TRAFFIC PROCESSES

Traffic process $(N(t),\ t \geq 0)$: $N(t)$ is the (cumulative) number of arrivals in $[0, t]$. Assume that $N(0) = 0$ and that no two calls can arrive at *exactly* the same time.

If N were a *Poisson process* (with rate γ), then

$$\mathrm{E}_s\left(N(t) - N(s)\right) = \gamma(t - s), \qquad 0 \leq s \leq t.$$

or, more generally,

$$\mathrm{E}_s\left(N(t) - N(s)\right) = \int_s^t \gamma(u)\, du, \qquad 0 \leq s \leq t.$$

Here E_s represents expectation conditional on the past at time s (formally, $\mathrm{E}_s(\cdot) = \mathrm{E}(\cdot\,|\mathcal{F}_s)$, where $\{\mathcal{F}_s\}$ is an increasing, right-continuous family of σ-algebras, with $N(s)$ being $\mathcal{F}_s$ measurable and $\mathcal{F}_0$ being complete).

We say that N has a *deterministic past-conditional arrival rate*.

WATANABE'S THEOREM

Now let N be an arbitrary traffic process.

Theorem If N has a *deterministic past-conditional arrival rate*, then N is a *Poisson process*.

Outline Proof For simplicity, suppose that

$$\mathsf{E}_s\left(N(t) - N(s)\right) = t - s, \quad 0 \leq s \leq t. \tag{1}$$

Now extend (1) to stopping times τ and σ with $\tau \geq \sigma$ as follows:

$$\mathsf{E}\left(N(\tau) - N(\sigma)\right) = \mathsf{E}(\tau - \sigma) \tag{2}$$

(recall that τ is a *stopping time* if $\{\tau \leq s\} \in \mathcal{F}_s$). Let $0 < T_1 < T_2 < \ldots$ be the sequence of arrival times, and, for fixed i, define

$$\sigma = \min(t, T_{i-1}) \quad \text{and} \quad \tau = \min(t, T_i).$$

Consider the three possible cases

$$T_{i-1} \le T_i \le t,\ T_{i-1} \le t < T_i,\ t < T_{i-1} \le T_i,$$

and observe that

$$N(\tau) - N(\sigma) = I(T_i \le t).$$

Thus,

$$\begin{aligned} P(T_i \le t) &= \mathsf{E}\left(N(\tau) - N(\sigma)\right) \\ &= \mathsf{E}(\tau - \sigma) \\ &= \mathsf{E}\left(\int_0^t I(N(s) = i-1)\, ds\right) \\ &= \int_0^t P(N(s) = i-1)\, ds. \end{aligned}$$

Setting $p_i(t) = P(N(t) = i)$ we find that

$$\begin{aligned} p_i(t) &= P(T_i \le t < T_{i+1}) \\ &= P(T_i \le t) - P(T_{i+1} \le t) \\ &= \int_0^t p_{i-1}(s)\, ds - \int_0^t p_i(s)\, ds, \end{aligned}$$

and hence that $p_i' = p_{i-1} - p_i$.

For $i = 0$ we get

$$p_0(t) = 1 - \int_0^t p_0(s)\,ds,$$

which gives $p_0' = -p_0$. Since $p_i(0) = \delta_{i0}$ we deduce that $N(t)$ has a Poisson distribution with parameter t.

We also require $N(t) - N(s)$ $(0 \le s \le t)$ to be independent of the past at s. Fix s and $A \in \mathcal{F}_s$, and let $M(t) = N(s+t) - N(s)$. Then, (2) gives

$$\mathsf{E}_t'(M(u) - M(t)) = u - t, \quad 0 \le t \le u,$$

where E_t' refers to expectation conditional on the past at time t, *including* the fixed event A. But the previous analysis then gives a Poisson(t) distribution for $M(t)$, that is, a Poisson distribution with parameter t for $N(s+t) - N(s)$, conditional on A. This establishes the required independence.

CONDITIONAL INTENSITY

Let $(N(t),\ t \geq 0)$ be an arbitrary traffic process.

Under mild regularity conditions, the limit

$$\gamma(s) = \lim_{t \downarrow 0} \frac{1}{t} (\mathsf{E}_s(N(t+s) - N(s)))$$

exists and defines the *conditional intensity (process)*, $(\gamma(s),\ s \geq 0)$, and, if we set

$$\Gamma(t) = \int_0^t \gamma(u)\, du,$$

then, for all $0 \leq s \leq t$,

$$\mathsf{E}_s(N(t) - N(s)) = \mathsf{E}_s(\Gamma(t) - \Gamma(s)).$$

For stopping times τ and σ such that $\tau \geq \sigma$,

$$\mathsf{E}_\sigma(N(\tau) - N(\sigma)) = \mathsf{E}_\sigma(\Gamma(\tau) - \Gamma(\sigma)),$$

where E_σ means expectation conditional on the past at time σ: formally, the σ-algebra of events, A, such that $A \cap [\sigma \leq t] \in \mathcal{F}_t$ for each t.

POISSON APPROXIMATIONS

Let N be a traffic process with conditional intensity γ. Fix $t \geq 0$ and let A be an event determined by N on $[0, t]$. It is proposed to approximate $P(A)$ by $\Pi(A)$, where $\Pi(A)$ is the probability that A would have if N were Poisson with given rate α.

The idea is to *construct* a Poisson process M using N. Define $\hat{\Gamma}(t) = \inf\{s : \Gamma(s) \geq t\}$, the (pseudo) inverse of the increasing (random) function $s \mapsto \Gamma(s)$. In general $\Gamma(\hat{\Gamma}(t)) = t$, but $\hat{\Gamma}(\Gamma(t)) \geq t$, with inequality occurring at only those t for which Γ is constant around an interval containing t. Certainly $\hat{\Gamma}(t) \geq \hat{\Gamma}(s)$ for $t \geq s$, and so we may apply the above (stopping time) result, namely

$$\mathsf{E}_\sigma(N(\tau) - N(\sigma)) = \mathsf{E}_\sigma(\Gamma(\tau) - \Gamma(\sigma)),$$

with $\sigma = \hat{\Gamma}(s)$ and $\tau = \hat{\Gamma}(t)$.

This gives

$$\mathsf{E}_{\hat{\Gamma}(s)}\{N(\hat{\Gamma}(t)) - N(\hat{\Gamma}(s))\}$$

$$= \mathsf{E}_{\hat{\Gamma}(s)}\{\Gamma(\hat{\Gamma}(t)) - \Gamma(\hat{\Gamma}(s))\} = t - s.$$

Thus, by the previous argument, $\{N(\hat{\Gamma}(t))\}$ is a unit-rate Poisson process and so M, where $M(t) = N(\hat{\Gamma}(\alpha t))$, is a Poisson process with rate α. A familiar coupling argument then leads to the following result:

Proposition For arbitrary A,

$$\begin{aligned} |P(A) - \Pi(A)| &\le \int_0^t \mathsf{E}|\alpha - \gamma(s)|\, ds \\ &\le \int_0^t \left(\mathsf{E}(\alpha - \gamma(s))^2\right)^{1/2} ds, \end{aligned}$$

where $\Pi(A)$ is the probability that A would have if the traffic process were Poisson with parameter α. If the traffic process is stationary with $\alpha = \mathsf{E}\gamma(s)$, then

$$|P(A) - \Pi(A)| \le t\,(\mathrm{Var}\,\gamma(s))^{1/2}.$$

A STAR NETWORK

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There are K outer nodes which communicate via a single central node. Thus, there are K links (circuit groups), and each route consists of a pair of links (i, j).

Assume that

- the links have C circuits,
- calls arrive as independent Poisson processes,
- these have the same rate, $\rho = \nu/(K-1)$,
- call lengths are exponential with mean 1,
- these are mutually independent.

Keep the total offered traffic, ν, (to the network) fixed and let the number of switching nodes, K, become large; what happens?

A POISSON APPROXIMATION FOR THE STAR NETWORK

Let $N_k(t)$ be the number of calls offered to link k in the time-interval $[0,t]$ and let $m_j(s)$ be the number of circuits in use on link j at time s. Clearly γ_k, the conditional intensity of N_k, is given by

$$\gamma_k(s) = \sum_{j \neq k} \rho I[m_j(s) < C].$$

Thus $\mathsf{E}\gamma_k(s) = \nu P(m_1 < C)$ and

$$\mathsf{Var}\,\gamma_k(s) =$$

$$\nu \left(\rho P(m_1 < C) + \left(\tfrac{K-1}{K-2}\right) \nu P(m_1 < C, m_2 < C) \right.$$

$$\left. -\nu (P(m_1 < C))^2 \right).$$

Recall that if A is any event determined by N_k on $[0,t]$, then

$$|P(A) - \Pi(A)| \leq t\,(\mathsf{Var}\,\gamma_k(s))^{1/2}.$$

ASYMPTOTICALLY INDEPENDENT BLOCKING

Proposition (Ziedins and Kelly (1989)): By keeping the total offered traffic, ν, fixed and letting the number of switching nodes, K, become large, the links are blocked *independently*. Specifically, as $K \to \infty$,

$$P(m_1 < C, m_2 < C) - (P(m_1 < C))^2 \to 0.$$

We deduce that the traffic offered to any given *link* is approximately Poisson, since

$$\text{Var}\,\gamma_k(s) =$$

$$\nu \left(\rho P(m_1 < C) + \left(\tfrac{K-1}{K-2}\right) \nu P(m_1 < C, m_2 < C) - \nu (P(m_1 < C))^2 \right)$$

$$\sim \frac{\nu^2}{K-1} P(m_1 < C) \left(1 + \left(\tfrac{K-1}{K-2}\right) P(m_1 < C) \right).$$

A MARKOVIAN QUEUEING NETWORK

Label the links $1, 2, \ldots, J$ and make the usual simplifying assumptions:

- the links are perfectly reliable and are not subject to noise, so that message transmission times are determined by their length.

- the time taken by the nodes to switch, and, if necessary, buffer, reassemble and acknowledge messages, is negligible in comparison with their transmission times.

- traffic entering the network from external sources is Poisson and message lengths are mutually independent and exponentially distributed with mean 1.

A total effort (or capacity) of $\phi_j(n)$ is provided by link j when there are n messages whose transmission is incomplete.

A POISSON APPROXIMATION

Let $N_k(t)$ be the net traffic offered to a given link, k, on the interval $[0, t]$. Since all messages have unit mean length, the rate at which messages are transmitted by link j is $\phi_j(n)$ when there are n messages present. Thus, if $n_j(s)$ is the number of messages present at link j at time s, then γ_k, the conditional intensity of N_k, is given by

$$\gamma_k(s) = \nu_k + \sum_{j=1}^{J} \phi_j(n_j(s))\lambda_{jk},$$

where ν_k is the rate of externally offered traffic and λ_{jk} is the proportion of messages emanating from link j which next use link k.

Let $\alpha_k = \mathrm{E}\gamma_k(s)$. Then (in equilibrium)

$$\alpha_k = \nu_k + \sum_{j=1}^{J} \alpha_j \lambda_{jk}, \qquad k = 1, 2, \ldots, J.$$

An equilibrium exists if and only if, for each j,

$$b_j^{-1} = \sum_{n=0}^{\infty} \frac{\alpha_j^n}{\prod_{l=1}^n \phi_j(l)} < \infty,$$

in which case, the states of the individual links are independent and the probability that there are n messages at link j is given by

$$\pi_j(n) = \frac{b_j \alpha_j^n}{\prod_{l=1}^n \phi_j(l)}.$$

We can then show that

$\mathrm{Var}\, \gamma_k(s)$

$$= \sum_{j=1}^{J} \alpha_j \left(\sum_{n=0}^{\infty} \phi_j(n+1)\pi_j(n) - \alpha_j \right) \lambda_{jk}^2.$$

This leads to a bound on the degree of deviation of N_k from a Poisson process with rate α_k.

STATE-INDEPENDENT CAPACITY

Suppose

$$\phi_j(n) = \phi_j I[n > 0].$$

Then

$$|P(A) - \Pi(A)| \leq t \left(\sum_{j=1}^{J} \phi_j^2 \lambda_{jk}^2 \rho_j (1 - \rho_j) \right)^{1/2},$$

where $\rho_j = \alpha_j / \phi_j$ (we must have that $\phi_j > \alpha_j$). Thus, if for each j, $\phi_j \lambda_{jk}$ is $O(1)$ and either

(1) $\phi_j \lambda_{jk}$ is $o(J^{-1/2})$,

(2) ρ_j is $o(J^{-1})$, or

(3) $1 - \rho_j$ is $o(J^{-1})$,

then the Poisson approximation is sure to be good.

SIMULTANEOUS APPROXIMATIONS BY INDEPENDENT POISSON PROCESSES

Let $N_1, N_2, \ldots, N_l$ be l traffic processes and let $\gamma_1, \gamma_2, \ldots, \gamma_l$ be their conditional intensities.

If A is an event which is determined by these processes on the interval $[0, t]$ and $\Pi(A)$ is the probability that A would have if these processes were *independent* Poisson processes with rates $\alpha_1, \alpha_2, \ldots, \alpha_l$, then

$$|P(A) - \Pi(A)| \leq \sum_{k=1}^{l} \int_0^t (\mathsf{E}(\alpha_k - \gamma_k(s))^2)^{1/2}\, ds.$$

If the system is in equilibrium and $\alpha_1, \alpha_2, \ldots, \alpha_l$ are chosen such that $\alpha_k = \mathsf{E}\gamma_k(s)$, then a simpler bound is given by

$$|P(A) - \Pi(A)| \leq t \sum_{k=1}^{l} (\mathrm{Var}\, \gamma_k(t))^{1/2}.$$