High-density limits for infinite occupancy processes

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This is joint work with . . .

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UC Berkeley

The basic model

An *infinite occupancy process* $\boldsymbol{X}_t = (X_{i,t})_{i=1}^{\infty}$ is a Markov chain on $\{0,1\}^{\mathbb{Z}_+}$ with the property that, conditional on $\boldsymbol{X}_t$, the occupancies $X_{1,t+1}, X_{2,t+1}, \ldots$, at time $t+1$, are mutually independent. In particular, the dynamics of an infinite occupancy process are determined by the collection of functions

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It will be convenient to write

$$P_i(\boldsymbol{x}) = S_i(\boldsymbol{x}) x_i + C_i(\boldsymbol{x})(1 - x_i), \qquad \boldsymbol{x} \in \{0,1\}^{\mathbb{Z}_+},$$

where $S_i, C_i : \{0,1\}^{\mathbb{Z}_+} \to [0,1]$; $C_i(\boldsymbol{x})$ and $1 - S_i(\boldsymbol{x})$ are the (configuration dependent) "flip" probabilities.

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Voter Model: $S_i(\boldsymbol{x}) = 1 - \sum_{j=1}^{\infty} p_{ij}(1 - x_j)$, $C_i(\boldsymbol{x}) = \sum_{j=1}^{\infty} p_{ij} x_j$ ($p_{ii} = 0$).

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Domany-Kinzel PCA on the discrete torus of length n: $S_i(\boldsymbol{x}) = (q_2 - q_1) x_{i+1}$, $C_i(\boldsymbol{x}) = q_1 x_{i+1}$, $q_1, q_2 \in [0,1]$.

A metapopulation model

The sites $i = 1, 2, \ldots$ are habitat patches, and $X_{i,t}$ is 1 or 0 according to whether patch i is occupied or unoccupied at time t. $S_i(\boldsymbol{x}) = s_i$ (patch i *survival probability*) is the same for all $\boldsymbol{x}$, and

$$C_i(\boldsymbol{x}) = f\left(a_i \sum_{j=1}^{\infty} d_{ij} x_j\right),$$

where $f : [0, \infty) \to [0, 1]$ (called the *colonisation function*) satisfies $f(0) = 0$ (so there is total extinction at $\boldsymbol{x} \equiv 0$), and is typically an increasing function, a_i is a weight that may be interpreted as the capacity, or area, of patch i, and d_{ij} is the migration potential from patch j to patch i. (Further assumptions will be added later.)

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This particular form is reminiscent of the *Hanski incidence function model*[1], but now there is *no fixed upper limit* on the number of patches that can be occupied.

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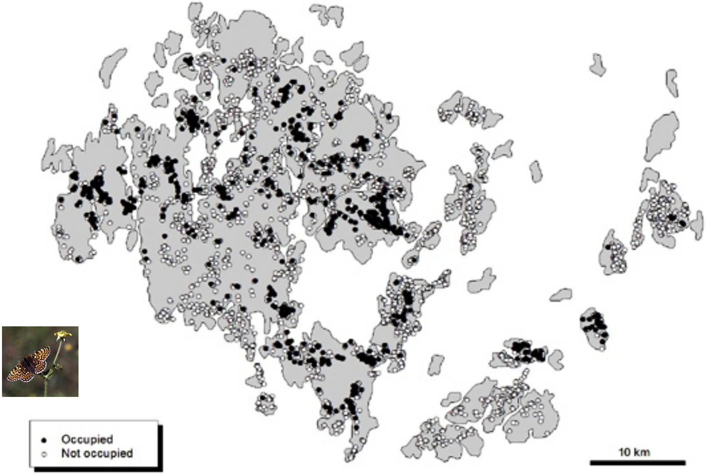

Glanville fritillary butterfly (Melitaea cinxia) in the Åland Islands in Autumn 2005.

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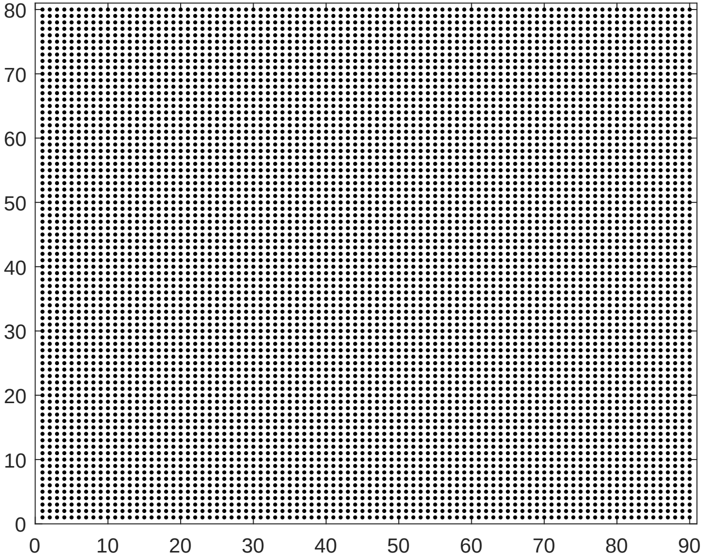

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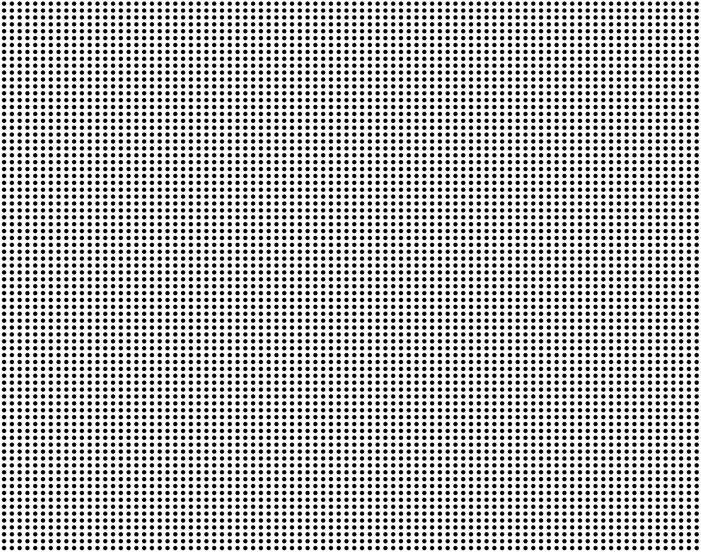
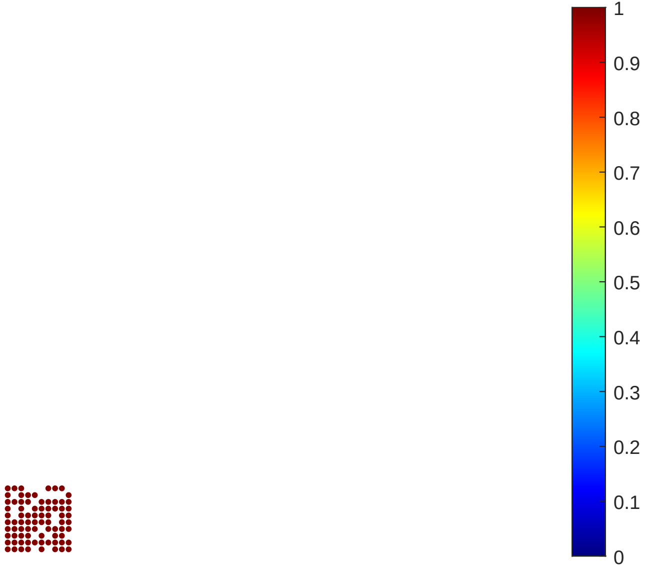
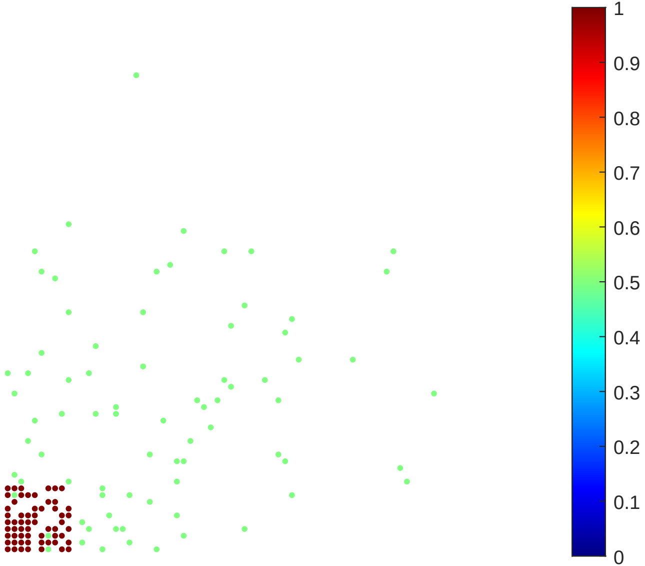


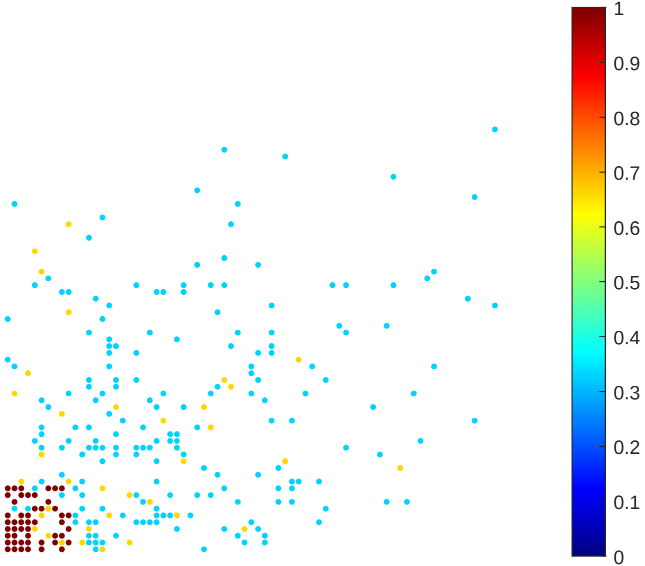
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 0)$



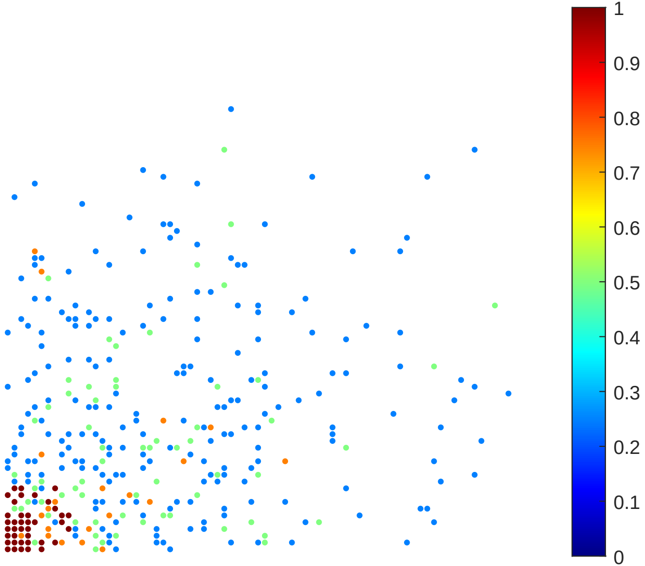
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 1)$



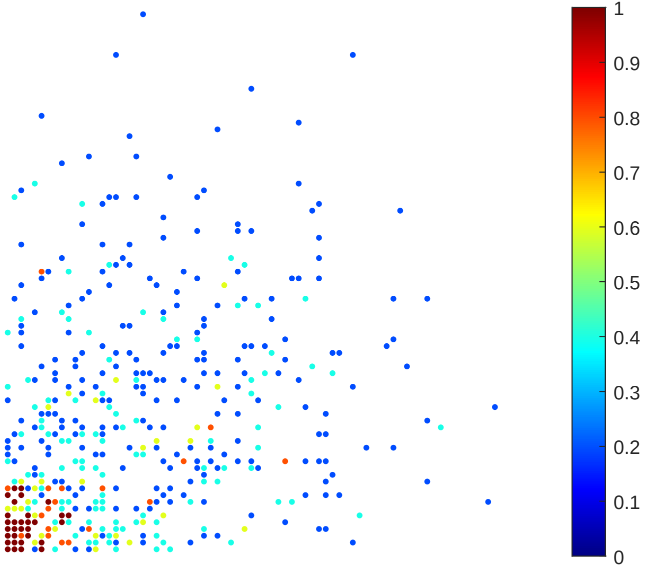
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ ($t = 2$)



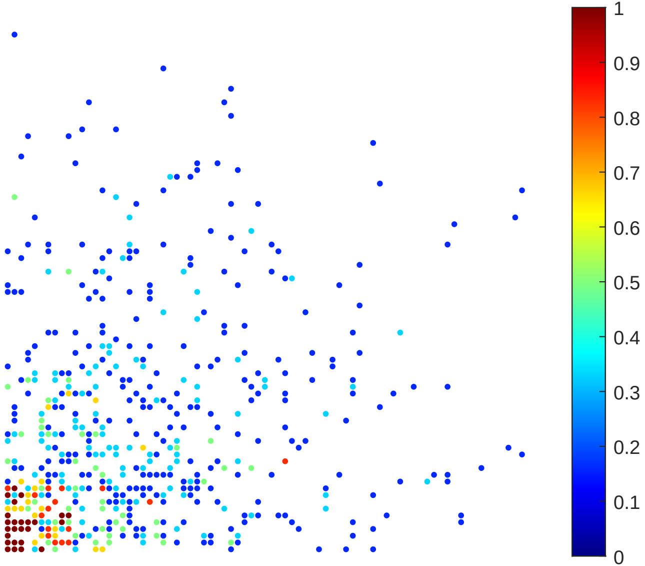
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ ($t = 3$)



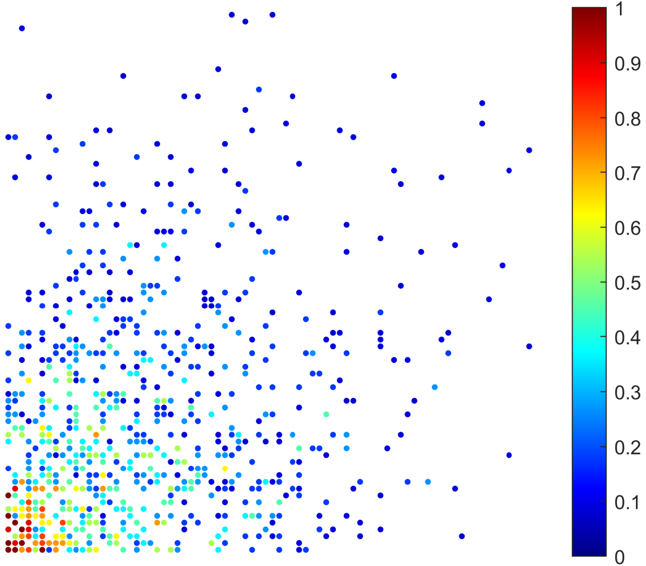
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 4)$



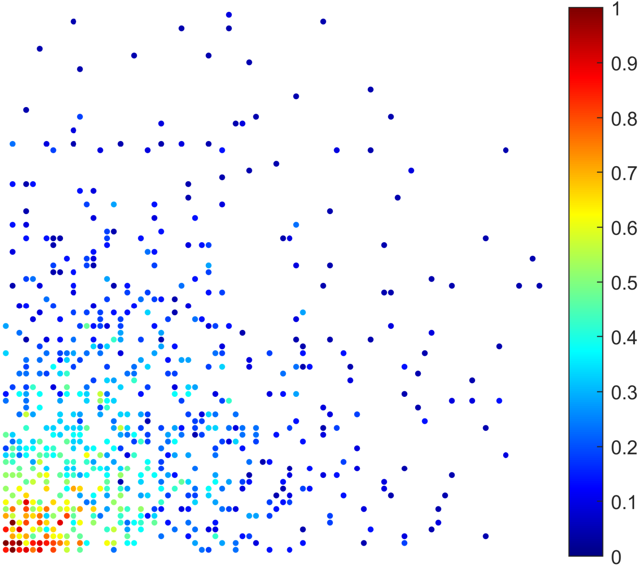
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ ($t = 5$)



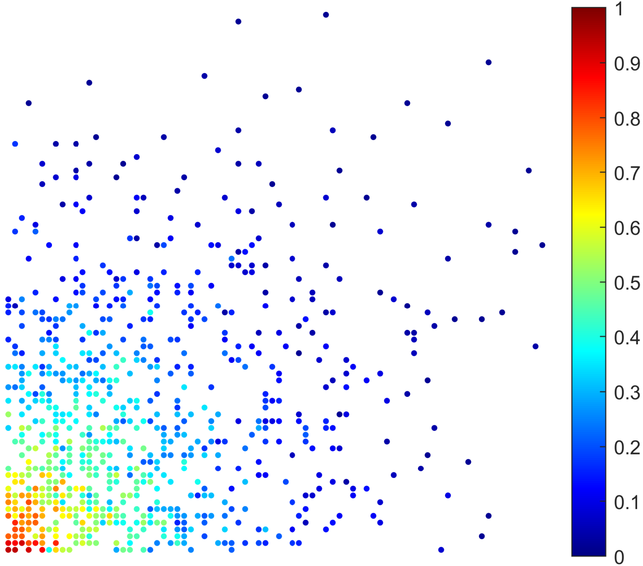
A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 10)$



A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ ($t = 20$)



A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ ($t = 50$)



Two approximating models

Returning to the general case

$$\mathbb{P}(X_{i,t+1} = 1 | \boldsymbol{X}_t) = S_i(\boldsymbol{X}_t) X_{i,t} + C_i(\boldsymbol{X}_t)(1 - X_{i,t}), \quad i = 1, 2, \ldots, \; t = 0, 1, \ldots$$

we will assume that, for some $M > 0$, $\sum_i C_i(\boldsymbol{x}) \leqslant M \sum_i x_i$ for all $\boldsymbol{x} \in E$, were E is the subset of $\{0,1\}^{\mathbb{Z}_+}$ with finitely many non-zero entries (finitely many sites occupied).

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Extending the domains of S_i and C_i to $[0,1]^{\mathbb{Z}_+}$, we consider a *deterministic analogue*[2] $\boldsymbol{p}_t = \{p_{i,t}\}_{i=1}^{\infty}$ that evolves according to

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The closeness of $\boldsymbol{X}_t$ and $\boldsymbol{p}_t$ (in a weak sense) is established by coupling $\boldsymbol{X}_t$ with an *independent site approximation* [2] $\boldsymbol{W}_t = \{W_{i,t}\}_{i=1}^{\infty}$ satisfying

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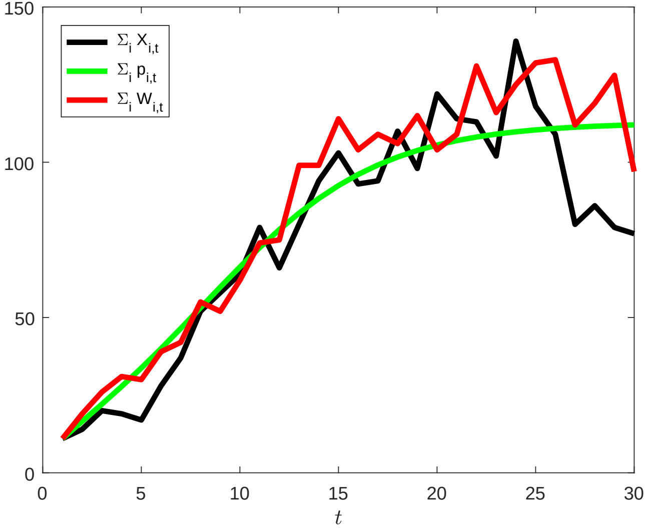

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We use common random numbers $\{U_{i,t}\}$ (independent, uniformly distributed on $[0,1]$):

$$\begin{aligned} X_{i,t+1} &= B_{i,t}(S_i(\boldsymbol{X}_t))X_{i,t} + B_{i,t}(C_i(\boldsymbol{X}_t))(1 - X_{i,t}) \\ W_{i,t+1} &= B_{i,t}(S_i(\boldsymbol{p}_t))W_{i,t} + B_{i,t}(C_i(\boldsymbol{p}_t))(1 - W_{i,t}) \end{aligned}$$

with $\boldsymbol{X}_0 = \boldsymbol{W}_0$, where $B_{i,t}(x) = \mathbb{1}\{U_{i,t} \geqslant 1 - x\}$, $x \in [0,1]$, is the quantile function of the Bernoulli distribution with success probability x.

The main result

To assess the quality of our approximations, we shall let [3]

$$\alpha = \sup_{j\in\mathbb{Z}_+} \sum_{i=1}^{\infty} \|\partial_j P_i\|_\infty \quad \beta = \sum_{i=1}^{\infty} \left(\sum_{j=1,\, j\neq i}^{\infty} \|\partial_j P_i\|_\infty^2 \right)^{1/2} \quad \gamma = \sum_{i,j=1}^{\infty} \|\partial_j^2 P_i\|_\infty$$

and assume these quantities are all finite. Here ∂_j and ∂_j^2 are the first and second partial derivative operators in the j-th coordinate.

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Theorem 1 There is a constant $C \in (0, 2\sqrt{\pi}]$ such that, for any $\boldsymbol{w} \in \ell^\infty$ and $t \geqslant 0$,

$$\mathbb{E} \left| \sum_{i=1}^{\infty} w_i (X_{i,t} - p_{i,t}) \right| \leqslant C \|\boldsymbol{w}\|_\infty (\beta + \gamma)(1 + 2\alpha)^t + \left(\sum_{i=1}^{\infty} w_i^2 p_{i,t} \right)^{1/2}.$$

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In our metapopulation model

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Recall that s_i is the patch i survival probability, a_i is the patch weight, d_{ij} is the migration potential from patch j to patch i, and $f : [0, \infty) \to [0, 1]$, the colonisation function, satisfies $f(0) = 0$.

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Assume that

(i) $\sum_i a_i < +\infty$ (the total weight of all patches is finite), and

(ii) there exists a $\bar{D}$ such that $\sup_j \sum_i d_{ij} \leqslant \bar{D}$ and $\sup_i \sum_j d_{ij} \leqslant \bar{D}$.

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A typically case for which Assumption (ii) holds is $d_{ij} = D(z_i, z_j) := \kappa(\|z_i - z_j\|)$, for patches located at points $\{z_i\}$ in $\mathbb{R}^d$, where κ is a smooth, non-negative, monotone decreasing function (typically $\kappa(x) = e^{-\psi x}$, or $\kappa(x) = e^{-\psi x^2}$, $\psi > 0$).

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Assumptions (i) and (ii) are enough to ensure that α, β, γ are all finite.

The metapopulation model

Let's check.

$$P_i(\boldsymbol{x}) := s_i x_i + f\left(a_i \sum_j d_{ij} x_j\right)(1 - x_i), \qquad \boldsymbol{x} \in [0,1]^{\mathbb{Z}_+}.$$

Since $\{a_i\}$ is necessarily bounded by some constant A, the Mean Value Theorem together with the assumption that $f(0) = 0$ gives

$$\alpha := \sup_j \sum_i \|\partial_j P_i\|_\infty \leqslant \|f'\|_\infty \sup_j \sum_i a_i d_{ij} \leqslant \|f'\|_\infty A\bar{D}.$$

Similarly,

$$\beta := \sum_i \left(\sum_{j \neq i} \|\partial_j P_i\|_\infty^2\right)^{1/2} \leqslant \|f'\|_\infty \sum_i a_i \left(\sum_{j \neq i} d_{ij}^2\right)^{1/2} \leqslant \|f'\|_\infty \bar{D} \sum_i a_i < \infty,$$

and

$$\gamma := \sum_i \sum_j \|\partial_j^2 P_i\|_\infty \leqslant \|f''\|_\infty \sum_i a_i^2 \sum_j d_{ij}^2 \leqslant \|f''\|_\infty A\bar{D}^2 \sum_i a_i < \infty.$$

We shall suppose that the patch locations are spaced according to some measure σ. In particular, for any bounded continuous function g,

$$\frac{1}{m^d}\sum_{i=1}^{\infty} g(m^{-1}z_i) \to \int_{\mathbb{R}^d} g(z)\sigma(\mathrm{d}z), \qquad \text{as } m \to \infty.$$

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Suppose that there is a sequence of models $\{\boldsymbol{X}_t^{(m)}\}_{m=1}^{\infty}$ with parameters $s_i^{(m)}, a_i^{(m)}, d_{ij}^{(m)}$, and the same colonisation function f, such that

$$s_i^{(m)} = s\left(m^{-1}z_i\right), \quad a_i^{(m)} = a\left(m^{-1}z_i\right), \quad d_{ij}^{(m)} = m^{-d}\kappa\left(m^{-1}\|z_i - z_j\|\right),$$

for smooth functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$, $a : \mathbb{R}^d \to \mathbb{R}_+$, and $s : \mathbb{R}^d \to [0,1]$.

In this way, the patch locations are effectively being drawn together as $m \to \infty$.

The metapopulation model - a high density limit

Define, for each m and t, a finite measure $\pi_t^{(m)}$ by

$$\pi_t^{(m)}(B) = \frac{1}{m^d} \sum_{i=1}^{\infty} p_{i,t}^{(m)} \mathbb{1}\{m^{-1} z_i \in B\}, \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

and assume that $\pi_0^{(m)} \to \pi_0$ for some finite measure π_0. Evidently, π_0 will be absolutely continuous with respect to σ, and so there exists a function p_0 such that, for any bounded continuous function g,

$$\int g(z) \pi_0^{(m)}(\mathrm{d}z) = \frac{1}{m^d} \sum_{i=1}^{\infty} g(m^{-1} z_i) p_{i,0}^{(m)} \to \int g(z) p_0(z) \sigma(\mathrm{d}z).$$

One can show[1], furthermore, that there exists a finite measure π_t, which is absolutely continuous with respect to σ, such that $\pi_t^{(m)} \to \pi_t$.

[1] McVinish, R. and Pollett, P.K. (2014) The limiting behaviour of Hanski's incidence function metapopulation model. *J. Appl. Probab.* 51, 297–316.

Consequently,

$$\int g(z)\pi_t^{(m)}(\mathrm{d}z) = \frac{1}{m^d}\sum_{i=1}^{\infty} g(m^{-1}z_i)p_{i,t}^{(m)} \to \int g(z)p_t(z)\sigma(\mathrm{d}z),$$

for some function p_t. In particular, the functions p_t satisfy the recursion

$$p_{t+1}(x) = s(x)p_t(x) + (1-p_t(x))f\left(a(x)\int \kappa(\|x-z\|)p_t(z)\sigma(\mathrm{d}z)\right).$$

Define, for each m and t, a random measure $\mu_t^{(m)}$ by

$$\mu_t^{(m)}(B) = \frac{1}{m^d}\sum_{i=1}^{\infty} X_{i,t}^{(m)}\mathbb{1}\{m^{-1}z_i \in B\}, \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Assuming that each $X_{i,0}^{(m)}$ is a Bernoulli random variable with success probability $p_{i,0}^{(m)}$, it is clear that $\mu_0^{(m)} \xrightarrow{\mathcal{D}} \pi_0$ as $m \to \infty$. Our objective is to show that $\mu_t^{(m)} \xrightarrow{\mathcal{D}} \pi_t$ for every $t \geqslant 1$.

The metapopulation model - a high density limit

Recall the bound in the earlier Theorem 2: there is a constant $C \in (0, 2\sqrt{\pi}]$ such that, for any $\boldsymbol{w} \in \ell^\infty$ and $t \geqslant 0$,

$$\mathbb{E}\left|\sum_{i=1}^{\infty} w_i (X_{i,t} - p_{i,t})\right| \leqslant C\|\boldsymbol{w}\|_\infty (\beta + \gamma)(1 + 2\alpha)^t + \left(\sum_{i=1}^{\infty} w_i^2 p_{i,t}\right)^{1/2}.$$

We apply this with weights $w_i^{(m)} = g(m^{-1} z_i)$, where g is any bounded continuous function, and all quantities are indexed by m. Since

$$\frac{1}{m^d} \sum_{i=1}^{\infty} \left(w_i^{(m)}\right)^2 p_{i,t}^{(m)} = \int g^2(z) \pi_t^{(m)}(\mathrm{d}z) \to \int g^2(z) \pi_t(\mathrm{d}z),$$

we conclude that if $\{\alpha_m\}_{m=1}^{\infty}$ is bounded and $m^{-d}\beta_m$, $m^{-d}\gamma_m \to 0$ as $m \to \infty$, then

$$\mathbb{E}\left|\int g(z) \mu_t^{(m)}(\mathrm{d}z) - \int g(z) \pi_t^{(m)}(\mathrm{d}z)\right| \to 0.$$

Then, $\pi_t^{(m)} \xrightarrow{\mathcal{D}} \pi_t$ will imply that $\mu_t^{(m)} \xrightarrow{\mathcal{D}} \pi_t$.

The metapopulation model - a high density limit

Let's check: (i) $\{\alpha_m\}$ is bounded because

$$\alpha \leqslant \|f'\|_\infty \sup_j \sum_i a_i d_{ij} = \|f'\|_\infty \sup_j \frac{1}{m^d} \sum_i a\left(\frac{z_i}{m}\right) \kappa(m^{-1}\|z_i - z_j\|)$$

$$\to \|f'\|_\infty \sup_x \int a(y)\kappa(\|x-y\|)\sigma(\mathrm{d}y).$$

(ii) $m^{-d}\beta_m \to 0$ because

$$m^{-d/2}\beta_m \leqslant \|f'\|_\infty \frac{1}{m^d} \sum_i a\left(\frac{z_i}{m}\right) \left(\sum_j \frac{1}{m^d} \kappa\left(m^{-1}\|z_i - z_j\|\right)^2\right)^{1/2}$$

$$\to \|f'\|_\infty \int a(x) \left(\int \kappa(\|x-y\|)^2 d\sigma(y)\right)^{1/2} d\sigma(x).$$

(iii) $m^{-d}\gamma_m \to 0$ because

$$\gamma_m \leqslant \|f''\|_\infty \sum_i a_i^2 \sum_j d_{ij}^2 = \|f''\|_\infty \frac{1}{m^d} \sum_i a^2\left(\frac{z_i}{m}\right) \sum_j \frac{1}{m^d} \kappa\left(m^{-1}\|z_i - z_j\|\right)^2$$

$$\to \|f''\|_\infty \int a^2(x)\kappa(\|x-y\|)^2 d\sigma(x) d\sigma(y).$$

Details

$d = 2$

Colonisation function: $f(x) = 1 - \exp(-\alpha x)$ with $\alpha = 0.01$.

Survival function: $s(\boldsymbol{z}) = \exp(-\phi\|\boldsymbol{z}\|)$ with $\phi = 0.25$.

Patch weight function: $a(\boldsymbol{z}) = \exp(-\theta\|\boldsymbol{z}\|)$ with $\theta = 0.25$.

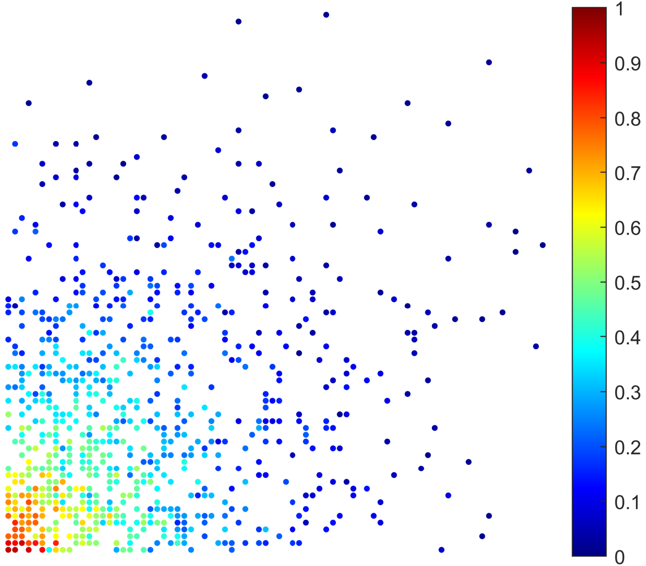
Easy of movement function: $d(\boldsymbol{x}, \boldsymbol{z}) = b\exp(-\psi\|\boldsymbol{x} - \boldsymbol{z}\|)$ with $b = 25$ and $\psi = 0.4$.

Scaling: $m = 8$

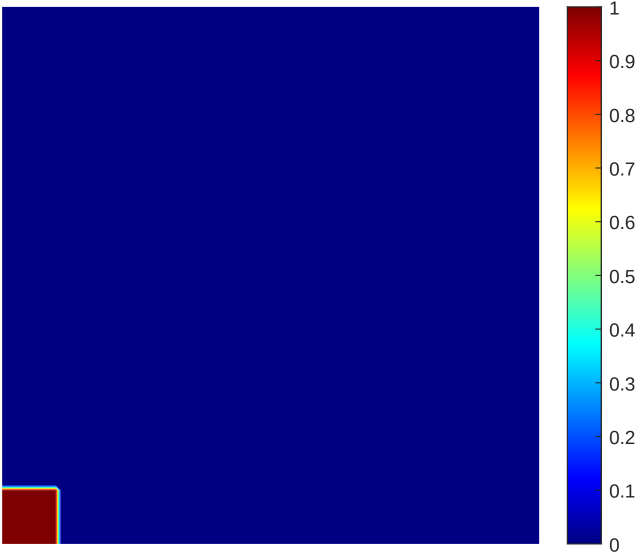
$$s_i^{(m)} = s\left(m^{-1} z_i\right), \quad a_i^{(m)} = a\left(m^{-1} z_i\right), \quad d_{ij}^{(m)} = m^{-2}\kappa\left(m^{-1}\|z_i - z_j\|\right)$$

Initially configuration: 70 percent of patches are occupied in $\{1, 2, \ldots, 10\}^2$.

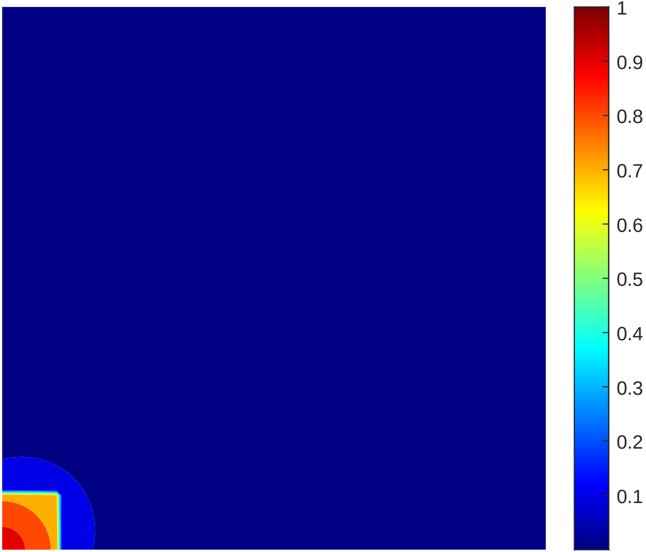
The earlier simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 50)$



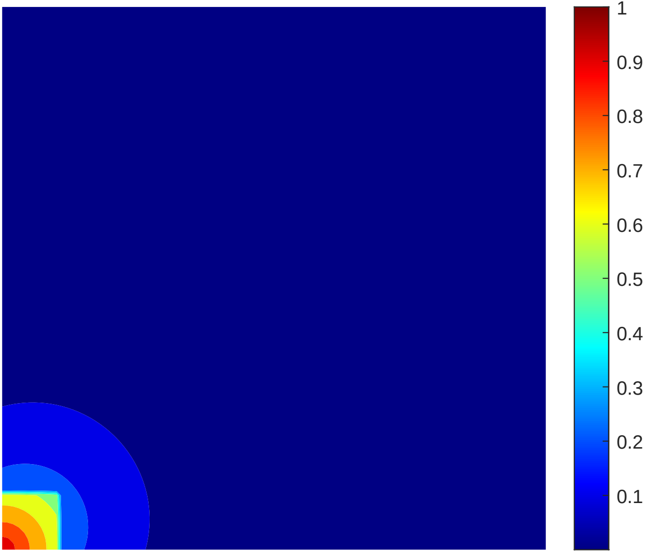
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 0)$



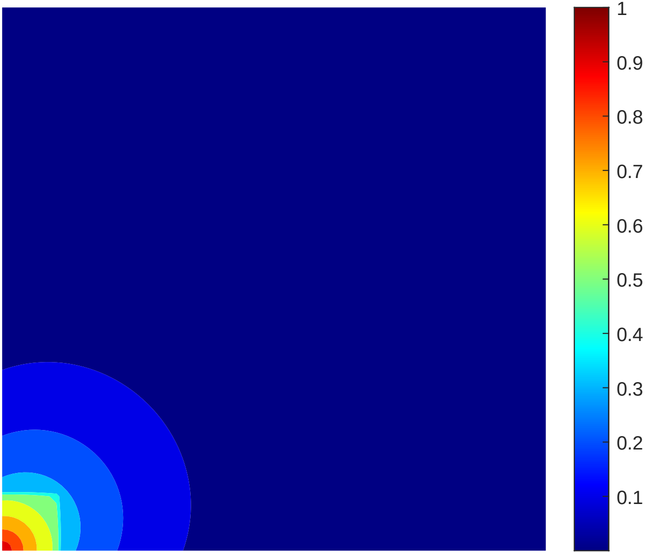
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 1)$



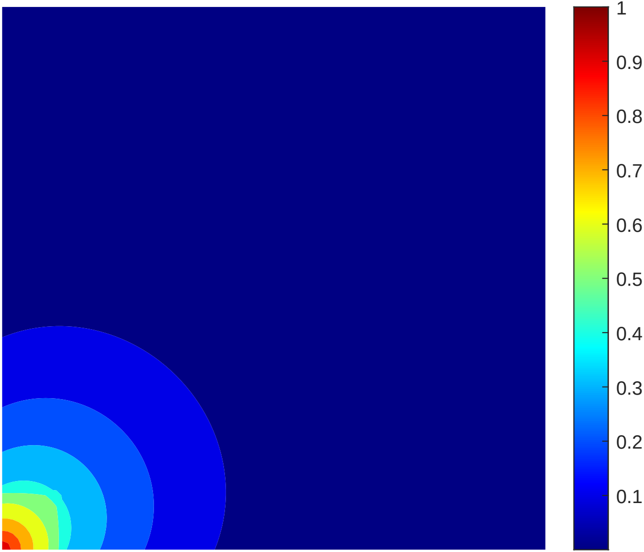
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 2)$



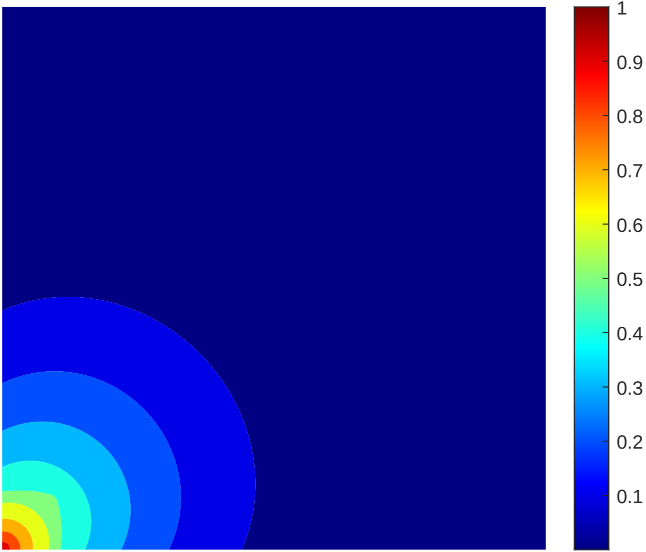
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 3)$



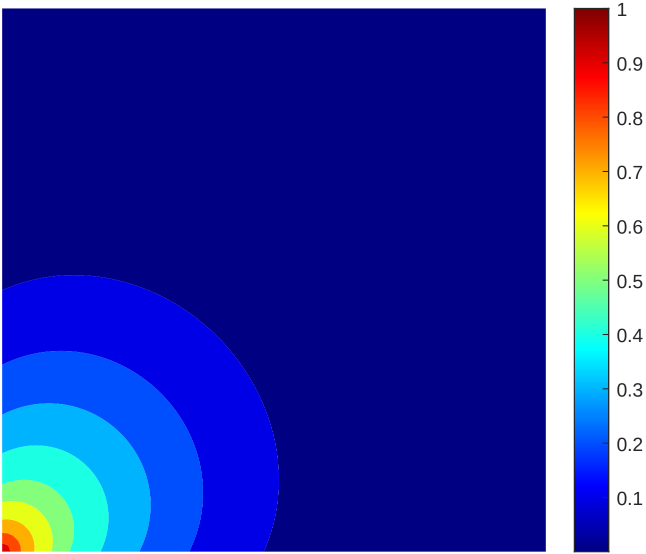
Occupancy probability heatmap $p_t(\boldsymbol{z})$ ($t = 4$)



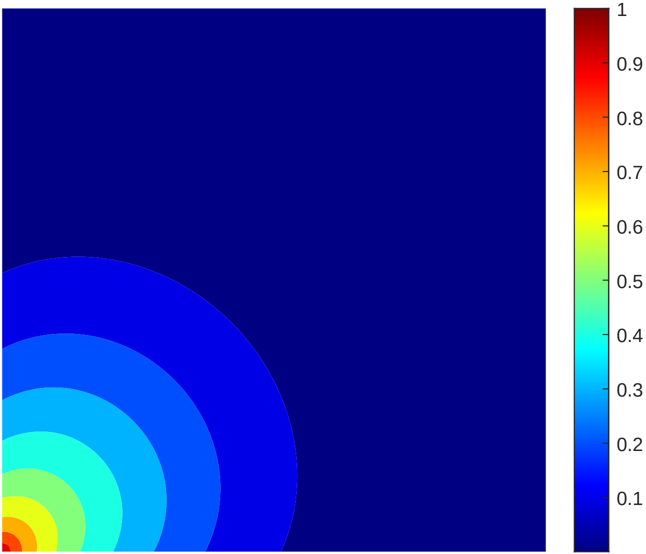
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 5)$



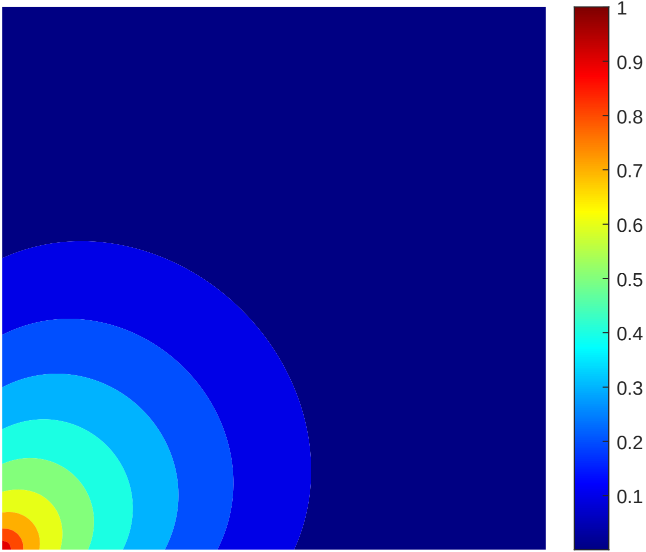
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 6)$



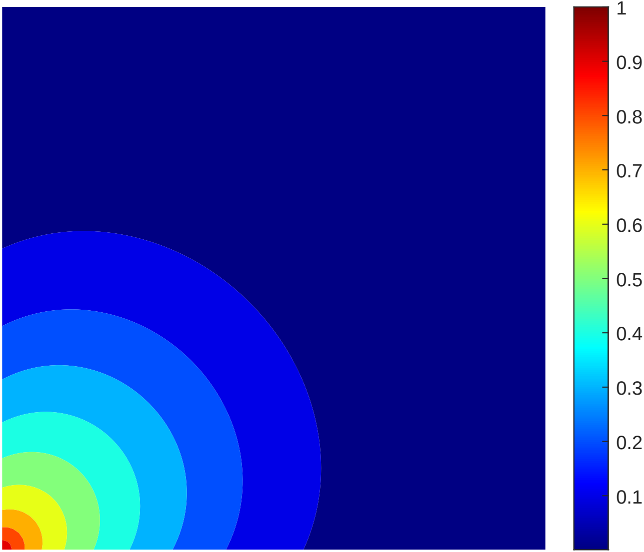
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 7)$



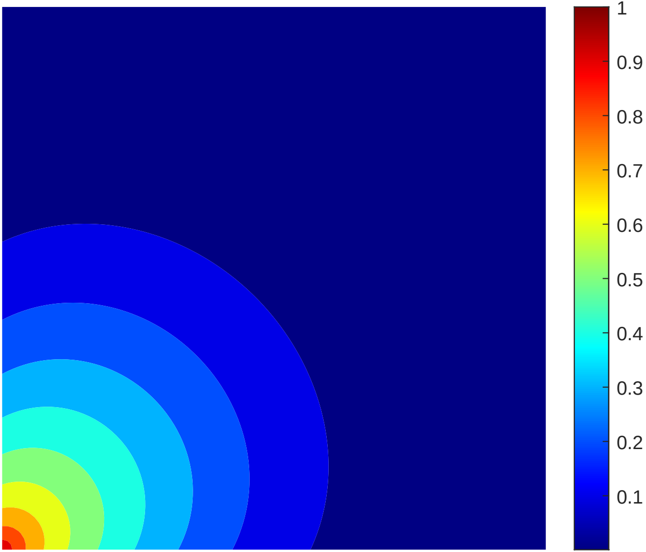
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 8)$



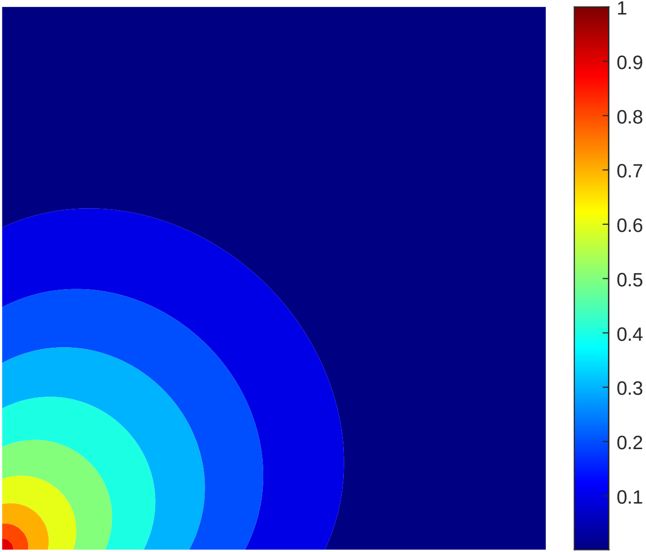
Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 9)$



Occupancy probability heatmap $p_t(\boldsymbol{z})$ $(t = 10)$



Occupancy probability heatmap $p_t(\boldsymbol{z})$ ($t = 50$)



A simulation - patches located on the integer lattice $\mathbb{Z}_+^2$ $(t = 50)$