Costs and Decisions in Population Control

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AUSTRALIAN RESEARCH COUNCIL
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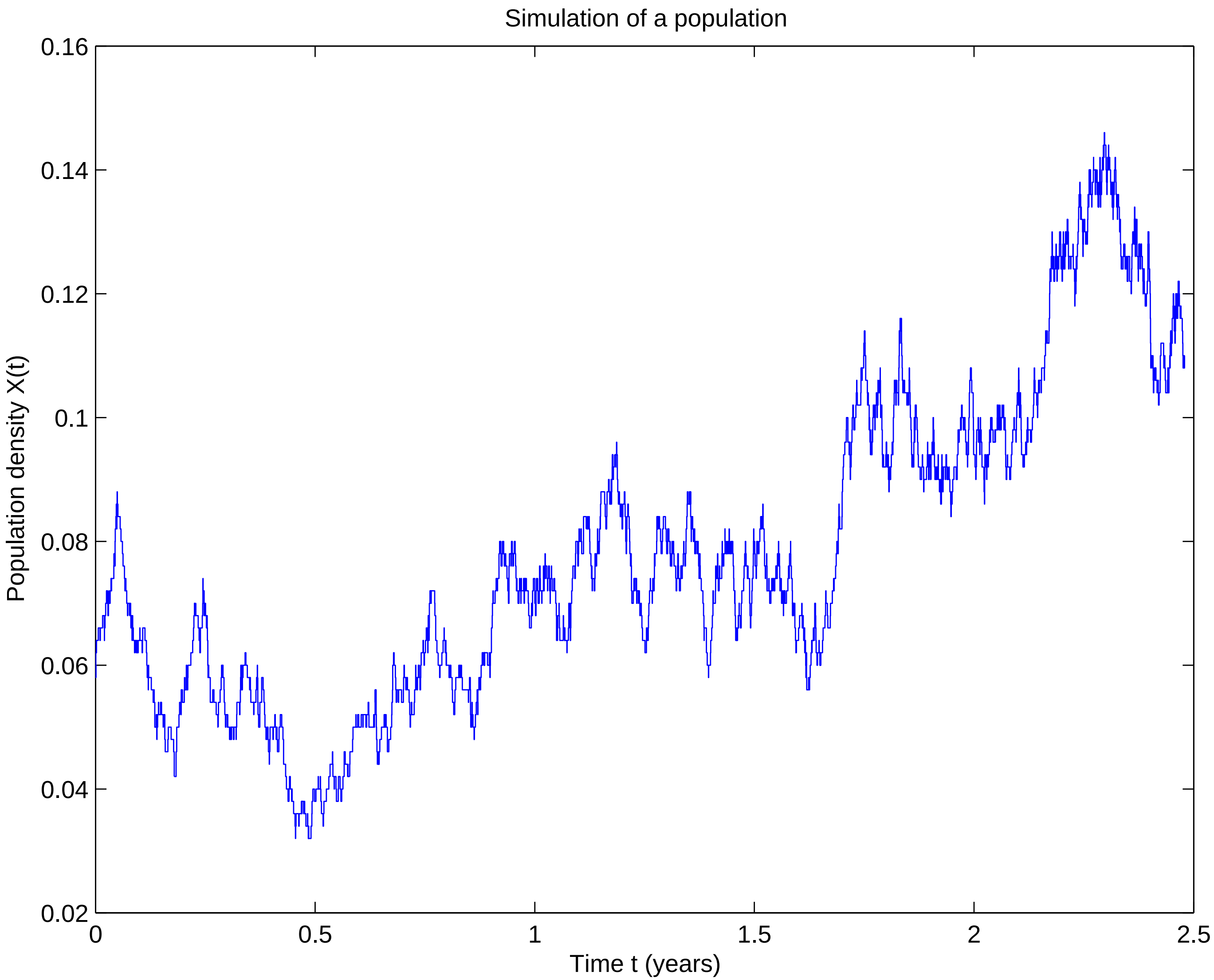
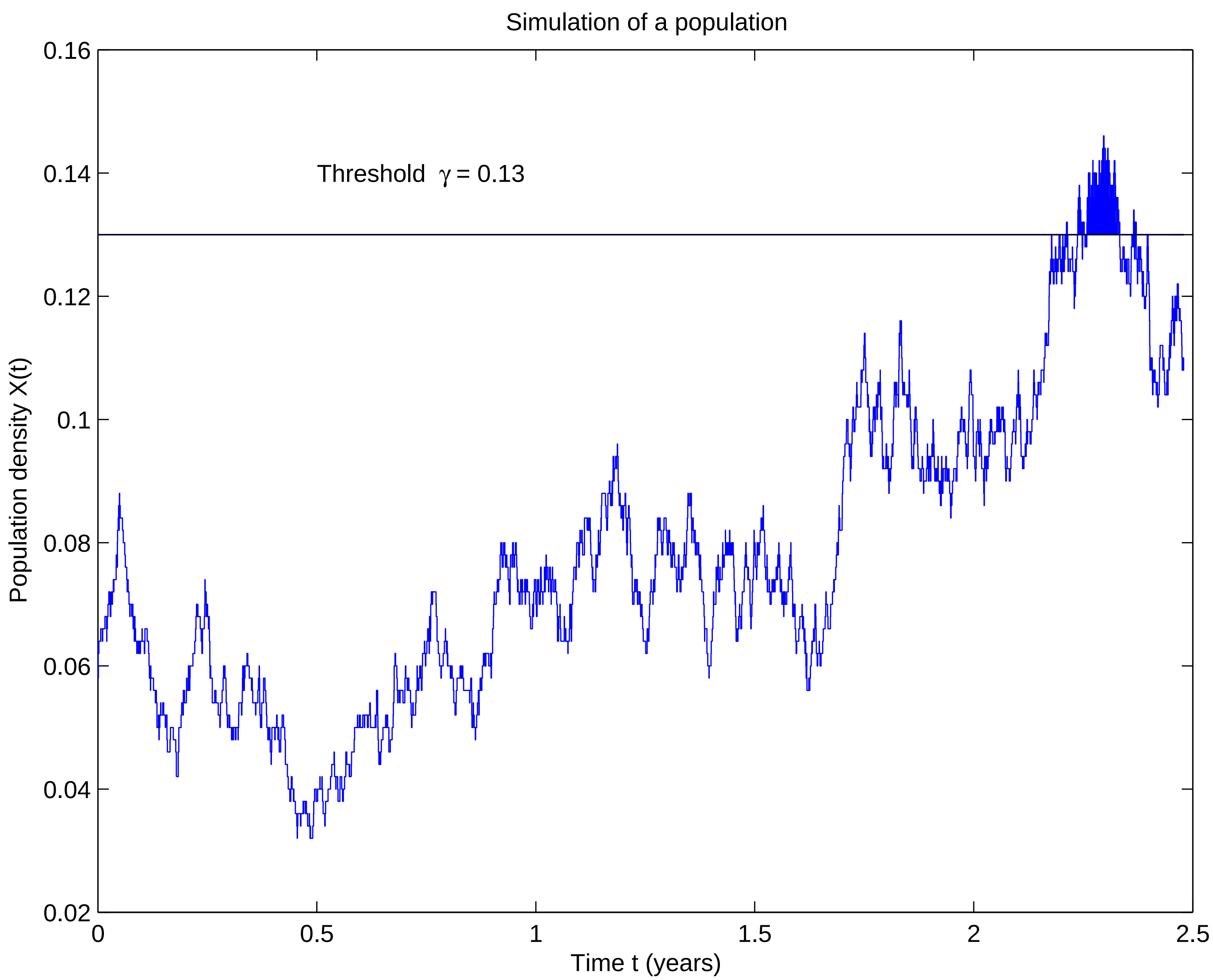
and Statistics of Complex Systems

A population process



A population process



Total cost

Let $X(t)$ be the population density at time t.

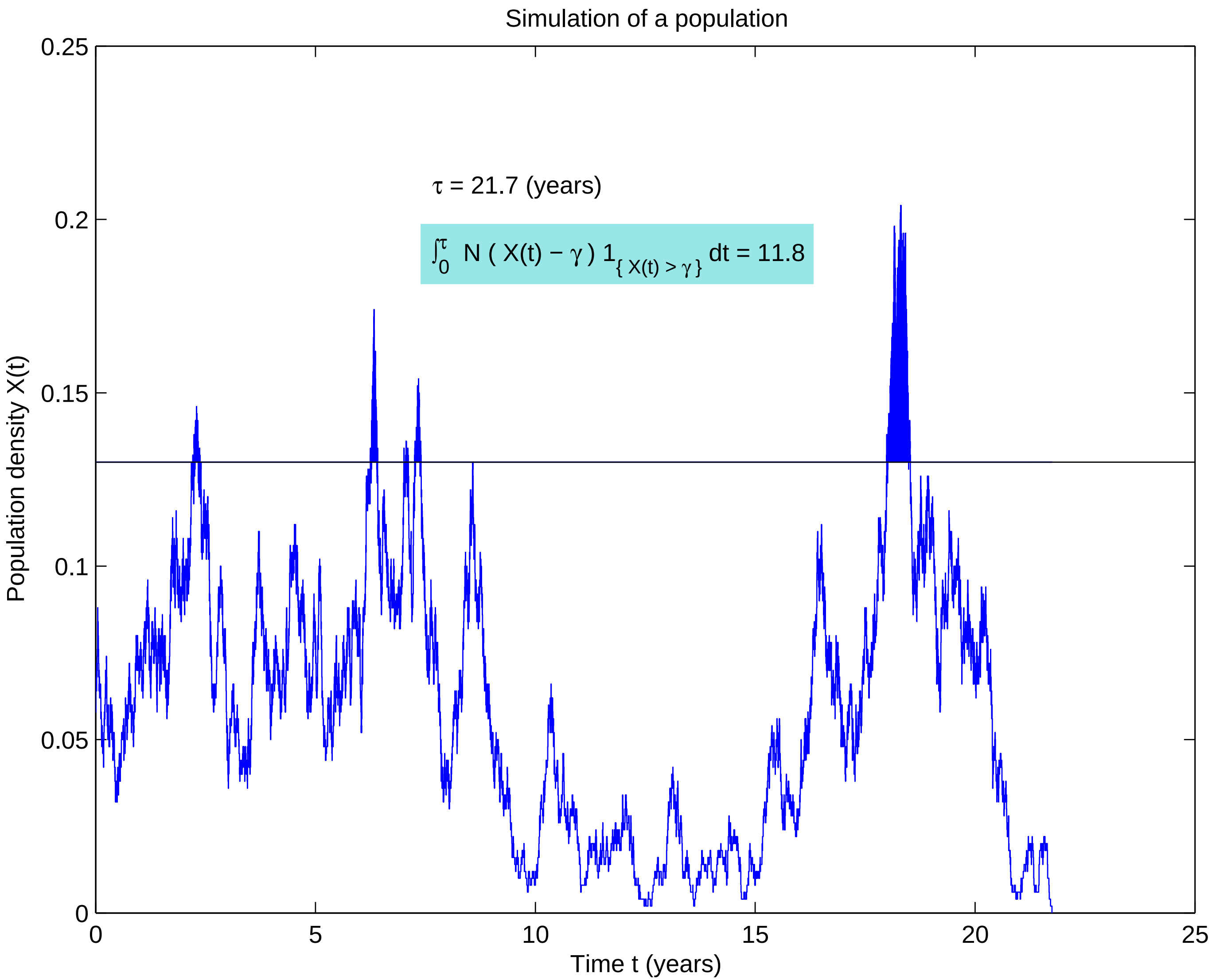
Let $c(x)$ be the cost per unit time of maintaining the population when its density is x units above a threshold γ.

Then, if τ is the time to extinction,

$$\int_0^\tau c(X(t) - \gamma) 1_{\{X(t) > \gamma\}} \, dt$$

is the total cost over the life of the population.

A population process



Ingredients

- A random process $(X(t),\ t \geq 0)$ in continuous time

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- A set of states A
- The (random) time τ to first exit from A
- The cost (per unit time) f_x of being in state x
- The "path integral"

$$\Gamma = \int_0^{\tau} f_{X(t)}\, dt,$$

the total cost incurred before leaving A (also random)

Other examples

- Consider a dam with finite capacity V, and let $X(t)$ be the water level at time t.

  We might wish to estimate the total time for which the level was below a given value γ,

$$\Gamma = \int_0^\tau 1_{\{X(t)<\gamma\}} \, dt,$$

  where τ is (say) the time to reach capacity or to empty (whichever occurs first).

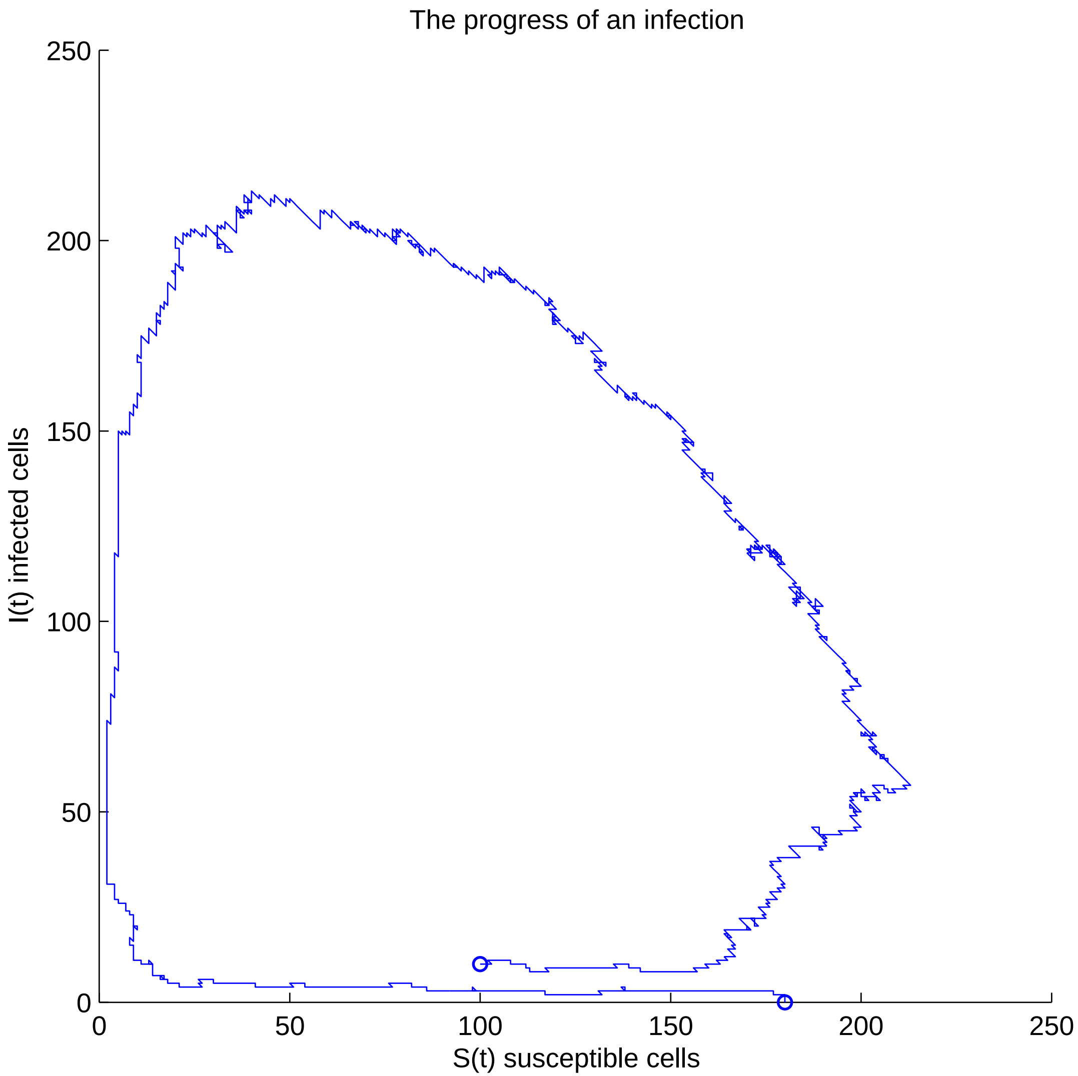
Other examples

- Let $(S(t), I(t))$ be the number of susceptibles and infectives in an epidemic at time t.

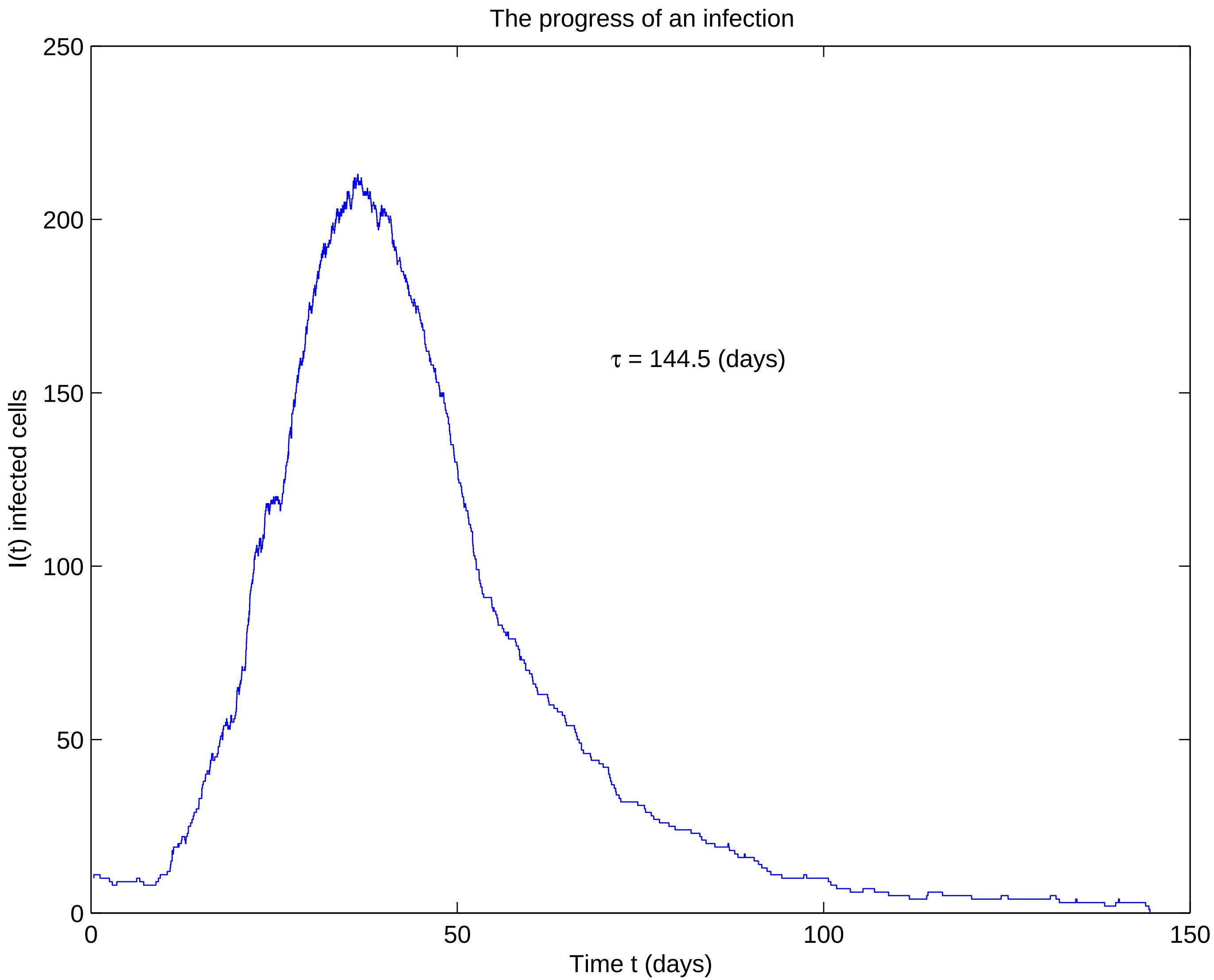
  If τ is the period of infection and $f_{(s,i)} = i$, then Γ is the total amount of infection:

$$\Gamma = \int_0^\tau I(t)\,dt.$$

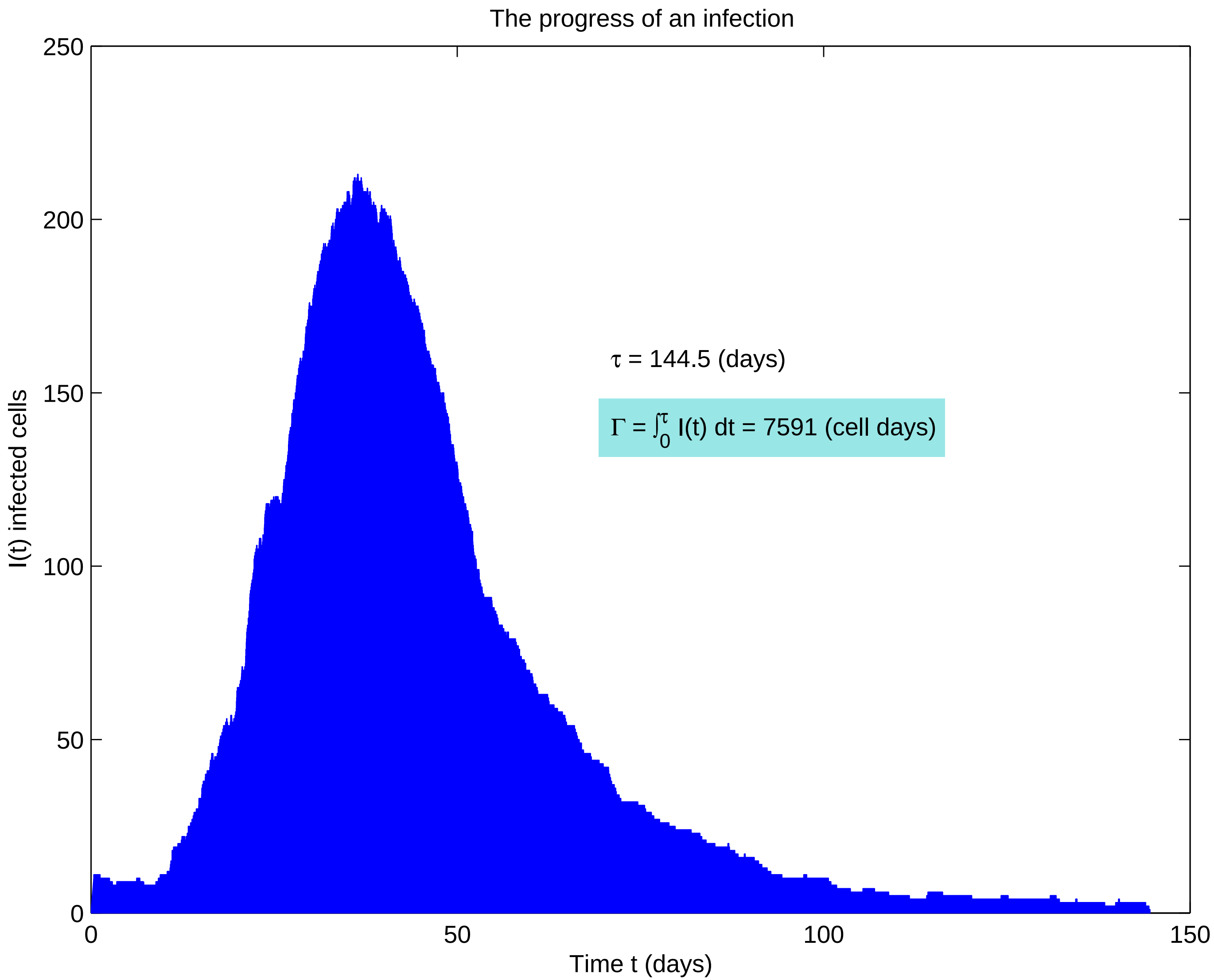
Epidemic


The progress of an infection
I(t) infected cells
S(t) susceptible cells
0
50
100
150
200
250
0
50
100
150
200
250

Epidemic


The progress of an infection
I(t) infected cells
250
200
150
100
50
0
0
50
100
150
Time t (days)
τ = 144.5 (days)

Epidemic



The progress of an infection
250
200
150
100
50
0
0
50
100
150
I(t) infected cells
Time t (days)
τ = 144.5 (days)
Γ = ∫₀^τ I(t) dt = 7591 (cell days)

The problem

Our problem is to determine the expected value, and the distribution of the total cost

$$\Gamma = \int_0^{\tau} f_{X(t)}\, dt,$$

where recall that τ is the time to first exit from a set A and f_x is cost per unit time of being in state x.

For simplicity, suppose that $X(t)$ takes values in $S = \{0, 1, \dots\}$.

For example, $X(t)$ might be the number in a population at time t, and $A = \{1, 2, \dots\}$, so that τ is the time to extinction.

Markovian models

We will assume that $(X(t),\, t \geq 0)$ is a Markov chain with transition rates

$$Q = (q_{ij},\, i, j \in S),$$

so that q_{ij} represents the rate of transition from state i to state j, for $j \neq i$, and $q_{ii} = -q_i$, where

$$q_i := \sum_{j \neq i} q_{ij} \; (< \infty)$$

represents the total rate out of state i.

Markovian models

An example is the birth-death process, which has

$$q_{i,i+1} = \lambda_i \qquad \text{(birth rates)}$$
$$q_{i,i-1} = \mu_i \qquad \text{(death rates)},$$

with $\mu_0 = 0$ and otherwise 0 ($q_i = \lambda_i + \mu_i$):

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Example

The Stochastic Logistic Model (simulated earlier) is a birth-death process on $S = \{0, 1, \ldots, N\}$, with

$$\lambda_i = \frac{\lambda}{N} i(N - i) \quad \text{and} \quad \mu_i = \mu i,$$

where $\lambda, \mu > 0$.

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The epidemic model mentioned earlier is a two-dimensional Markov chain with transition rates

$$q_{(s\ i),(s+1\ i)} = \alpha s, \qquad q_{(s\ i),(s\ i-1)} = \gamma i,$$

$$q_{(s\ i),(s-1\ i+1)} = \beta s i,$$

$\alpha, \gamma, \beta > 0$ are the splitting, removal and infection rates.

The expected value of Γ

Returning to our general Markov chain, let $e_i = E_i(\Gamma) := E(\Gamma | X(0) = i)$, and condition on the time of the first jump and the state visited at that time, to get

$$E_i\left(\Gamma\right) = \int_0^\infty \sum_{k \neq i} \left(\frac{f_i}{q_i} + E_k\left(\Gamma\right) \right) \frac{q_{ik}}{q_i} q_i e^{-q_i u} \, du,$$

which leads to

$$q_i e_i = f_i + \sum_{k \neq i} q_{ik} e_k,$$

so that

$$\sum_k q_{ik} e_k + f_i = 0.$$

The expected value of Γ

We can do better:

Theorem 1 $\boldsymbol{e} = (e_i,\, i \in A)$, where $e_i = E_i(\Gamma)$, is the minimal non-negative solution to

$$\sum_{k \in A} q_{ik} z_k + f_i = 0, \quad i \in A,$$

in the sense that $\boldsymbol{e}$ satisfies these equations, and, if $\boldsymbol{z} = (z_i,\, i \in A)$ is any non-negative solution, then $e_i \le z_i$ for all $i \in A$.

Birth-death processes

Let's apply this to birth-death processes:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Assume that the birth rates $(\lambda_i,\, i \geq 1)$ and the death rates $(\mu_i,\, i \geq 0)$ are all strictly positive, except that $\lambda_0 = 0$. So, all states in $A = \{1, 2, \dots\}$ intercommunicate, and 0 is an absorbing state (corresponding to population extinction).

Birth-death processes

Define $(\pi_i,\, i \geq 1)$ by $\pi_1 = 1$ and

$$\pi_i = \prod_{j=2}^{i} \frac{\lambda_{j-1}}{\mu_j}, \qquad i \geq 2,$$

and assume that

$$\sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} = \infty,$$

a condition that corresponds to extinction being certain.

Birth-death processes

On applying Theorem 1 we get:

Proposition The expected cost up to the time of extinction, starting in state i (≥ 1), is given by

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{\infty} f_k \pi_k,$$

this being finite if and only if $\sum_{k=1}^{\infty} f_k \pi_k < \infty$.

Birth-death processes

In the finite state-space case ($S = \{0, 1, \ldots, N\}$), we get

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{N} f_k \pi_k, \qquad i = 1, 2, \ldots, N.$$

For the Stochastic Logistic Model,

$$E_i(\Gamma) = \frac{1}{\mu} \sum_{j=1}^{i} \sum_{k=0}^{N-j} \left(\frac{1}{N\rho} \right)^k \frac{f_{j+k}}{j+k} \frac{(N-j)!}{(N-j-k)!},$$

where $\rho = \mu/\lambda$. If $\rho < 1$ (the interesting case),

$$E_i(\Gamma) \sim \frac{\rho}{\mu(1-\rho)} \left(\frac{e^{-(1-\rho)}}{\rho} \right)^N \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{i} f_j \rho^j \quad \text{as } N \to \infty.$$

The distribution of Γ

Can we evaluate the distribution of Γ, that is,

$$\Pr(\Gamma \leq x | X(0) = i)\,?$$

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I will explain how to evaluate $y_i(\theta) = E_i(e^{-\theta\Gamma})$, the Laplace-Steiltjes Transform (LST) of the distribution:

$$y_i(\theta) = \int_0^\infty e^{-\theta x}\, d\Pr(\Gamma \leq x | X(0) = i).$$

The distribution of Γ

An argument similar to that used to evaluate $E_i(\Gamma)$ leads to:

Theorem 2 For each $\theta > 0$, $\boldsymbol{y}(\theta) = (y_i(\theta), i \in S)$ is the maximal solution to

$$\sum_{k \in S} q_{ik} z_k = \theta f_i z_i, \quad i \in A,$$

with $0 \leq z_i \leq 1$ for $i \in A$ and $z_i = 1$ for $i \notin A$.

A catastrophe process

Assume that the transition rates have the form

$$q_{ij} = \begin{cases} i\rho a, & i \geq 0,\ j = i+1, \\ -i\rho, & i \geq 0,\ j = i, \\ i\rho d_{i-j}, & i \geq 2,\ 1 \leq j < i, \\ i\rho \sum_{k \geq i} d_k, & i \geq 1,\ j = 0, \end{cases}$$

with all other transition rates equal to 0. Here ρ and a are positive, d_i is positive for at least one i in $A = \{1, 2, \ldots\}$ and $a + \sum_{i=1}^{\infty} d_i = 1$.

Clearly 0 is an absorbing state for the process and A is a communicating class.

A catastrophe process

We will consider only the subcritical case, where the drift D, given by $D = a - \sum_{i=1}^{\infty} i d_i$, is strictly negative and extinction is certain.

Let $b(s) = d(s) - s$, where d is the probability generating function $d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}$, $|s| < 1$.

There is a unique solution, σ, to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

We can evaluate $E_i(e^{-\theta\Gamma})$ for specific choices of f.

For example, take $f_i = i$.

We seek the maximal solution to

$$\sum_{j=0}^{\infty} q_{ij} z_j = \theta i z_i, \qquad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

We can evaluate $E_i(e^{-\theta\Gamma})$ for specific choices of f.

For example, take $f_i = i$.

We seek the maximal solution to

$$\rho a z_{i+1} - \rho z_i + \rho \sum_{j=1}^{i-1} d_{i-j} z_j + \rho z_0 \sum_{j=i}^{\infty} d_j = \theta z_i, \quad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

Multiplying by s^{i-1} and summing over i gives

$$\sum_{i=1}^{\infty} E_i(e^{-\theta\Gamma})s^{i-1} = \frac{1}{1-s} - \frac{\theta(\gamma_\theta - s)}{(1-\gamma_\theta)(1-s)(\rho b(s) - \theta s)},$$

where γ_θ is the unique solution to $\rho b(s) = \theta s$ on the interval $0 < s < \sigma$, where σ itself is the unique solution to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

In the case of "geometric catastrophes" ($d_i = d(1-q)q^{i-1}$, $i \geq 1$, where $d > 0$ satisfies $a + d = 1$, and $0 \leq q < 1$), we get

$$E_i(e^{-\theta\Gamma}) = \frac{\beta(\theta) - q}{1-q}\left(\beta(\theta)\right)^{i-1}, \quad i \geq 1,$$

where $\beta(\theta)$ is the smaller of the two zeros of $a\rho s^2 - (\rho(1+qa)+\theta)s + \rho(d+qa) + q\theta$.

Koalas (*Phascolarctos cinereus*)

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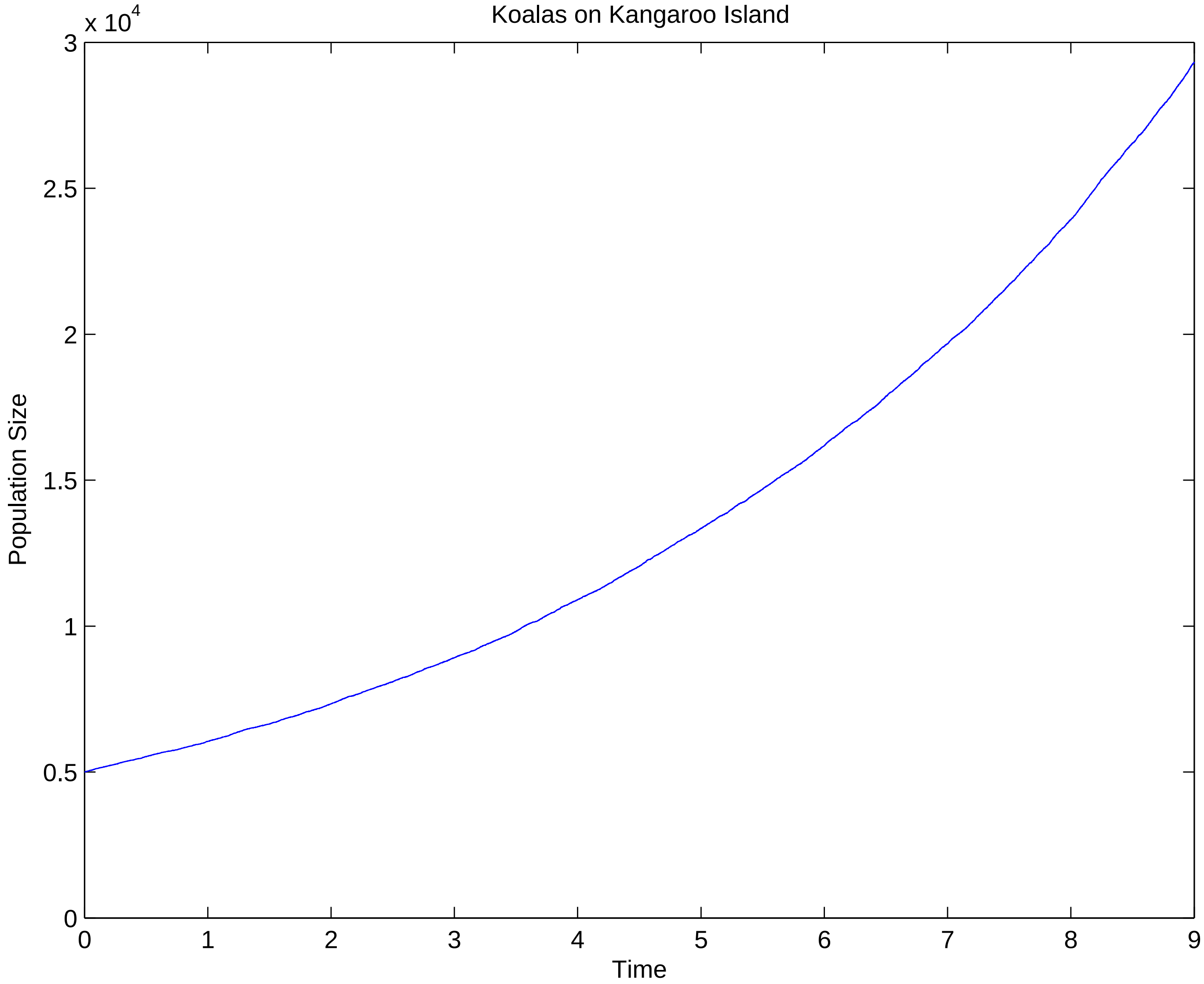
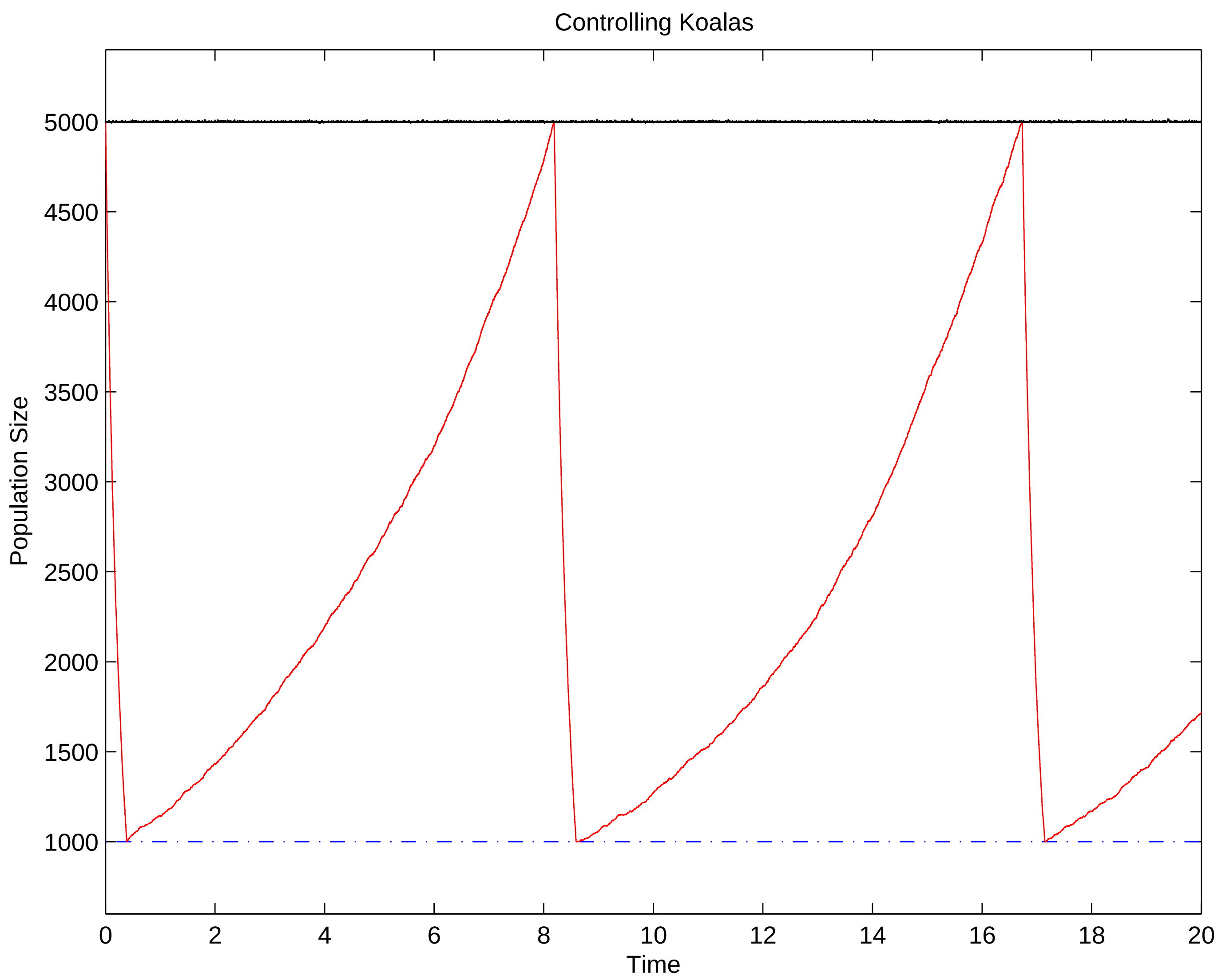
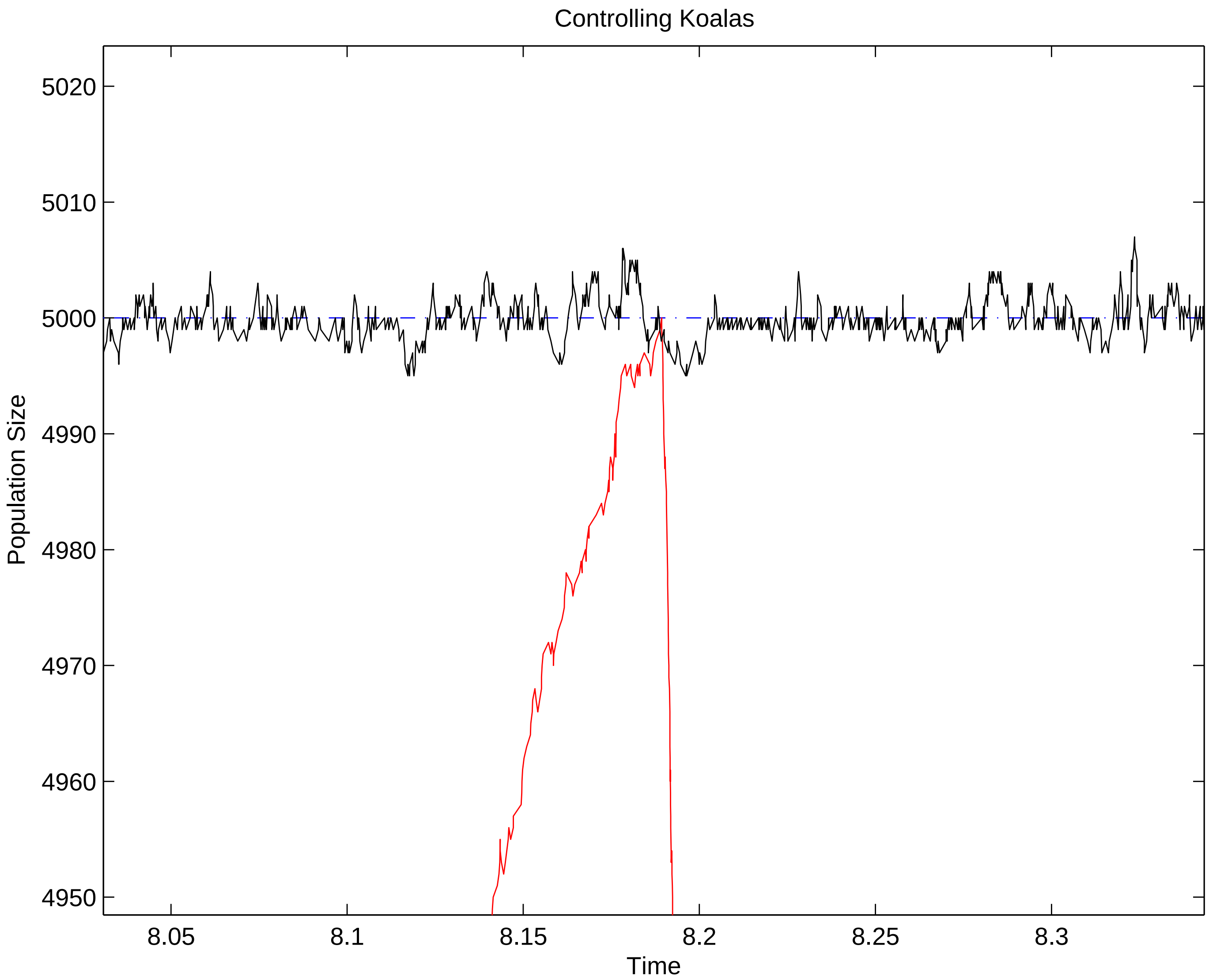


Controlling Koalas



Controlling Koalas



Outline

- Stochastic Models

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- Selection of Rates and Reduction Level

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- Selection of Control Policy

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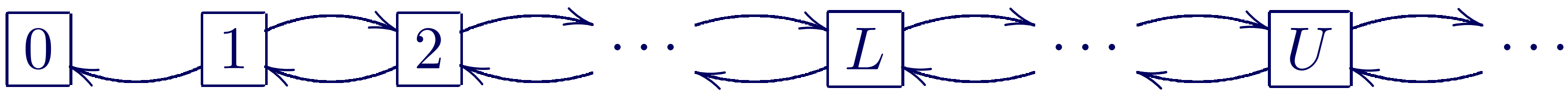
- Selection of Rates and Reduction Level
- Selection of Control Policy
- - Total cost of control

Models - Controlled Populations

The birth-and-death process - Transition Diagram



Models - Controlled Populations

The birth-and-death process is a continuous-time Markov chain taking values in $S = \{0, 1, \dots\}$ with non-zero transition rates

$$q_{x,x+1} = \lambda_x$$

and

$$q_{x,x-1} = \mu_x$$

where λ_x and μ_x are the population birth and death rates respectively.

Models - Controlled Populations

The linear birth-and-death process is a continuous-time Markov chain taking values in $S = \{0, 1, \dots\}$ with non-zero transition rates

$$q_{x,x+1} = \lambda x$$

and

$$q_{x,x-1} = \mu x$$

where λ and μ are the per individual birth and death rates respectively.

Models - Controlled Populations

Linear birth-and-death process with suppression and constant culling

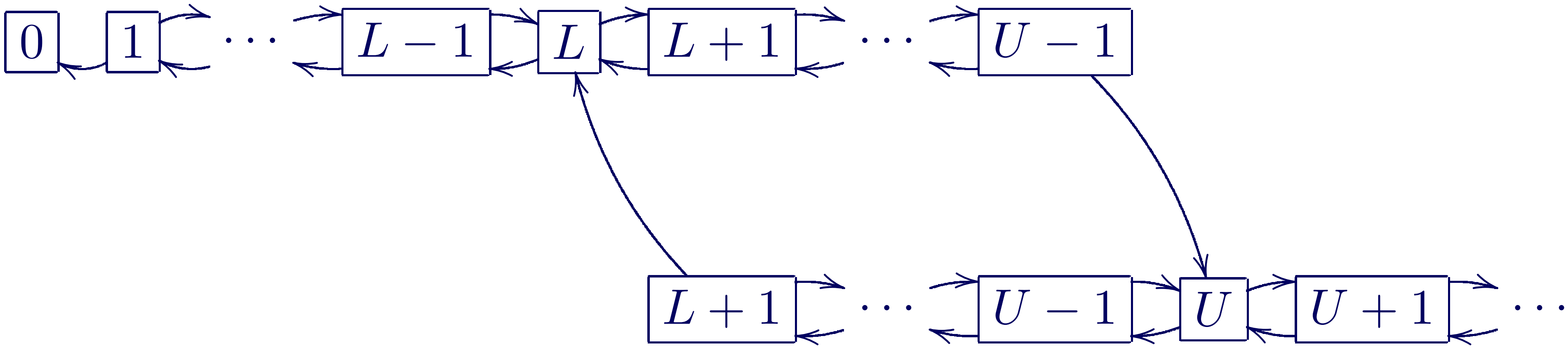
$$q_{x,x+1} = \lambda x \qquad \text{for all } x$$

$$q_{x,x-1} = \begin{cases} \mu x & x \leq U \\ \mu x + \kappa & x > U \end{cases}$$

where κ is the rate of culling (control).

Models - Controlled Populations

Transition Diagram for Reduction Regime Models



Models - Controlled Populations

Linear birth-and-death process with reduction and per-capita culling

$$q_{(x,0),(x+1,0)} = \lambda x \qquad x < U - 1$$

$$q_{(x,0),(x-1,0)} = \mu x \qquad x < U$$

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$$q_{(U-1,0),(U,1)} = \lambda(U - 1)$$

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$$q_{(U-1,0),(U,1)} = \lambda(U-1)$$

$$q_{(x,1),(x+1,1)} = \lambda x \qquad x \in \{L+1, L+2, \ldots\}$$

$$q_{(x,1),(x-1,1)} = (\mu + \psi)x \qquad x \in \{L+2, L+3, \ldots\}$$

$$q_{(L+1,1),(L,0)} = (\mu + \psi)(L+1)$$

where ψ is the rate of culling (control).

Some Decisions of Controlling

- Which control regime?

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- How much culling (control) to perform?

  i.e. What level should L be set to?

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  i.e. What level should L be set to?

- What rate of culling to use?

  i.e. How large should κ and/or ψ be?

Koala Parameters

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Koala Parameters

- Per-koala birth rate: $\lambda = 0.3$
- Per-koala death rate: $\mu = 0.1$
- Culling level: $U = 5,000$
- Reduction level: $L =$?
- Reduction per-koala culling rate: $\psi =$?
- Suppression constant culling rate: $\kappa =$?

Choice of Reduction Level L

- Probability of the population “persisting”.

Choice of Reduction Level L

- Probability of the population reaching the culling level U before 0 starting from reduction level L

$$\alpha_L = \Pr(\text{hit } U \text{ before } 0 \mid X(0) = L) \geq \rho.$$

Choice of Reduction Level L

- Probability of the population reaching the culling level U before 0 starting from reduction level L

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- Expected time between culling phases.

Choice of Reduction Level L

- Probability of the population reaching the culling level U before 0 starting from reduction level L

$$\alpha_L = \Pr(\text{hit } U \text{ before } 0 \mid X(0) = L) \geq \rho.$$

- Expected time to hit U starting from L conditional on hitting U before 0

$$\text{E}(T_U \mid X(0) = L, \text{ hit } U \text{ before } 0).$$

Choice of Reduction Level L

For a birth-and-death process

$$\alpha_i = \Pr(\text{hit } U \text{ before } 0 | X(0) = i) = \frac{s_i}{s_U}$$

where $s_0 = 0$, $s_1 = 1$ and for $2 \leq i \leq U$

$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}.$$

Choice of Reduction Level L

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$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}.$$

Therefore we have

$$\alpha_i = \frac{1 - \left(\frac{\mu}{\lambda}\right)^i}{1 - \left(\frac{\mu}{\lambda}\right)^U}.$$

Choice of Reduction Level L

After choosing a suitable minimum probability ρ, the minimum reduction level L is given by

$$L := \left\lceil \frac{\ln\{1 - \rho[1 - (\mu/\lambda)^U]\}}{\ln(\mu/\lambda)} \right\rceil .$$

Koalas - Minimum L

L	ρ
4	0.9876543209877
6	0.9986282578875
8	0.9998475842097
10	0.9999830649122
12	0.9999981183236
14	0.9999997909248
16	0.9999999767694
18	0.9999999974188
20	0.9999999997132

Expected Phase Times

- Phase 1 - Time between culling phases.

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- - Monitoring and assessment periods.

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$$\tau_L = \mathrm{E}(T_U \,|\, \text{hit } U \text{ before } 0, X(0) = L) = \sum_{i=L}^{U-1} \frac{1}{\lambda_i \pi_i s_i s_{i+1}} \sum_{j=1}^{i} s_j^2 \pi_j$$

Expected Phase Times

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where $s_0 = 0$, $s_1 = 1$ and for $2 \le i \le U$,

$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}$$

and $\pi_1 = 1,\ \pi_j = \prod_{i=2}^{j} \frac{\lambda_{i-1}}{\mu_i}$ for $j \ge 2$.

Koalas - Expected Phase 1 Time

L	Expected Time (yrs)
20	27.868
500	11.522
1000	8.051
2000	4.583
3000	2.555

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L	Expected Time (yrs)
20	27.868
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$L = 1000$.

Expected Phase Times

- Phase 2 - Duration of culling phase.

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- - Planning and choice of culling rates.

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For our model

$$\tau_U^L = \frac{1}{\mu+\psi} \sum_{k=L+1}^{U} \sum_{j=0}^{\infty} \frac{1}{j+k} \left(\frac{\lambda}{\mu+\psi} \right)^j .$$

Choice of Culling Rates κ and ψ

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For a birth-death process

$$c_U = \sum_{k=L+1}^{U} \frac{1}{\mu_k \pi_k} \sum_{j=k}^{\infty} f_j \pi_j$$

where $\pi_j = \prod_{i=L+1}^{j} \frac{\lambda_{i-1}}{\mu_i}$ and f_j is the cost per unit time of culling a population of size j.

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- - Cost function $f_j = d\psi^{1+\delta} j$ or $f_j = d\psi^{1+\delta}\left(b + \frac{c}{j}\right) j$.

Choice of Culling Rates κ and ψ

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- - Minimise the cost of the culling phase.
- - Cost function $f_j = d\psi^{1+\delta} j$ or $f_j = d\psi^{1+\delta}\left(b + \frac{c}{j}\right) j$.

Therefore we have

$$c_U = \frac{d\psi^{1+\delta}(U - L)}{\mu + \psi - \lambda}.$$

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$\delta = 0.05 \implies \psi = 4.2$ and $\tau_U^L = 246$ hrs $\implies 41$ days.

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For a birth-death process

$$\alpha_{U+1} = \frac{s_{U+1}}{s_{\left\lceil \frac{\kappa}{\lambda-\mu} \right\rceil}}$$

and $s_{U+1} = 1$, $s_i = 1 + \sum_{j=U+1}^{i-1} \prod_{k=U+1}^{j} \frac{\mu_k}{\lambda_k}$ for $i > U+1$.

Koalas - Choice of Culling Rates

κ	α_{U+1}
1010	1.7825×10^{-2}
1020	6.280×10^{-3}
1050	1.5501×10^{-4}
1070	3.3711×10^{-6}
1100	1.1707×10^{-9}
1120	1.3957×10^{-12}
1200	6.3347×10^{-29}

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General

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Reduction model with per-capita culling

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- $\psi = 4.2$

Choice of Control Policy

How do we choose the “best" control regime?

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How do we choose the “best" control regime?

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- Total Costs

Total Costs

Cost Functions

Linear Birth-death Suppression Model with Constant Culling

$$f_j = K1_{\{j>U\}} + M.$$

Linear Birth-death Reduction Model with Per-capita Culling

$$f_{(j,0)} = N \text{ and } f_{(j,1)} = Cj + N.$$

Total Costs

Cost Functions

Linear Birth-death Suppression Model with Constant Culling

$$f_j = K 1_{\{j>U\}} + M.$$

$$K = \$50,000 \text{ and } M = \$10,000.$$

Linear Birth-death Reduction Model with Per-capita Culling

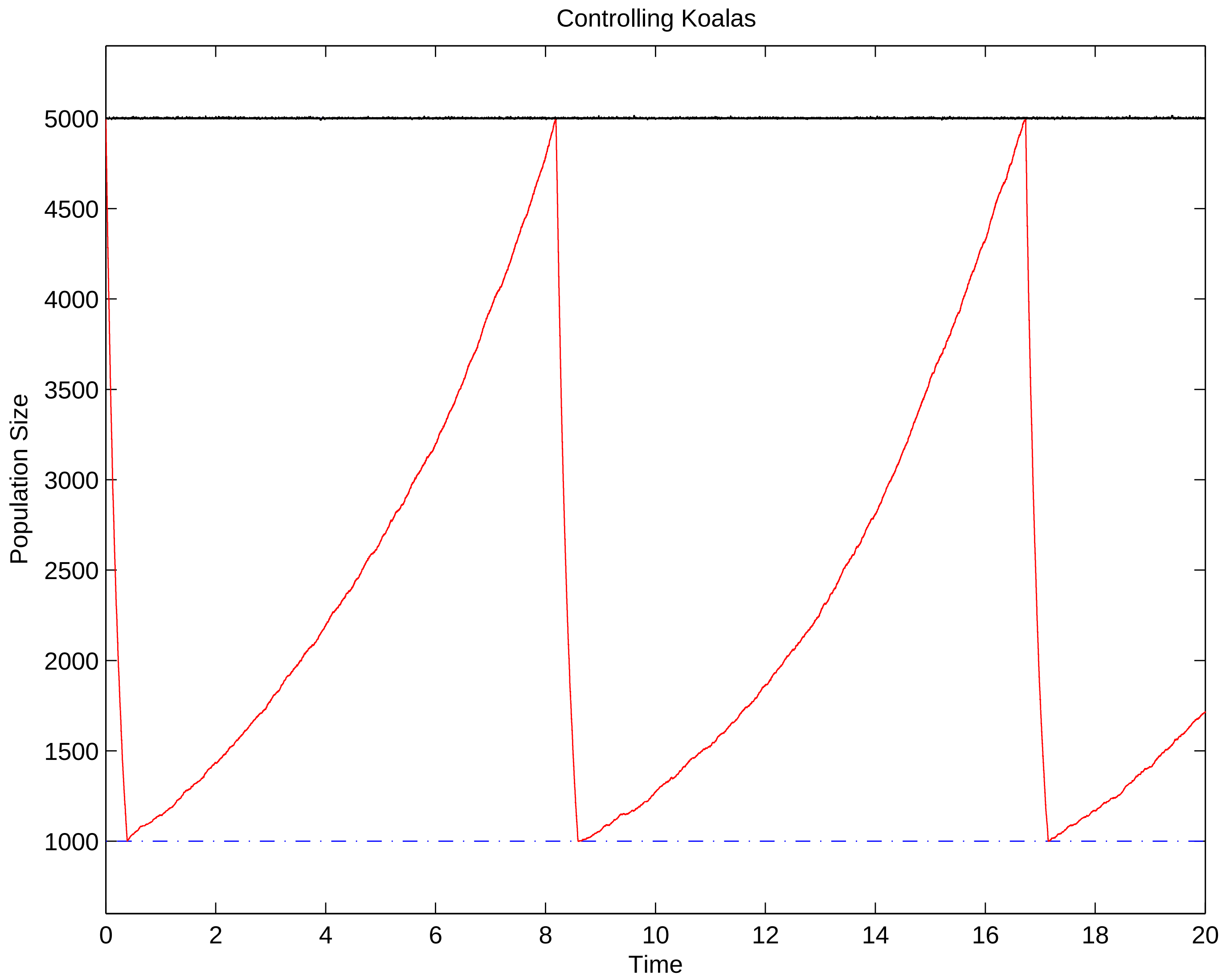
$$f_{(j,0)} = N \text{ and } f_{(j,1)} = Cj + N.$$

$$C = \$100 \text{ and } N = \$7,000.$$

Choice of Control Policy

Decision Tool	*Supp. & Const.*	*Red. & Per-capita*
Cost/Time	$\$36,470$ per year	$\$7,841$ per year

Conclusion



Controlling Koalas
5000
4500
4000
3500
3000
2500
2000
1500
1000
Population Size
0
2
4
6
8
10
12
14
16
18
20
Time

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