-matrix, describing a birth-death process incorporating catastrophes to state 0 and instantaneous resurrection from state 0:

\[
Q = \begin{pmatrix}
-d_1 & h_1 & h_2 & h_3 & h_4 & \cdots \\
-(d_1 + b_1) & b_1 & 0 & 0 & 0 & \cdots \\
-d_2 & a_2 & -(a_2 + b_2 + d_2) & b_2 & 0 & \cdots \\
0 & a_3 & -(a_3 + b_3 + d_3) & b_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(33)

where $d_i > 0$, $b_i > 0$, $i \geq 0$, $a_i > 0$, $i \geq 1$, $h_j \geq 0$, $j \geq 1$, and $\sum_{j=1}^{\infty} h_j = \infty$. Define $\pi = (\pi_i, i \geq 1)$ by $\pi_1 = 1$ and, for $i \geq 2$,

\[
\pi_i = \prod_{j=2}^{i} \frac{b_{j-1}}{a_j}.
\]

It is easy to show that if $\mu$ satisfies $0 \leq \mu \leq \inf_{i \geq 1} d_i$, and $h_i = c\pi_i(d_i - \mu)$, $i \geq 1$, where $c$ is a positive constant, then $m = (m_i, i \geq 0)$ given by

\[
m_0 = 1, \quad m_i = c\pi_i, \quad i \geq 1,
\]

(34)
is a $\mu$-invariant measure for $Q$.

Proposition 2 If $\mu$ satisfies $0 \leq \mu \leq \inf_{i \geq 1} d_i$, and $Q$ defined in (33) satisfies $\sum_{i=1}^{\infty} \pi_i < \infty$ and $\sum_{i=1}^{\infty} \pi_i d_i = \infty$, then there exists a $Q$-process for which $m$, defined by (34), is a $\mu$-invariant probability measure.

Proof. The condition $\sum_{i=1}^{\infty} \pi_i < \infty$ implies that $m$ is a finite measure, and $\sum_{i=1}^{\infty} \pi_i d_i = \infty$ and $\sum_{i=1}^{\infty} \pi_i < \infty$ together imply that $\sum_{j=1}^{\infty} h_j = \infty$. Hence, the result follows from Theorem 3.

6 Necessary conditions

In both of the examples above our finite measure $m$ satisfied

\[
\sum_{i \neq b} m_i q_{ib} = \infty,
\]

(35)

and hence was invariant for $Q$ (that is, (5) holds for all $j \in S$). We have established that only almost $\mu$-invariance is needed for the existence of a $Q$-process for which the given (finite) measure is $\mu$-invariant. It would therefore be of interest to know whether (35) is actually necessary for a measure $m$ (finite or infinite) to be $\mu$-invariant for $P$. We shall content ourselves with the following result, which shows that (35) is necessary in the $\mu = 0$ case under the condition that $P$ is reversible.

Theorem 4 Let $Q$ be a uni-instantaneous $q$-matrix with instantaneous state $b$ and let $P$ be a $Q$-process with invariant measure $m$. If $P$ is reversible with respect to $m$, that is,

\[
m_i p_{ij}(t) = m_j p_{ji}(t), \quad i, j \in S,
\]

(36)

then (35) holds.
Proof. On dividing (36) by $t$ and letting $t \downarrow 0$, we get $m_i q_{ij} = m_j q_{ji}, \ j \neq i$. Hence,

$$\sum_{i \neq j} m_i q_{ij} = m_j \sum_{i \neq j} q_{ji}, \quad j \in S,$$

so that, in particular, since $Q$ is conservative,

$$\sum_{i \neq b} m_i q_{ib} = m_b \sum_{i \neq b} q_{bi} = \infty.$$

We gain some insight into the general case by way of the following simple result, which follows directly from the proof of Theorem 1.

**Theorem 5** Let $Q$ be a uni-instantaneous $q$-matrix with instantaneous state $b$ and let $P$ be a $Q$-process with $\mu$-invariant measure $m$. Let $P^*$ and $Q^*$ be, respectively, the $\mu$-reverse of $P$ with respect to $m$ and the $\mu$-reverse of $Q$ with respect to $m$. Then, $P^*$ is honest. In particular, $b$ is an honest state for $P^*$, whilst being instantaneous for $Q^*$. Moreover,

$$m_b \sum_{j \neq b} q^*_{bj} = \sum_{j \neq b} m_j q_{jb},$$

so that, in particular, $b$ is a conservative state for $Q^*$ if and only if (35) holds.

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**References**


