Quasi-stationary distributions

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Quasi stationarity

Simulation of a metapopulation model ($c = 0.1625$, $e = 0.0325$)
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Quasi stationarity

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$X(t)$ (occupied patches at time $t$)

t (years)
A “quasi-stationary” distribution

Simulation of a metapopulation model ($c = 0.1625, e = 0.0325$)
Quasi stationary Distributions

We consider a single transient class of a markov chain states 0, 1, 2, 3, ... . Probability is ultimately absorbed (with prob 1) in a state -1.

If we shifted state labels up by 1, absorbing state 0, then this could for instance represent an absorbing state in a population which ultimately becomes extinct.

We consider the chain as being described by \( Q_{ij} = i, j \geq 0 \)

So \( Q_{ij} > 0 \) if \( Q_{ij} < 1 \)

If we consider sub-stochastic matrices

For simplicity we will assume that all states have period 1.

Then, we have a sub-stochastic irreducible chain \( (Q_{ij}) \). Then for all \( i, j \), the series \( \sum_{n=0}^{\infty} Q_{ij}^{(n)} \) have a common vachine of convergence. Sufficientes to prove it for any \( i, j \), the series \( \sum_{n=0}^{\infty} Q_{ii}^{(n)} \), \( \sum_{n=0}^{\infty} Q_{ij}^{(n)} \), \( \sum_{n=0}^{\infty} Q_{ji}^{(n)} \) have same vachine of convergence.
A “quasi-stationary” distribution

Simulation of a metapopulation model ($c = 0.1625$, $e = 0.0325$)
A “quasi-stationary” distribution

Think of an observer who at some time $t$ is aware of the occupancy of some patches, yet cannot tell exactly which of $n$ patches are occupied.
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What is the chance of there being precisely $i$ patches occupied?
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What is the chance of there being precisely \( i \) patches occupied?

If we were equipped with the full set of state probabilities

\[
p_i(t) = \mathbb{P}(X(t) = i), \quad i \in \{0, 1, \ldots, n\},
\]

we would evaluate the *conditional probability*

\[
u_i(t) = \mathbb{P}(X(t) = i|X(t) \neq 0) = \frac{p_i(t)}{1 - p_0(t)},
\]

for \( i \) in the set \( S = \{1, \ldots, n\} \) of transient states.
A “quasi-stationary” distribution

\[ u_i(t) = \mathbb{P}(X(t) = i | X(t) \neq 0) = \frac{p_i(t)}{1 - p_0(t)}, \quad i \in S. \]

Then, in view of the behaviour observed in our simulation, it would be natural for us to seek a distribution \( u = (u_i, i \in S) \) over \( S \) such that if \( u_i(t) = u_i \) for a particular \( t > 0 \), then \( u_i(s) = u_i \) for all \( s > t \).
A quasi-stationary distribution

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Such a distribution is called a *stationary conditional distribution* or *quasi-stationary distribution* (QSD).
A quasi-stationary distribution

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Such a distribution $\mathbf{u}$ is called a *stationary conditional distribution* or *quasi-stationary distribution* (QSD).

**Key message:** $\mathbf{u}$ can usually be determined from the transition rates of the process and $\mathbf{u}$ might then also be a *limiting conditional distribution* (LCD) in that $u_i(t) \to u_i$ as $t \to \infty$, and thus be of use in modelling the long-term behaviour of the process.
A quasi-stationary distribution

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**Key message:** $u$ can usually be determined from the transition rates of the process and $u$ might then also be a limiting conditional distribution (LCD) in that $u_i(t) \to u_i$ as $t \to \infty$, and thus be of use in modelling the long-term behaviour of the process.

**Note:** There are other approaches to modelling quasi stationarity, and indeed other “quasi-stationary” distributions.
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The Yaglom limit

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If the expected number $\lambda$ of offspring is less than 1, then

$$u_i = \lim_{n \to \infty} \mathbb{P}(X_n = i | X_n \neq 0, X_0 = 1), \quad i \in S,$$

exists and defines a proper probability distribution $\mathbf{u} = (u_i, i \in S)$ over $S$. 
Simulation a branching process ($\lambda = 1.1$, $X_0 = 30$)
Simulation a branching process ($\lambda = 0.96, X_0 = 30$)
The Yaglom limit

QSD of the branching process ($\lambda = 0.96$)
Subcritical (only just) - quasi stationarity?

Simulation a branching process ($\lambda = 0.99$, $X_0 = 30$)
The Yaglom limit

QSD of the branching process ($\lambda = 0.99$)
Simulation of a metapopulation model ($c = 0.1625$, $e = 0.0325$)
The quasi-stationary distribution

Simulation of a metapopulation model \((c = 0.1625, e = 0.0325)\)

\[ X(t) \text{ (occupied patches at time } t) \]

\( t \) (years)
The idea of a limiting conditional distribution goes back much further than Yaglom, at least to Wright* in his discussion of gene frequencies in finite populations:

“As time goes on, divergences in the frequencies of factors may be expected to increase more and more until at last some are either completely fixed or completely lost from the population. The distribution curve of gene frequencies should, however, approach a definite form if the genes which have been wholly fixed or lost are left out of consideration.”

The idea of “quasi stationarity” was crystallized by Bartlett*:

“While presumably on the above model [for the interactions between active and passive forms of flour beetle] extinction of the population will occur after a long enough time, this may (for a deterministic ‘ceiling’ population not too small, but fluctuations relatively small) be so long delayed as to be negligible and an effective or quasi-stationarity be established.”

Bartlett* later coined the term “quasi-stationary distribution”:

“It still may happen that the time to extinction is so long that it is still of more relevance to consider the effectively ultimate distribution (called a ‘quasi-stationary’ distribution) of [the process] $N$.”

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