Limits of large metapopulations with patch dependent extinction probabilities

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AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

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MASCOS
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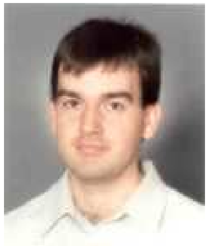
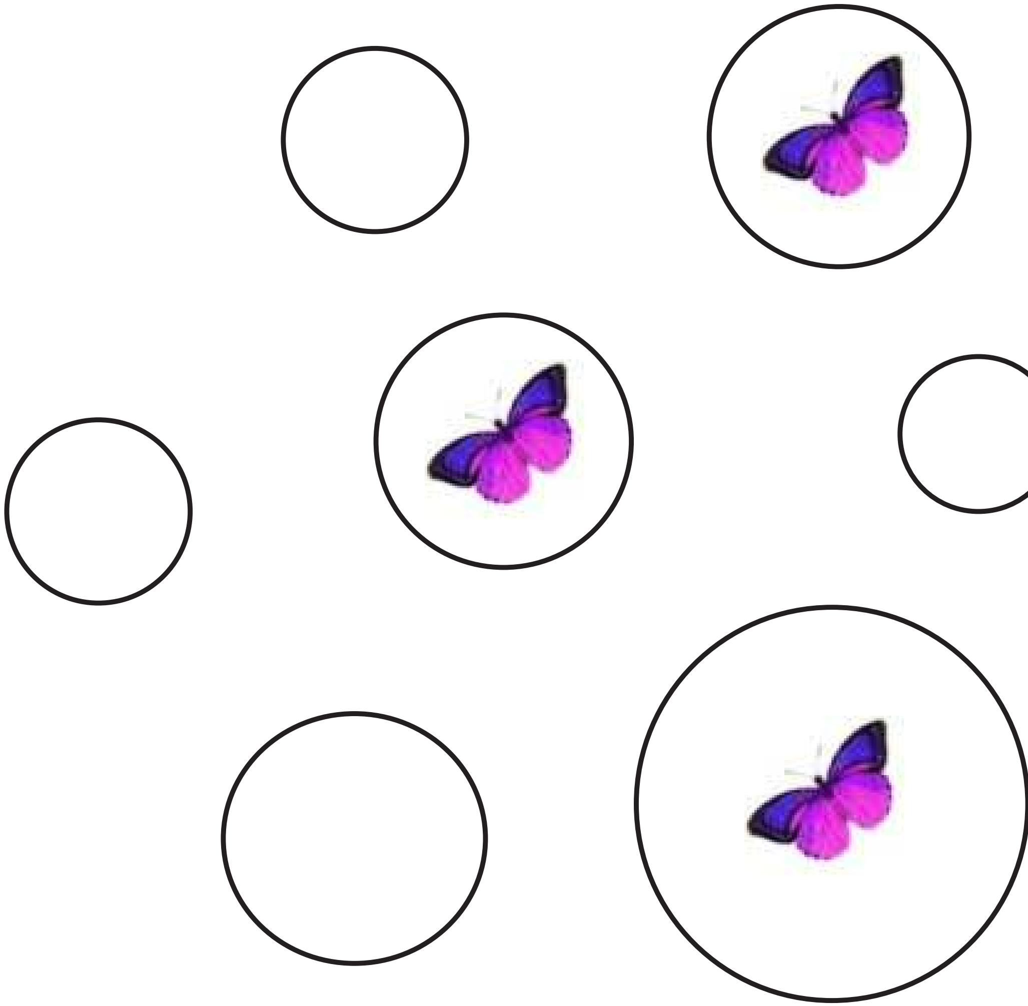


*McVinish, R. and Pollett, P.K. (2010) Limits of large metapopulations with patch dependent extinction probabilities. Advances in Applied Probability 42 (in press, accepted 02/09/10).

Metapopulations



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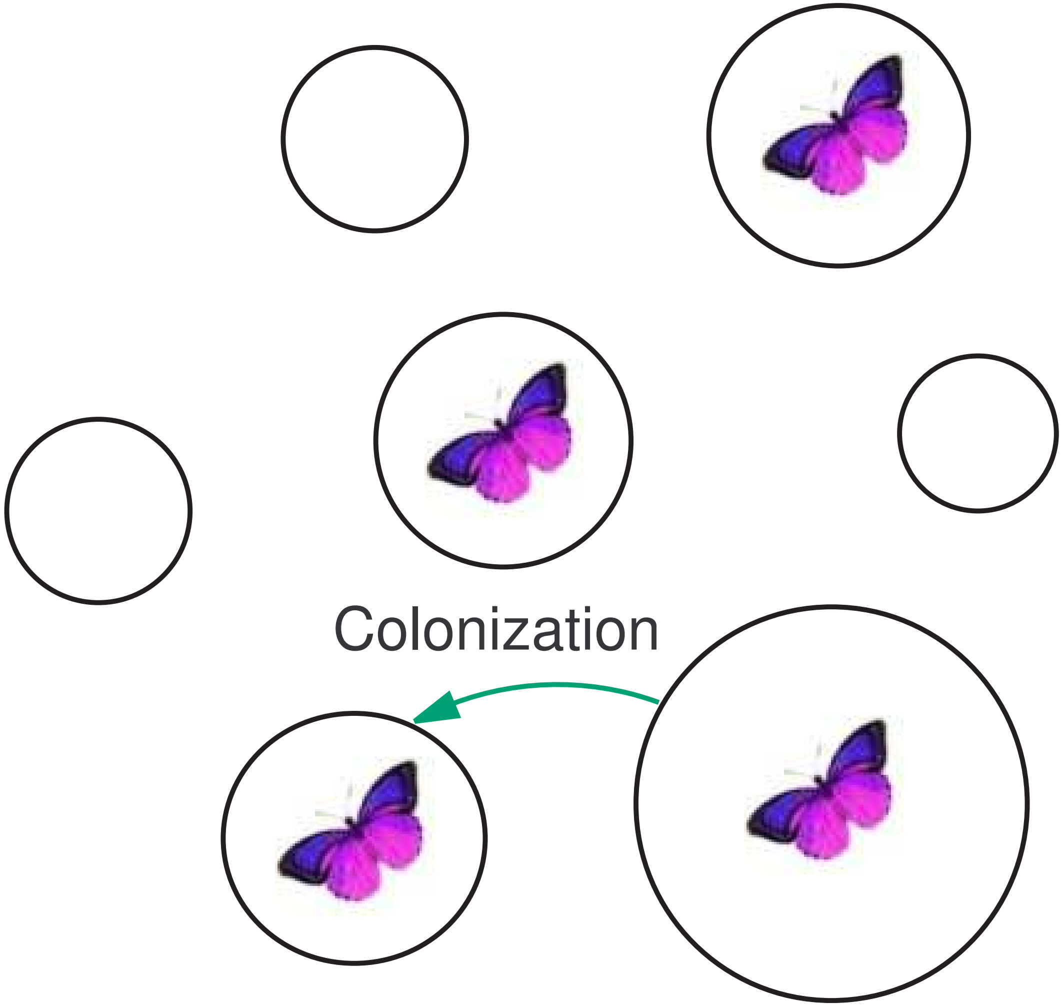

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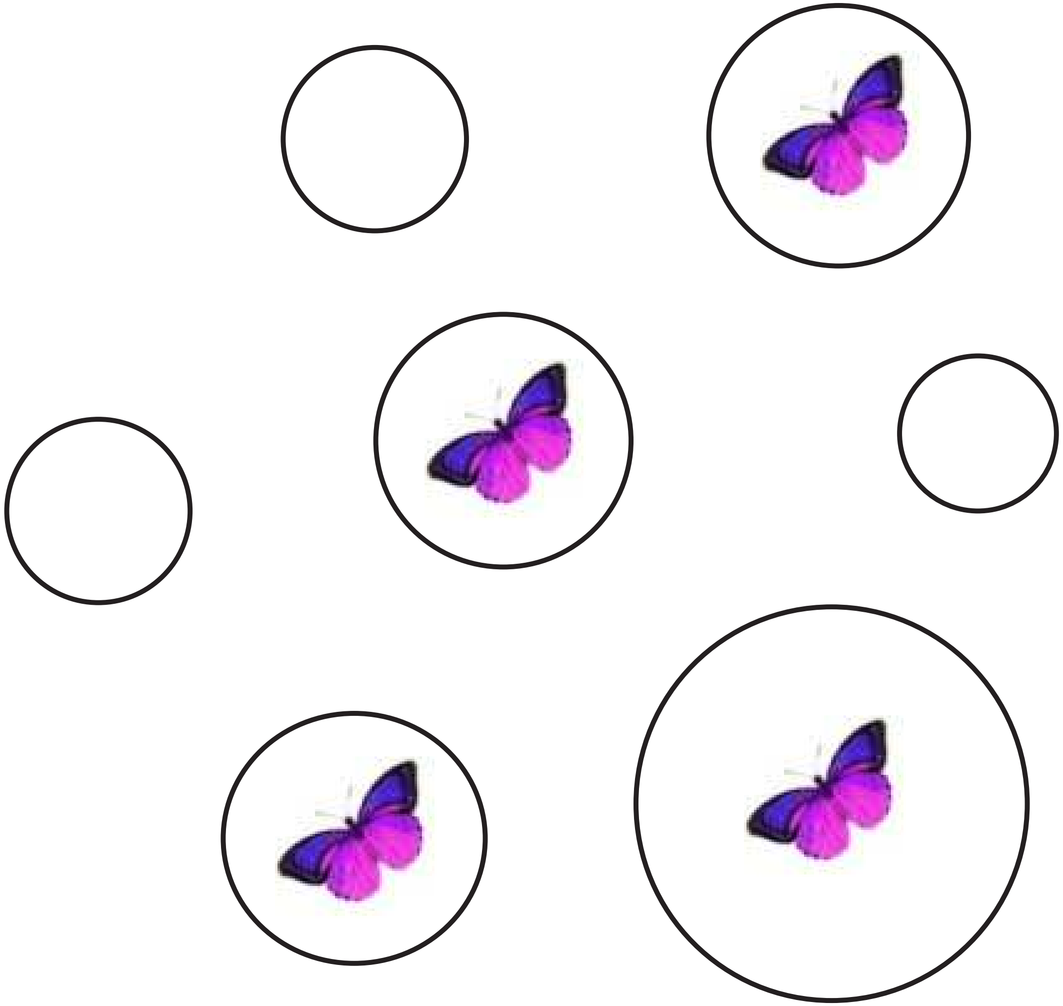

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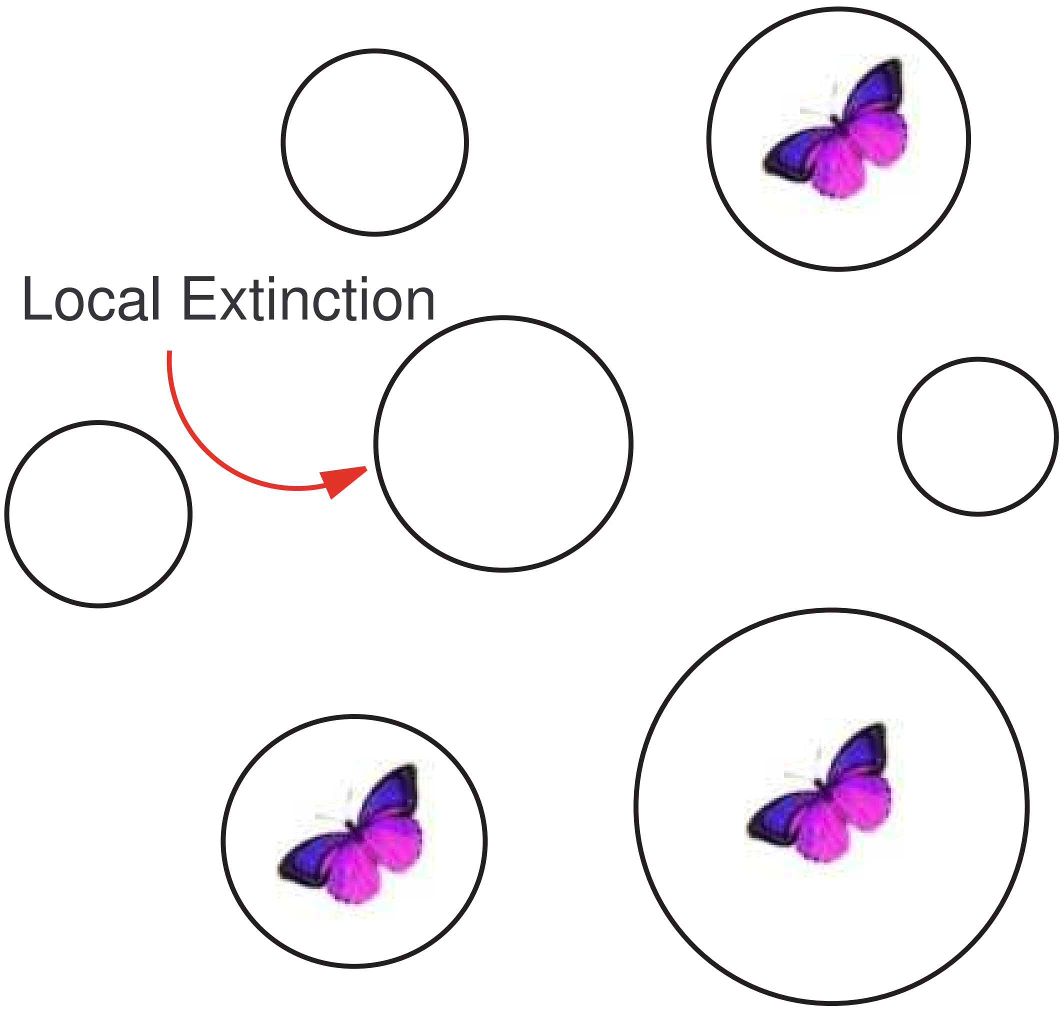

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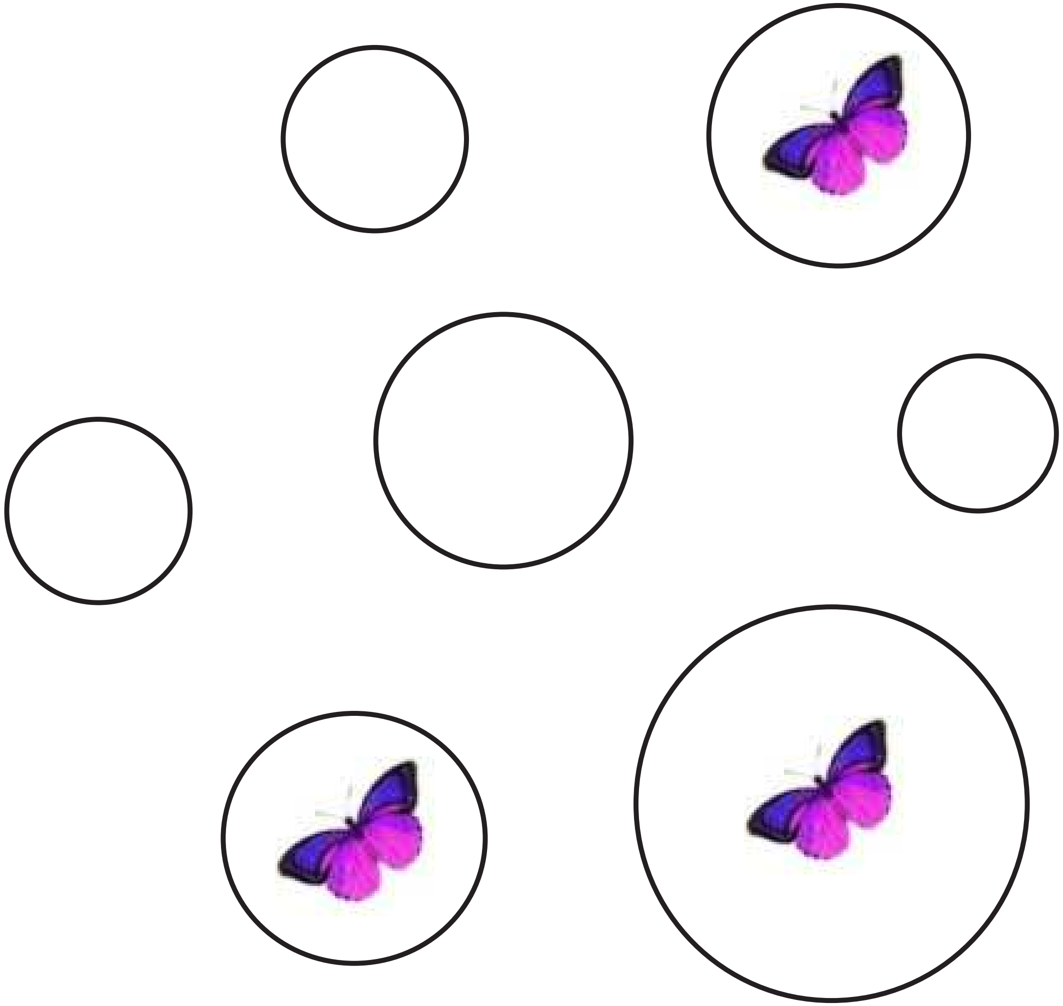

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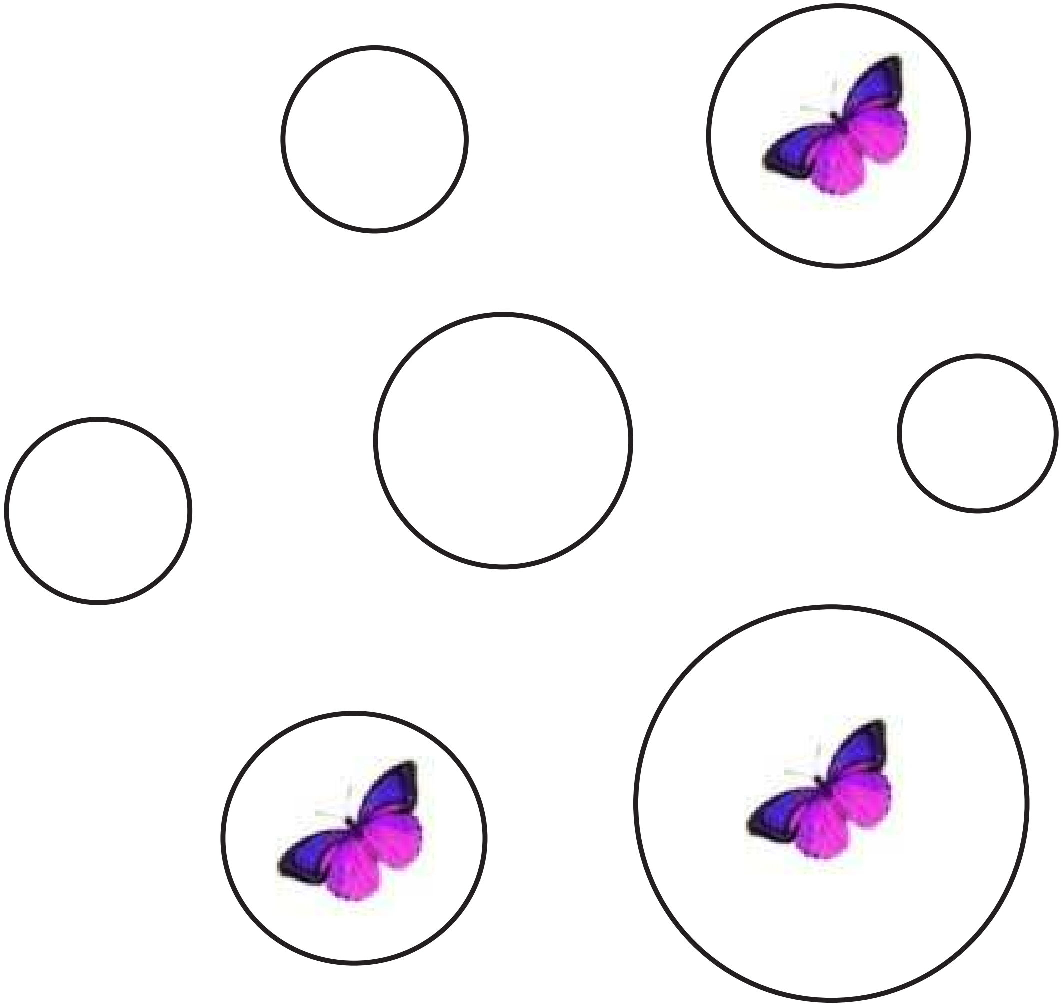

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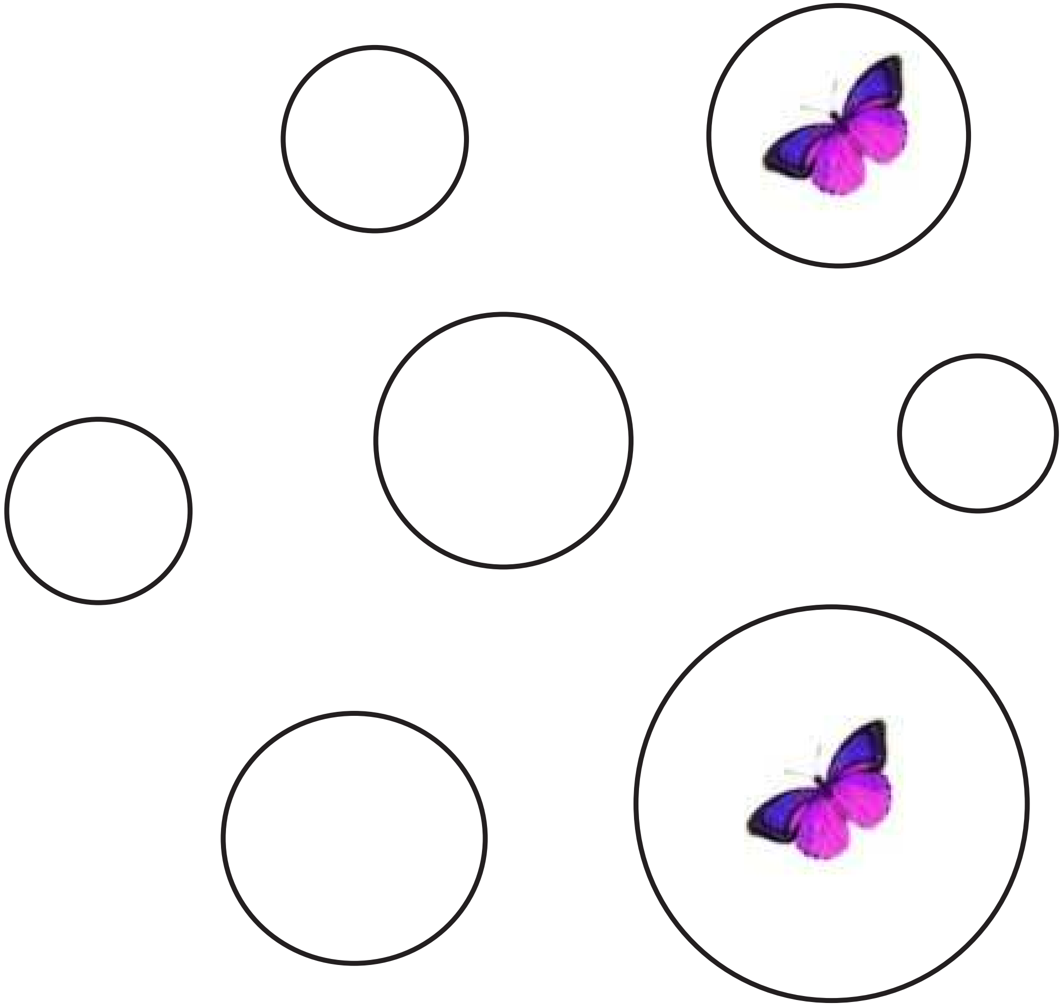

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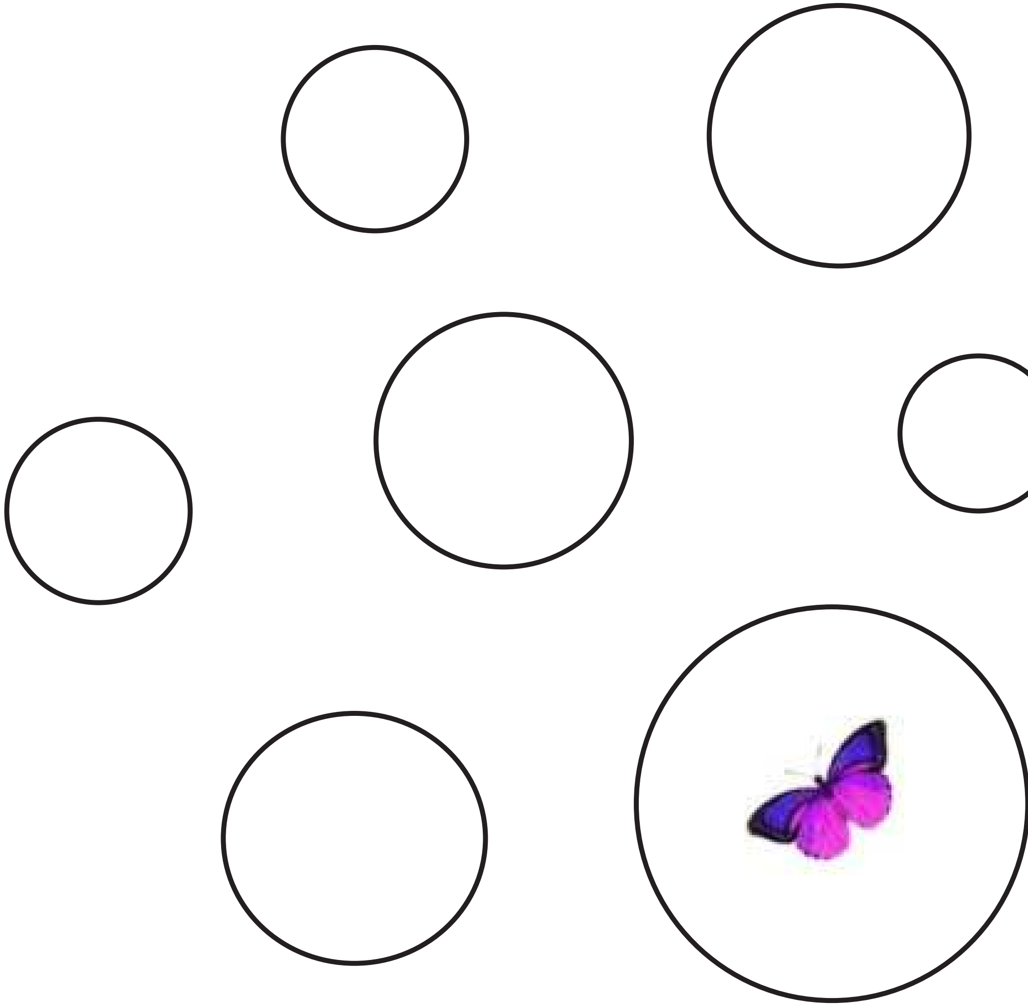

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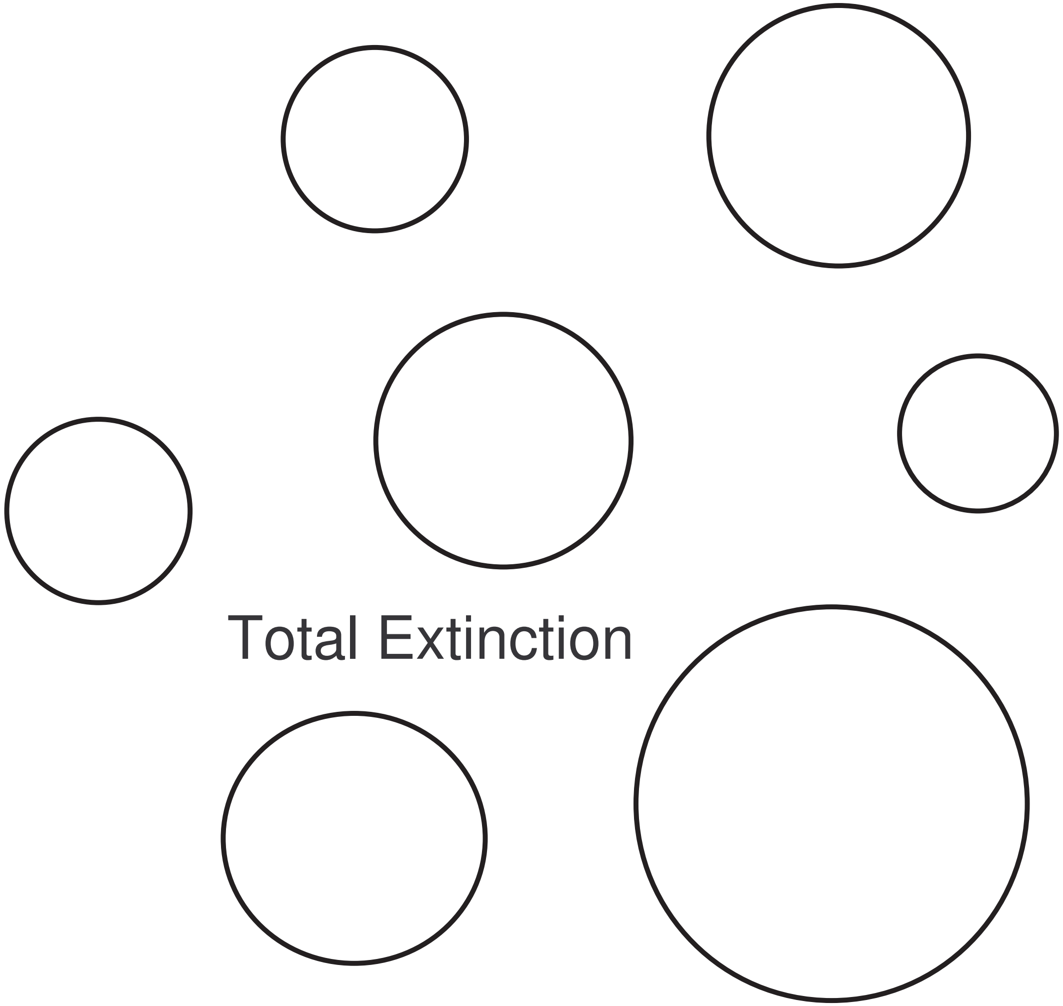

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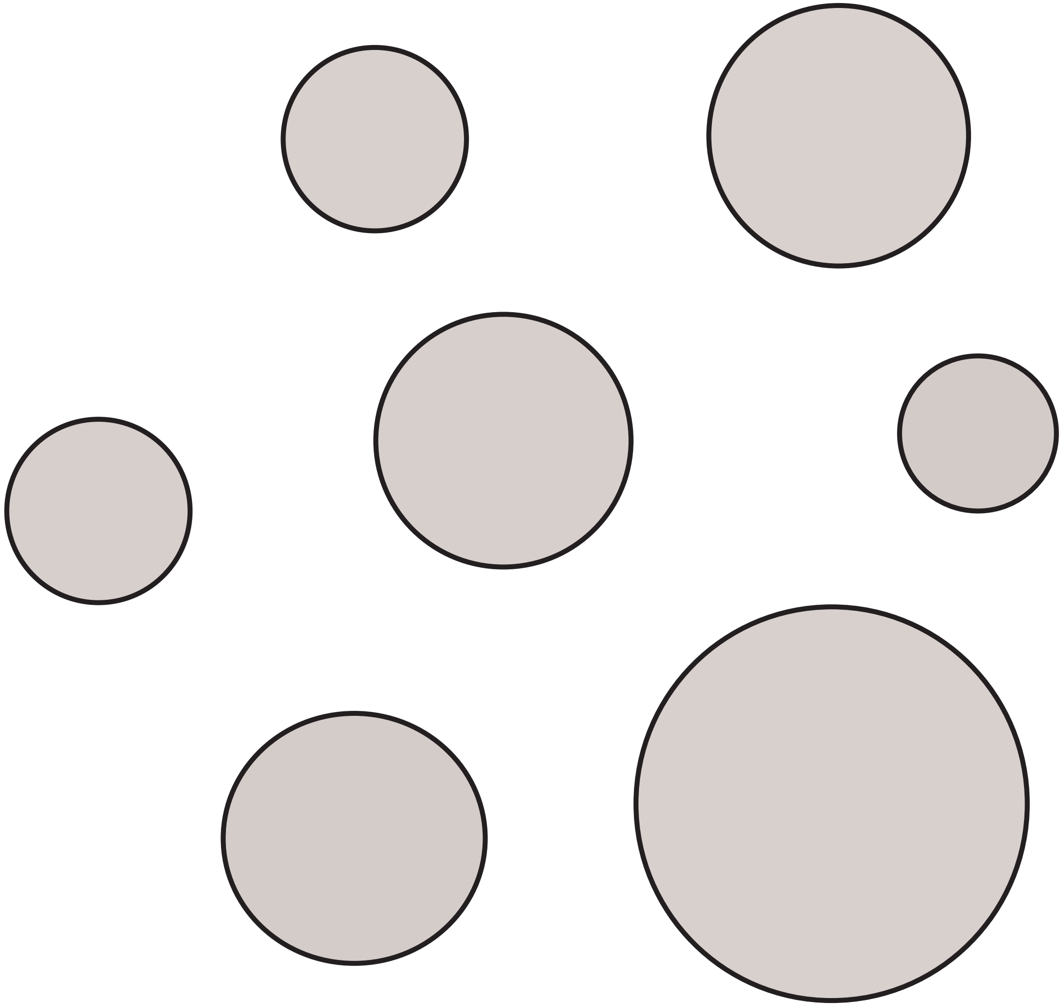

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SPOM - Phase structure

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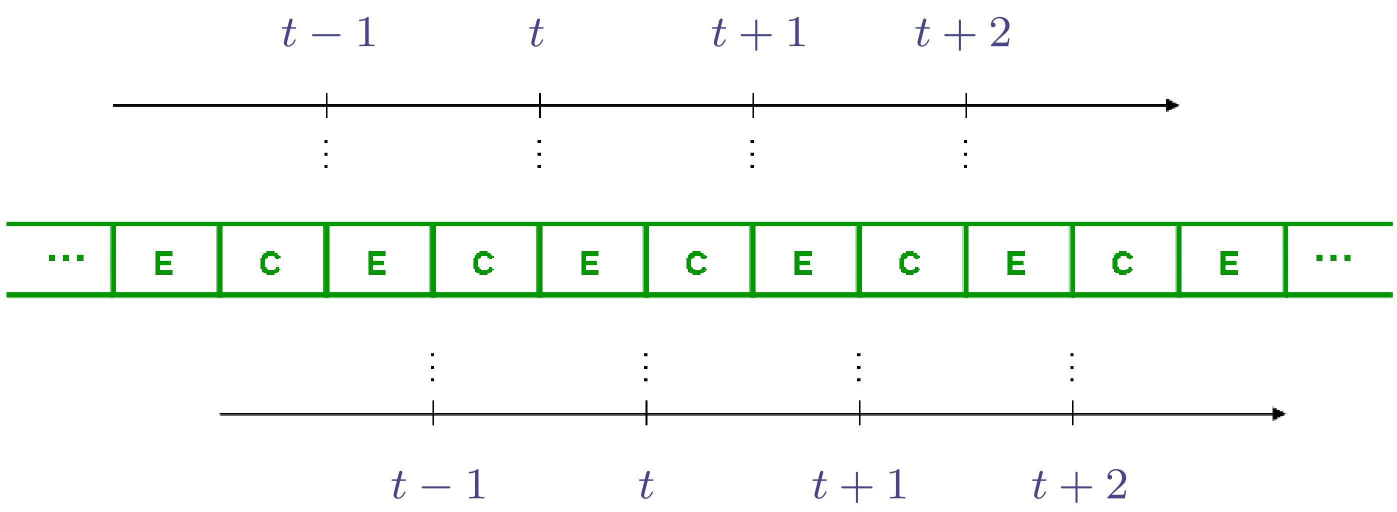

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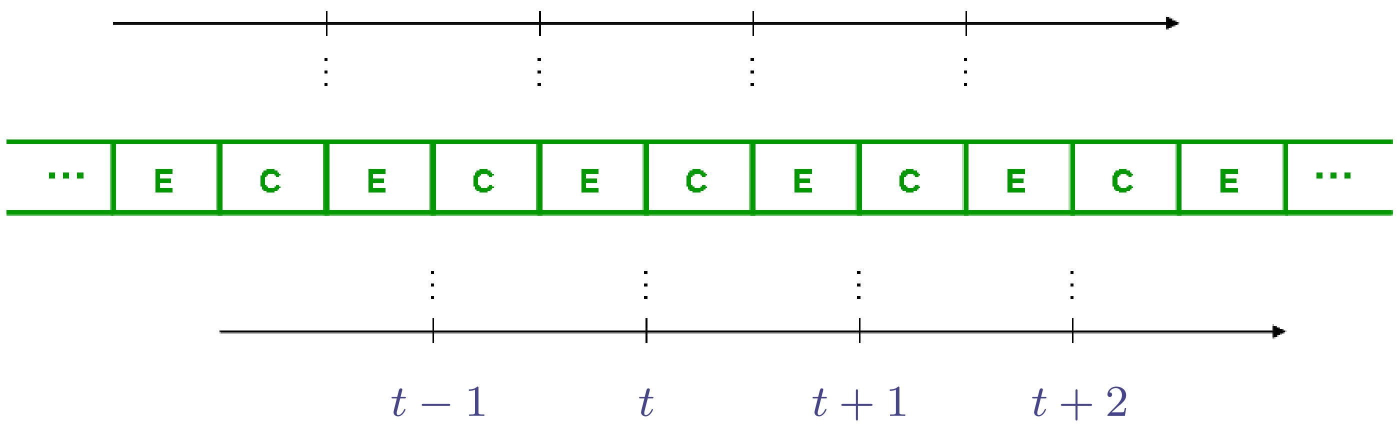

We will we assume that the population is *observed after successive extinction phases* (CE Model).

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Colonization: unoccupied patches become occupied independently with probability $c(n^{-1}\sum_{i=1}^{n} X_{i,t}^{(n)})$, where $c : [0,1] \to [0,1]$ is continuous, increasing and concave, and $c'(0) > 0$.

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Extinction: occupied patch i remains occupied independently with probability S_i (random).

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Thus, we have a *Chain Bernoulli* structure:

$$X_{i,t+1}^{(n)} \stackrel{d}{=} \mathit{Bin}\left(X_{i,t}^{(n)} + \mathit{Bin}\left(1 - X_{i,t}^{(n)}, c\left(\frac{1}{n}\sum_{j=1}^{n} X_{j,t}^{(n)}\right)\right), S_i\right)$$

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Notation: $\mathit{Bin}(m, p)$ is a binomial random variable with m trials and success probability p.

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Compare this with the *homogenous case*, where $S_i = s$ (non-random) is the same for each i, and we merely count the *number* $N_t^{(n)}$ of occupied patches at time t.

We have the following *Chain Binomial* structure:

$$N_{t+1}^{(n)} \stackrel{d}{=} \mathit{Bin}\Big(N_t^{(n)} + \mathit{Bin}\Big(n - N_t^{(n)}, c\big(\tfrac{1}{n}N_t^{(n)}\big)\Big), s\Big)$$

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A deterministic limit

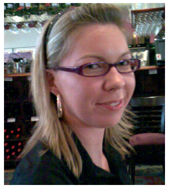
Theorem* If $N_0^{(n)}/n \xrightarrow{p} x_0$ (a constant), then

$$N_t^{(n)}/n \xrightarrow{p} x_t, \quad \text{for all } t \geq 1,$$

with (x_t) determined by $x_{t+1} = f(x_t)$, where

$$f(x) = s(x + (1 - x)c(x)).$$

*Buckley, F.M. and Pollett, P.K. (2010) Limit theorems for discrete-time metapopulation models. Probability Surveys 7, 53-83.

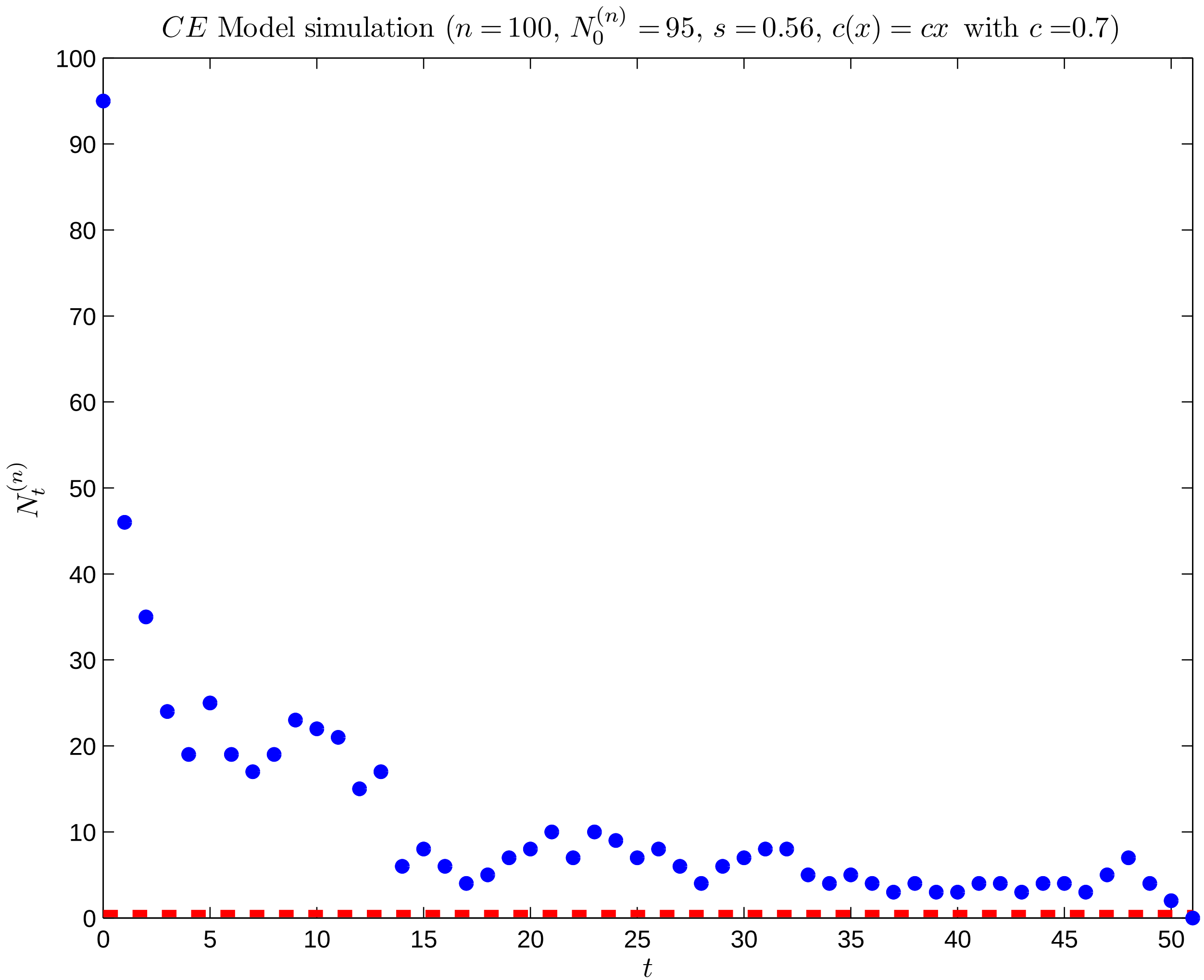


Stability

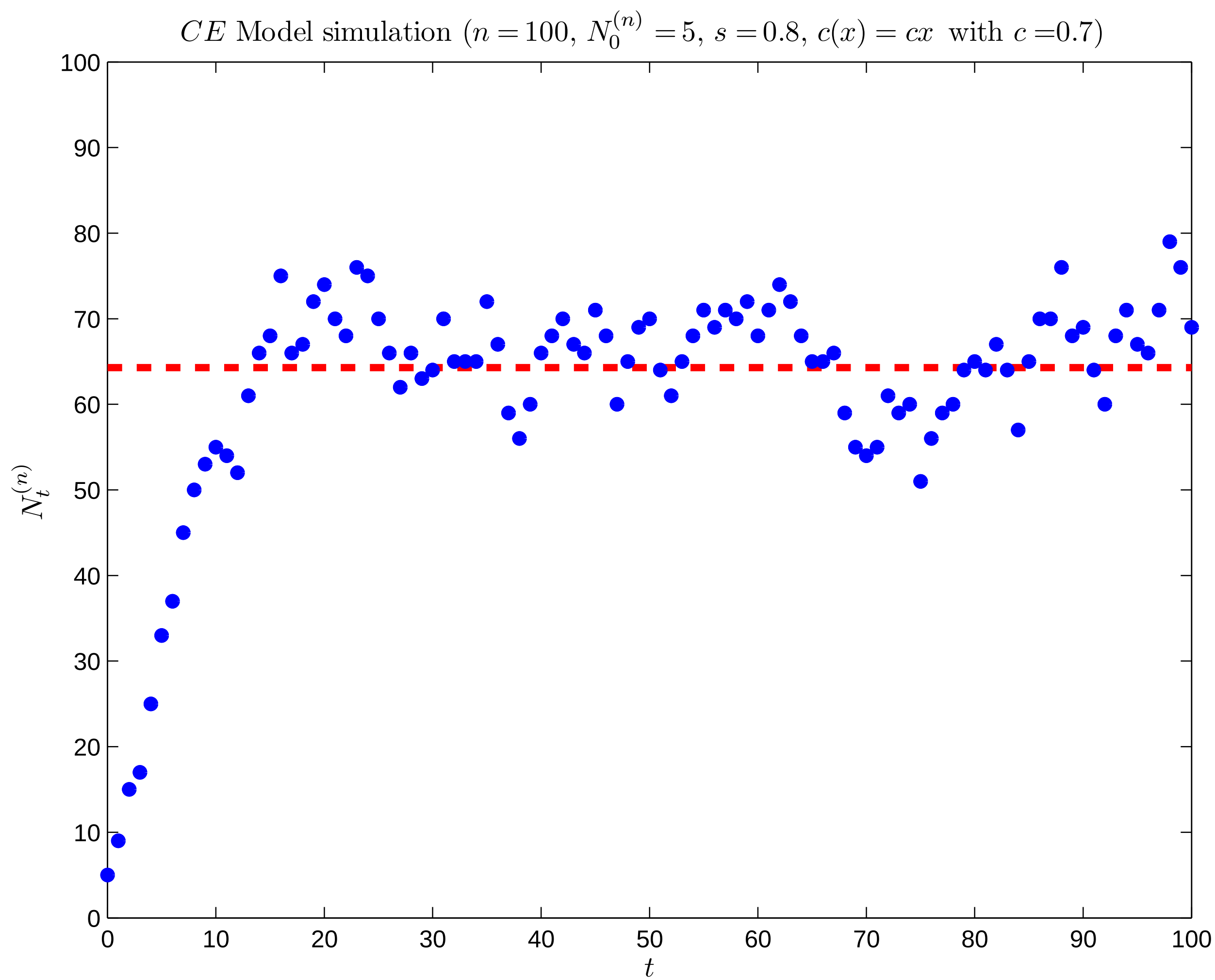
$x_{t+1} = f(x_t)$, where $f(x) = s(x + (1-x)c(x))$.

- *Stationarity*: $c(0) > 0$. There is a unique fixed point $x^* \in [0,1]$. It satisfies $x^* \in (0,1)$ and is stable.
- *Evanescence*: $c(0) = 0$ and $1 + c'(0) \leq 1/s$. Now 0 is the unique fixed point in $[0,1]$. It is stable.
- *Quasi stationarity*: $c(0) = 0$ and $1 + c'(0) > 1/s$. There are two fixed points in $[0,1]$: 0 (unstable) and $x^* \in (0,1)$ (stable).

CE Model - Evanescence



CE Model - Quasi stationarity



A deterministic limit

Returning to the general case, where patch survival probabilities are *random* and *patch dependent*, and we keep track of which patches are occupied ...

$$X_{i,t+1}^{(n)} \stackrel{d}{=} \mathit{Bin}\left(X_{i,t}^{(n)} + \mathit{Bin}\left(1 - X_{i,t}^{(n)}, c\left(\tfrac{1}{n}\sum_{j=1}^{n} X_{j,t}^{(n)}\right)\right), S_i\right)$$

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First, ...

Notation: If σ is a probability measure on $[0, 1)$ and let $\bar{s}_k$ denote its k-th moment, that is,

$$\bar{s}_k = \int_0^1 \lambda^k \sigma(d\lambda).$$

A deterministic limit

Theorem Suppose there is a probability measure σ and deterministic sequence $\{d(0,k)\}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} S_i^k \xrightarrow{p} \bar{s}_k \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} S_i^k X_{i,0}^{(n)} \xrightarrow{p} d(0,k)$$

for all $k = 0, 1, \ldots, T$. Then, there is a (deterministic) triangular array $\{d(t,k)\}$ such that, for all $t = 0, 1, \ldots, T$ and $k = 0, 1, \ldots, T-t$,

$$\frac{1}{n}\sum_{i=1}^{n} S_i^k X_{i,t}^{(n)} \xrightarrow{p} d(t,k),$$

where

$$d(t+1,k) = d(t,k+1) + c\,(d(t,0))\,(\bar{s}_{k+1} - d(t,k+1))\,.$$

Remarks

- Typically, we are only interested in $d(t,0)$, being the asymptotic proportion of occupied patches.
- However, we may still interpret the ratio $d(t,k)/d(t,0)$ $(k \geq 1)$ as the k-th moment of the conditional distribution of the patch survival probability given that the patch is occupied. (From these moments, the conditional distribution could then be reconstructed.)

Remarks

- When $\bar{s}_k = \bar{s}_1^k$ for all k, that is the patch survival probabilities are the same, then it is possible to simplify

$$d(t+1,k) = d(t,k+1) + c\left(d(t,0)\right)\left(\bar{s}_{k+1} - d(t,k+1)\right).$$

We can show by induction that $d(t,k) = \bar{s}_1^k x_t$, where

$$x_{t+1} = \bar{s}_1\left(x_t + (1-x_t)\, c(x_t)\right).$$

(Compare this with the earlier result.)

Stability

Theorem The fixed points are given by

$$d(k) = \int_0^1 \frac{c(\psi)\lambda^{k+1}}{1-\lambda+c(\psi)\lambda}\sigma(d\lambda),$$

where ψ solves

$$R(\psi) = \int_0^1 \frac{c(\psi)\lambda}{1-\lambda+c(\psi)\lambda}\sigma(d\lambda) = \psi. \quad (1)$$

If $c(0) > 0$, there is a unique $\psi > 0$. If $c(0) = 0$ and

$$c'(0)\int_0^1 \frac{\lambda}{1-\lambda}\sigma(d\lambda) \leq 1,$$

then $\psi = 0$ is the unique solution to (1). Otherwise, (1) has two solutions, one of which is $\psi = 0$.

Stability

Theorem If $c(0) = 0$ and

$$c'(0) \int_0^1 \frac{\lambda}{1-\lambda} \sigma(d\lambda) \leq 1,$$

then $d(k) \equiv 0$ is a stable fixed point. Otherwise, the non-zero solution to

$$R(\psi) = \int_0^1 \frac{c(\psi)\lambda}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda) = \psi$$

provides the stable fixed point through

$$d(k) = \int_0^1 \frac{c(\psi)\lambda^{k+1}}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda).$$