Limits of large metapopulations with patch dependent extinction probabilities

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Metapopulations
Metapopulations

Colonization

[Diagram showing various circles with butterflies, indicating the concept of colonization within a metapopulation framework.]
Metapopulations

Local Extinction
Metapopulations
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Total Extinction
A Stochastic Patch Occupancy Model (SPOM)
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We will assume that the population is observed after successive extinction phases (CE Model).
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**Extinction**: occupied patch \( i \) remains occupied independently with probability \( S_i \) (random).
Thus, we have a *Chain Bernoulli* structure:

\[
X_{i,t+1}^{(n)} \overset{d}{=} Bin\left(X_{i,t}^{(n)} + Bin\left(1 - X_{i,t}^{(n)}, c\left(\frac{1}{n} \sum_{j=1}^{n} X_{j,t}^{(n)}\right)\right), S_i\right)
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**Notation:** $Bin(m, p)$ is a binomial random variable with $m$ trials and success probability $p$. 
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Compare this with the *homogenous case*, where $S_i = s$ (non-random) is the same for each $i$, and we merely count the *number* $N_t^{(n)}$ of occupied patches at time $t$.

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A deterministic limit

**Theorem** If \( N_0^{(n)}/n \xrightarrow{p} x_0 \) (a constant), then

\[
N_t^{(n)}/n \xrightarrow{p} x_t, \quad \text{for all } t \geq 1,
\]

with \( (x_t) \) determined by \( x_{t+1} = f(x_t) \), where

\[
f(x) = s(x + (1 - x)c(x)).
\]

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- **Stationarity**: \( c(0) > 0 \). There is a unique fixed point \( x^* \in [0, 1] \). It satisfies \( x^* \in (0, 1) \) and is stable.

- **Evanescence**: \( c(0) = 0 \) and \( 1 + c'(0) \leq 1/s \). Now \( 0 \) is the unique fixed point in \([0, 1]\). It is stable.

- **Quasi stationarity**: \( c(0) = 0 \) and \( 1 + c'(0) > 1/s \). There are two fixed points in \([0, 1]\): 0 (unstable) and \( x^* \in (0, 1) \) (stable).
CE Model simulation ($n = 100$, $N_0^{(n)} = 95$, $s = 0.56$, $c(x) = cx$ with $c = 0.7$)
CE Model simulation ($n = 100$, $N_0^{(n)} = 5$, $s = 0.8$, $c(x) = cx$ with $c = 0.7$)
Returning to the general case, where patch survival probabilities are *random* and *patch dependent*, and we keep track of which patches are occupied . . .

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X_{i,t+1}^{(n)} \overset{d}{=} Bin\left(X_{i,t}^{(n)} + Bin\left(1 - X_{i,t}^{(n)}, c\left(\frac{1}{n} \sum_{j=1}^{n} X_{j,t}^{(n)}\right)\right), S_i\right)
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\]

First, . . .

**Notation:** If \( \sigma \) is a probability measure on \([0, 1)\) and let \( \bar{s}_k \) denote its \( k \)-th moment, that is,

\[
\bar{s}_k = \int_0^1 \lambda^k \sigma(d\lambda).
\]
Theorem  Suppose there is a probability measure $\sigma$ and deterministic sequence \( \{d(0, k)\} \) such that

$$
\frac{1}{n} \sum_{i=1}^{n} S_i^k \xrightarrow{p} \bar{s}_k \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} S_i^k X_{i,0}^{(n)} \xrightarrow{p} d(0, k)
$$

for all $k = 0, 1, \ldots, T$. Then, there is a (deterministic) triangular array \( \{d(t, k)\} \) such that, for all $t = 0, 1, \ldots, T$ and $k = 0, 1, \ldots, T - t$,

$$
\frac{1}{n} \sum_{i=1}^{n} S_i^k X_{i,t}^{(n)} \xrightarrow{p} d(t, k),
$$

where

$$
d(t + 1, k) = d(t, k + 1) + c(d(t, 0)) (\bar{s}_{k+1} - d(t, k + 1)).
$$
Typically, we are only interested in \( d(t, 0) \), being the asymptotic proportion of occupied patches.

However, we may still interpret the ratio \( d(t, k)/d(t, 0) \) \((k \geq 1)\) as the \( k \)-th moment of the conditional distribution of the patch survival probability given that the patch is occupied. (From these moments, the conditional distribution could then be reconstructed.)
When \( \bar{s}_k = \bar{s}_1^k \) for all \( k \), that is the patch survival probabilities are the same, then it is possible to simplify

\[
d(t + 1, k) = d(t, k + 1) + c(d(t, 0)) (\bar{s}_{k+1} - d(t, k + 1))
\]

We can show by induction that \( d(t, k) = \bar{s}_1^k x_t \), where

\[
x_{t+1} = \bar{s}_1 (x_t + (1 - x_t) c(x_t))
\]

(Compare this with the earlier result.)
Theorem  The fixed points are given by

\[ d(k) = \int_0^1 \frac{c(\psi)\lambda^{k+1}}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda), \]

where \( \psi \) solves

\[ R(\psi) = \int_0^1 \frac{c(\psi)\lambda}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda) = \psi. \]  \hspace{1cm} (1)

If \( c(0) > 0 \), there is a unique \( \psi > 0 \). If \( c(0) = 0 \) and

\[ c'(0) \int_0^1 \frac{\lambda}{1-\lambda} \sigma(d\lambda) \leq 1, \]

then \( \psi = 0 \) is the unique solution to (1). Otherwise, (1) has two solutions, one of which is \( \psi = 0 \).
Theorem  If \( c(0) = 0 \) and

\[
c'(0) \int_0^1 \frac{\lambda}{1-\lambda} \sigma(d\lambda) \leq 1,
\]

then \( d(k) \equiv 0 \) is a stable fixed point. Otherwise, the non-zero solution to

\[
R(\psi) = \int_0^1 \frac{c(\psi) \lambda}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda) = \psi
\]

provides the stable fixed point through

\[
d(k) = \int_0^1 \frac{c(\psi) \lambda^{k+1}}{1-\lambda+c(\psi)\lambda} \sigma(d\lambda).
\]