A Method for Analysing
Complex Markovian Models

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\[ E(a, c) = \frac{a^c}{c!} \left( \sum_{i=0}^{c} \frac{a^i}{i!} \right)^{-1} \]
Erlang’s Formula

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Arrival rate
Erlang’s Formula

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- Arrival rate
- Number of circuits
\[ l_j = E \left( \sum_{r \in R} \lambda_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - l_i), c_j \right) \]

\[ j = 1, 2, \ldots, J \]
Erlang fixed point approximation

\[ l_j = E \left( \sum_{r \in \mathcal{R}} \lambda_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - l_i), c_j \right) \]

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\[ j = 1, 2, \ldots, J \]

Effective arrival rate at link \( j \)
Erlang fixed point approximation

\[ l_j = E \left( \sum_{r \in R} \lambda_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - l_i) \right), \quad (1) \]

\[ j = 1, 2, \ldots, J \]
The problem

Consider any large-scale stochastic system whose natural state description is Markovian, yet its behaviour (equilibrium or time-dependent behaviour) is difficult to analyze.

Can we find an alternative state description, together with an approximating transition structure, that can be analyzed more simply?

Our goal is to approximate quantities of interest and to assess the quality of the approximation.
Competition for resources

A collection of resources of different types and differing amounts (capacities)

\[ c_1 = 2 \quad c_2 = 4 \quad c_3 = 6 \quad c_4 = 4 \quad c_5 = 3 \]  
(Capacity)
Competition for resources

Customers of different types arrive as independent Poisson streams and request groups of resources

$c_1 = 2 \quad c_2 = 4 \quad c_3 = 6 \quad c_4 = 4 \quad c_5 = 3 \quad \text{(Capacity)}$
Competition for resources

Customers are identified by *which* resources they require

Type A

$c_1 = 2$
$c_2 = 4$
$c_3 = 6$
$c_4 = 4$
$c_5 = 3$  (Capacity)
Competition for resources

Customers are identified by *which* resources they require

Type B

\[ c_1 = 2 \quad c_2 = 4 \quad c_3 = 6 \quad c_4 = 4 \quad c_5 = 3 \]

(Capacity)
Competition for resources

Customers are identified by *which* resources they require

![Diagram showing competition for resources with Type C]
Competition for resources

Resources are captured and held for a random period and released simultaneously.

\[ u_1 = 0 \quad u_2 = 0 \quad u_3 = 0 \quad u_4 = 0 \quad u_5 = 0 \]  
(Usage)
Competition for resources

Type A \( (n_A = 0) \)
Type B \( (n_B = 0) \)
Type C \( (n_C = 0) \)

\[
\begin{align*}
  u_1 &= 0 \\
  u_2 &= 0 \\
  u_3 &= 0 \\
  u_4 &= 0 \\
  u_5 &= 0 \\
\end{align*}
\]

(Usage)
Competition for resources

Type A \( (n_A = 1) \)
Type B \( (n_B = 0) \)
Type C \( (n_C = 0) \)

\[ u_1 = 1 \quad u_2 = 1 \quad u_3 = 1 \quad u_4 = 0 \quad u_5 = 0 \] (Usage)
Competition for resources

Type A \( (n_A = 1) \)
Type B \( (n_B = 0) \)
Type C \( (n_C = 0) \)

\[ u_1 = 1 \quad u_2 = 1 \quad u_3 = 1 \quad u_4 = 0 \quad u_5 = 0 \]  
(Usage)
Competition for resources

Type A  \( (n_A = 1) \)
Type B  \( (n_B = 1) \)
Type C  \( (n_C = 0) \)

\[
\begin{array}{l}
\begin{aligned}
\text{\( u_1 = 1 \)} & \quad \text{\( u_2 = 2 \)} & \quad \text{\( u_3 = 2 \)} & \quad \text{\( u_4 = 1 \)} & \quad \text{\( u_5 = 0 \)} \\
\end{aligned}
\end{array}
\]

(Usage)
Competition for resources

Type A \( (n_A = 1) \)
Type B \( (n_B = 1) \)
Type C \( (n_C = 0) \)

\[
\begin{align*}
u_1 &= 1 \\
u_2 &= 2 \\
u_3 &= 2 \\
u_4 &= 1 \\
u_5 &= 0
\end{align*}
\]

(Usage)
Competition for resources

Type A \( (n_A = 1) \)  
Type B \( (n_B = 2) \)  
Type C \( (n_C = 0) \)

\[
\begin{align*}
 u_1 &= 1 \\
 u_2 &= 3 \\
 u_3 &= 3 \\
 u_4 &= 2 \\
 u_5 &= 0 \\
\end{align*}
\]

(Usage)
Competition for resources

Type A \((n_A = 1)\)

Type B \((n_B = 2)\)

Type C \((n_C = 1)\)

\(u_1 = 1\)

\(u_2 = 3\)

\(u_3 = 4\)

\(u_4 = 3\)

\(u_5 = 1\)

(Usage)
Competition for resources

Type A \( (n_A = 0) \)
Type B \( (n_B = 2) \)
Type C \( (n_C = 1) \)

\[ u_1 = 0 \quad u_2 = 2 \quad u_3 = 3 \quad u_4 = 3 \quad u_5 = 1 \]  
(Usage)
Competition for resources

Type A \( (n_A = 0) \)
Type B \( (n_B = 3) \)
Type C \( (n_C = 1) \)

\[ \begin{align*}
  u_1 &= 0 \\
  u_2 &= 3 \\
  u_3 &= 4 \\
  u_4 &= 4 \\
  u_5 &= 1 \\
\end{align*} \]

(Usage)
Competition for resources

Type A \( (n_A = 0) \)

BLOCKED Type B \( (n_B = 3) \)

BLOCKED Type C \( (n_C = 1) \)

Type-B and Type-C customers are now blocked because there is no more

\( u_1 = 0 \quad u_2 = 3 \quad u_3 = 4 \quad u_4 = 4 \quad u_5 = 1 \) (Usage)
Let $\mathcal{R}$ be the set of customer types and let $\mathbf{n} = (n_r, r \in \mathcal{R})$, where $n_r$ is the number of type-$r$ customers in the system.

Let $\mathbf{c} = (c_j, j = 1, \ldots, J)$ be the resource capacities, and $\Lambda = (\lambda_{jr})$ be the $J \times \mathcal{R}$ design matrix with $\lambda_{jr} = 1$ if resource $j$ is used by type-$r$ customers. The set of all states is then $S = \{ \mathbf{n} \in \mathbb{Z}^\mathcal{R}_+: \Lambda \mathbf{n} \leq \mathbf{c} \}$.

If type-$r$ customers arrive at rate $\nu_r$ (independent Poisson streams) and hold sets of resources for independent exponentially distributed times (with unit mean say), then $(\mathbf{n}_t, t \geq 0)$ is a Markov chain with transition rates:

\[ q(\mathbf{n}, \mathbf{n} + e_r) = \nu_r \quad q(\mathbf{n}, \mathbf{n} - e_r) = n_r \]

(here $e_r$ is the unit vector with a 1 as its $r$-th entry).
Competition for resources

The chain has stationary distribution

\[ p(n) = B \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \quad n \in S, \]

where \( B = B(c) \) is a normalizing constant.
Competition for resources

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\[ p(n) = B \prod_{r \in R} \frac{\nu_{nr}^{n_r}}{n_r!}, \quad n \in S, \]

where \( B = B(c) \) is a normalizing constant.

The chance that an arriving type-\( r \) customer is blocked is

\[ b_r = 1 - \frac{B(c)}{B(c - \Lambda e_r)}. \]
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The chance that an arriving type-\( r \) customer is blocked is

\[ b_r = 1 - \frac{B(c)}{B(c - \Lambda e_r)}. \]

However, this cannot (usually) be computed in polynomial time.
Expected rates

We have a large-scale stochastic system whose natural state description is Markovian, yet its behaviour (equilibrium or time-dependent behaviour) is difficult to analyze.

Idea (Peter Taylor, 1996). Find an alternative state description, together with an approximating transition structure, that can be analyzed more simply. For this description, impose a Markovian assumption: the rates of transition are given by the expected rates of the corresponding transitions of the original chain:

$$q'(u, v) = E_p \left( \sum_{m \in A(n(t))} q(n(t), m) \right),$$

where $n \rightarrow A(n)$ is all transitions out of $n$ that give rise to a $u \rightarrow v$ transition in the new structure.
Expected rates

We have a large-scale stochastic system whose natural state description is Markovian, yet its behaviour (equilibrium or time-dependent behaviour) is difficult to analyze.

Idea (Peter Taylor, 1996). Find an alternative state description, together with an approximating transition structure, that can be analyzed more simply. For this description, impose a Markovian assumption: the rates of transition are given by the expected rates of the corresponding transitions of the original chain:

$$q'(u, v) = E_p \left( \sum_{m \in A(n(t))} q(n(t), m) \big| n(t) \in A_u \right),$$

where $n \rightarrow A(n)$ is all transitions out of $n$ that give rise to a $u \rightarrow v$ transition in the new structure.
A general fixed point method

Modified idea.

\[ q'(u, v) = E_{\pi(0)} \left( \sum_{m \in A(n(t))} q(n(t), m) \bigg| n(t) \in A_u \right), \]

where \( \pi(0) \) is our best guess for the stationary distribution of the new Markov chain.
A general fixed point method

Modified idea.

\[ q'(u, v) = \mathbb{E}_{\pi^{(0)}} \left( \sum_{m \in A(n(t))} q(n(t), m) \bigg| n(t) \in A_u \right), \]

where \( \pi^{(0)} \) is our best guess for the stationary distribution of the new Markov chain.

Now evaluate the stationary distribution \( \pi^{(1)} \) of the new chain using transition rates \( q'(u, v) \).
A general fixed point method

Modified idea.

\[ q'(u, v) = \mathbb{E}_{\pi^{(0)}} \left( \sum_{m \in A(n(t))} q(n(t), m) \bigg| n(t) \in A_u \right), \]

where \( \pi^{(0)} \) is our best guess for the stationary distribution of the new Markov chain.

Now evaluate the stationary distribution \( \pi^{(1)} \) of the new chain using transition rates \( q'(u, v) \).

Repeat the procedure.
Hopes

$\pi^{(n)}$ will converge (to $\pi$ say)
A general fixed point method

Hopes

- $\pi^{(n)}$ will converge (to $\pi$ say)
- $\pi$ will provide good estimates of quantities of interest
A general fixed point method

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- $\pi^{(n)}$ will converge (to $\pi$ say)
- $\pi$ will provide good estimates of quantities of interest
- $\pi$ will provide the best estimate of a particular quantity of interest among members of a class of distributions (for example, product-form distributions)
A general fixed point method

Hopes

- $\pi^{(n)}$ will converge (to $\pi$ say)
- $\pi$ will provide good estimates of quantities of interest
- $\pi$ will provide the best estimate of a particular quantity of interest among members of a class of distributions (for example, product-form distributions)
- to delimit conditions under which the approximations are good
Application to the resource model

Focus on the usage \( u = (u_j, j = 1, \ldots, J) \). The process \((u_t, t \geq 0)\) is not (usually) Markovian. Let

\[
\pi(u) = \prod_{j=1}^{J} \pi_j(u_j), \quad \pi_j(u) = \frac{a_j^u}{u!} \left( \sum_{v=0}^{c_j} \frac{a_j^v}{v!} \right)^{-1} (u = 0, \ldots, c_j)
\]

where the \( a_j \)'s are to be determined. Then,

\[
q'(u, u + e_k) = E_{\pi} \left( \sum_{r \in R: k \in r} \nu_r \prod_{i \in r - \{k\}} 1\{U_i < c_i\} \bigg| U_k = u_k \right).
\]
Focus on the usage $u = (u_j, j = 1, \ldots, J)$. The process $(u_t, t \geq 0)$ is not (usually) Markovian. Let

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$$(u = 0, \ldots, c_j)$$

where the $a_j$'s are to be determined. Then,

$$q'(u, u + e_k) = \mathbb{E}_\pi \left( \sum_{r \in \mathcal{R}} \lambda_{kr} \nu_r \prod_{i \in r \setminus \{k\}} 1\{U_i < c_i\} \left| U_k = u_k \right. \right).$$
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\pi(u) = \prod_{j=1}^{J} \pi_j(u_j), \quad \pi_j(u) = \frac{a_j^u}{u!} \left( \sum_{v=0}^{c_j} \frac{a_j^v}{v!} \right)^{-1} (u = 0, \ldots, c_j)
$$

where the $a_j$'s are to be determined. Then,

$$
q'(u, u + e_k) = \sum_{r \in \mathcal{R}} \lambda_{kr} \nu_r
$$

$$
\Pr_{\pi}\left( U_i < c_i, \forall i \in r - \{k\} \mid U_k = u_k \right).
$$
Focus on the *usage* $\mathbf{u} = (u_j, j = 1, \ldots, J)$. The process $(\mathbf{u}_t, t \geq 0)$ is not (usually) Markovian. Let

$$
\pi(\mathbf{u}) = \prod_{j=1}^{J} \pi_j(u_j), \quad \pi_j(u) = \frac{a_j^u}{u!} \left( \sum_{v=0}^{c_j} \frac{a_j^v}{v!} \right)^{-1} (u = 0, \ldots, c_j)
$$

where the $a_j$’s are to be determined. Then,

$$
q'(\mathbf{u}, \mathbf{u} + \mathbf{e}_k) = \sum_{r \in \mathcal{R}} \lambda_{kr} \nu_r \prod_{i \in r \setminus \{k\}} (1 - l_i),
$$

where $l_i = \pi_i(c_i)$. 
Similarly,

\[ q'(u, u - e_k) = E_\pi \left( u_k 1\{U_k = u_k\} \mid U_k = u_k \right) = u_k. \]
Application to the resource model

Similarly,

\[ q'(u, u - e_k) = E_\pi \left( u_k 1 \{ U_k = u_k \} \middle| U_k = u_k \right) = u_k. \]

The limiting set of resource blocking probabilities \( l = (l_j, j = 1, \ldots, J) \) will satisfy

\[ l_j = E \left( \sum_{r \in R} \lambda_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - l_i), c_j \right), \]

where

\[ E(a, c) = \frac{a^c}{c!} \left( \sum_{v=0}^{c} \frac{a^v}{v!} \right)^{-1}. \]  

\((Erlang's\ formula)\)
Conclusion

We have a method that has the potential to be applied to any large-scale Markovian model.
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- In the case of resource systems, it gives the widely used, and highly accurate, Erlang fixed point approximation.
- Plenty of scope for applications (for example, queueing networks with blocking).
- Plenty of scope for mathematical developments (for example, fixed point theory).