# ESTIMATING BLOCKING PROBABILITIES IN TELECOMMUNICATIONS NETWORKS WITH LINEAR STRUCTURE

by

Mark Bebbington, Massey University
Phil Pollett, The University of Queensland
Ilze Ziedins, The University of Auckland

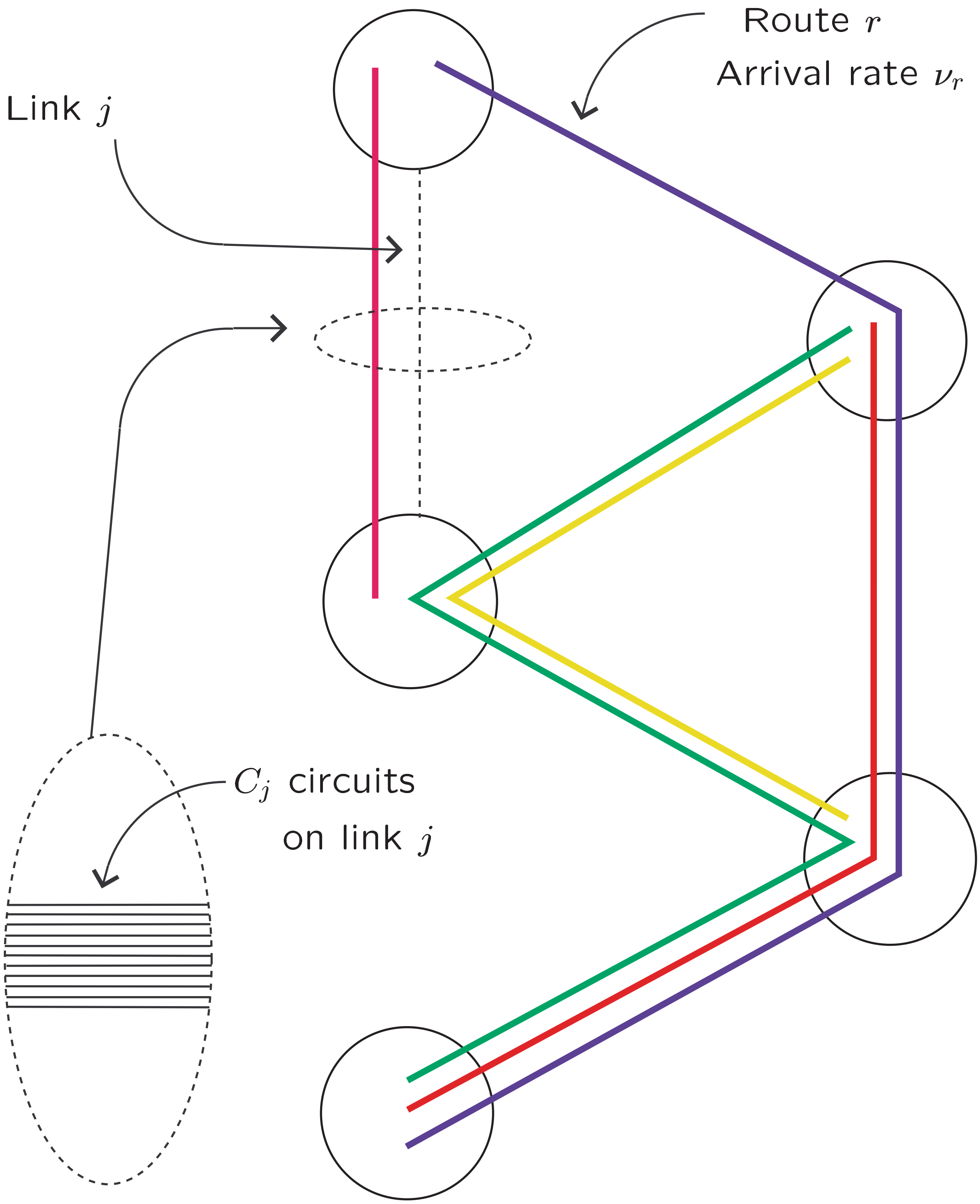


Fig 1. A typical circuit-switched network
(5 nodes, 6 links and 5 routes)

## A circuit-switched network

This consists of a group of nodes (locations),

$K$ links (circuit groups),

$C_j$ circuits comprising link $j$, and

$\mathcal{R}$ – a collection of routes.

Each route $r \in \mathcal{R}$ is a set of links. Calls using route $r$ are *offered* at rate $\nu_r$ as a Poisson stream, and use $a_{jr}(\geq 0)$ circuits from link $j$. $\mathcal{R}$ indexes *independent* Poisson processes.

Calls requesting route $r$ are *blocked* and lost if, on *any* link $j$, there are fewer than $a_{jr}$ free circuits. Otherwise, the call is *connected* and simultaneously holds $a_{jr}$ circuits on each link $j$ for the duration of the call. For simplicity, we shall take $a_{jr} \in \{0, 1\}$.

Call durations are iid exponential random variables with unit mean, and are *independent* of the arrival processes.

## The usual Markovian model

Let $\boldsymbol{n} = (n_r,\ r \in \mathcal{R})$, where $n_r$ is the number of calls in progress using route $r$, let $\boldsymbol{C} = (C_j,\ j = 1, \dots, K)$, and let $\boldsymbol{A} = (a_{jr},\ r \in \mathcal{R},\ j = 1, \dots, K)$.

The continuous-time Markov chain $(\boldsymbol{n}(t),\ t \geq 0)$ takes values in

$$S = S(\boldsymbol{C}) = \left\{ \boldsymbol{n} \in \mathsf{Z}_+^{\mathsf{R}} \ :\ \boldsymbol{A}\boldsymbol{n} \leq \boldsymbol{C} \right\}$$

and has a unique stationary distribution given by

$$\pi(\boldsymbol{n}) = G^{-1} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \qquad \boldsymbol{n} \in S,$$

where $G = G(\boldsymbol{C})$ is given by

$$G(\boldsymbol{C}) = \sum_{\boldsymbol{n} \in S(\boldsymbol{C})} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}.$$

## Blocking probabilities

The stationary probability that a route-$r$ call is blocked is given by

$$L_r = 1 - \frac{G(\boldsymbol{C} - \boldsymbol{A}\boldsymbol{e}_r)}{G(\boldsymbol{C})},$$

where $\boldsymbol{e}_r$ is the unit vector from $S(\boldsymbol{C})$ describing just one call in progress on route $r$

$$\left(\text{recall that } G(\boldsymbol{C}) = \sum_{\boldsymbol{n} \in S(\boldsymbol{C})} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}\right).$$

An explicit expression for $L_r$!

However, the bad news is that $G(\boldsymbol{C})$ *can't usually be computed* in polynomial time.

For example, consider the trivial case of a fully-connected network with all possible single-link routes ($|\mathcal{R}| = K$, $\boldsymbol{A} = \boldsymbol{I}$) and $C_j = C$. Clearly, $|S| = C^K$.

## Erlang Fixed Point Approximation (EFP)

**Theorem** *(Kelly (1986)) There is a unique vector* $(B_1, \ldots, B_K) \in [0,1]^K$ *satisfying*

$$B_j = E(\rho_j, C_j), \;\; j = 1, \ldots, K,$$

$$\rho_j = (1 - B_j)^{-1} \sum_r a_{jr} \nu_r (1 - L_r), \;\; j = 1, \ldots, K,$$

*and*

$$L_r = 1 - \prod_i (1 - B_i)^{a_{ir}}, \;\; r \in \mathcal{R},$$

*where*

$$E(\nu, C) = \frac{\nu^C}{C!} \left( \sum_{n=0}^{C} \frac{\nu^n}{n!} \right)^{-1} .$$

$E(\nu, C)$ is *Erlang's Formula* for the loss probability on a *single link* with $C$ circuits and Poisson traffic offered at rate $\nu$.

The celebrated *Erlang Fixed Point Approximation* is obtained by using $B_j$ to estimate the probability that link $j$ is full, and using $L_r$ to estimate the route-$r$ blocking probability.

## The rationale

(Independent blocking)

If links along route $r$ were blocked *independently* (they are clearly not) and if $B_j$ *were* the link-$j$ blocking probability, then $L_r$ would be the route-$r$ blocking probability:

$$L_r = 1 - \prod_{i \in r}(1 - B_i) = 1 - \prod_{i}(1 - B_i)^{a_{ir}}.$$

Carrying this rationale further, the traffic offered to link $j$ would be Poisson (at rate $\rho_j$, say) and the *carried traffic* (that which is accepted) on link $j$ would be

$$\sum_r a_{jr}\nu_r(1 - L_r)\ \Big(= (1 - B_j)\rho_j\Big)\ .$$

The Erlang Fixed Point Approximation stipulates that the blocking probabilities $(B_1, \ldots, B_K)$ should be consistent with this level of carried traffic:

$$B_j = E(\rho_j, C_j), \quad j = 1, \ldots, K.$$

## The Erlang Fixed Point

The (unique) *fixed point* of the system

$$B_j = E\left(\left(\sum_r a_{jr}\nu_r \prod_{i\in r-\{j\}} (1 - B_i)\right), C_j\right)$$

is called the *Erlang Fixed Point*.

**Remarks** The existence of a fixed point is easy to prove using the *Brouwer fixed point theorem*; these equations determine a continuous mapping from a compact convex set $[0,1]^K$ into itself. The uniqueness (Kelly) is difficult to prove.

For more complex systems, there may be more than one fixed point. This may be associated with multiple stable states for the *network*. For example, in networks with *Random Alternative Routing* the system can fluctuate between a *low blocking* state, where calls are accepted readily, and a *high blocking* state, where the likelihood of a call being accepted can be quite low.

## When does the EFP perform well?

There are two limiting regimes under which the EFP is asymptotically valid:

**Moderate loading** (Kelly (1986)) Consider a sequence of networks indexed by $N$ (arbitrary), and index the capacities and arrival rates accordingly:

$$\boldsymbol{C}^{(N)} = (C_j^{(N)},\ j = 1, \ldots, K),$$
$$\boldsymbol{\nu}^{(N)} = (\nu_r^{(N)},\ r \in \mathcal{R}).$$

**Theorem** *Suppose that as $N \to \infty$*

$$\frac{1}{N}\boldsymbol{C}^{(N)} \to \boldsymbol{C} \quad \text{and} \quad \frac{1}{N}\boldsymbol{\nu}^{(N)} \to \boldsymbol{\nu}\,.$$

*(Network topology fixed.) If $L_r^{(N)}$ is the route-$r$ loss probability, then, for each $r \in \mathcal{R}$, $L_r^{(N)} \to L_r$, where*

$$L_r = 1 - \prod_i (1 - B_i)^{a_{ir}},$$

*and $(B_1, \ldots, B_K)$ is the Erlang Fixed Point determined by $\boldsymbol{\nu}$ and $\boldsymbol{C}$.*

**Diverse routing** (General formulation - Hunt (1990)) Consider a sequence of networks indexed by $K$ (the number of links), and index the routing matrix and arrival rates accordingly:

$$\boldsymbol{A}^{(K)} = (a_{jr}^{(K)},\ r \in \mathcal{R},\ j = 1, \ldots, K),$$
$$\boldsymbol{\nu}^{(K)} = (\nu_r^{(K)},\ r \in \mathcal{R}).$$

Suppose that

$$\sum_r a_{jr}^{(K)} a_{kr}^{(K)} \nu_r^{(K)} \to 0, \quad j, k = 1, \ldots, K,$$

and

$$\sum_r a_{jr}^{(K)} \nu_r^{(K)} \to \nu_j > 0, \quad j = 1, \ldots, K.$$

(*Capacities fixed*.) The traffic along link $j$ is moderate ($\sim \nu_j$), but that which is common to *any two* links becomes small.

There are no general results. Examples include *star networks* (Ziedins and Kelly (1989)) and networks with *alternative routing* (Gibbens, Hunt and Kelly (1990)); there are many more.

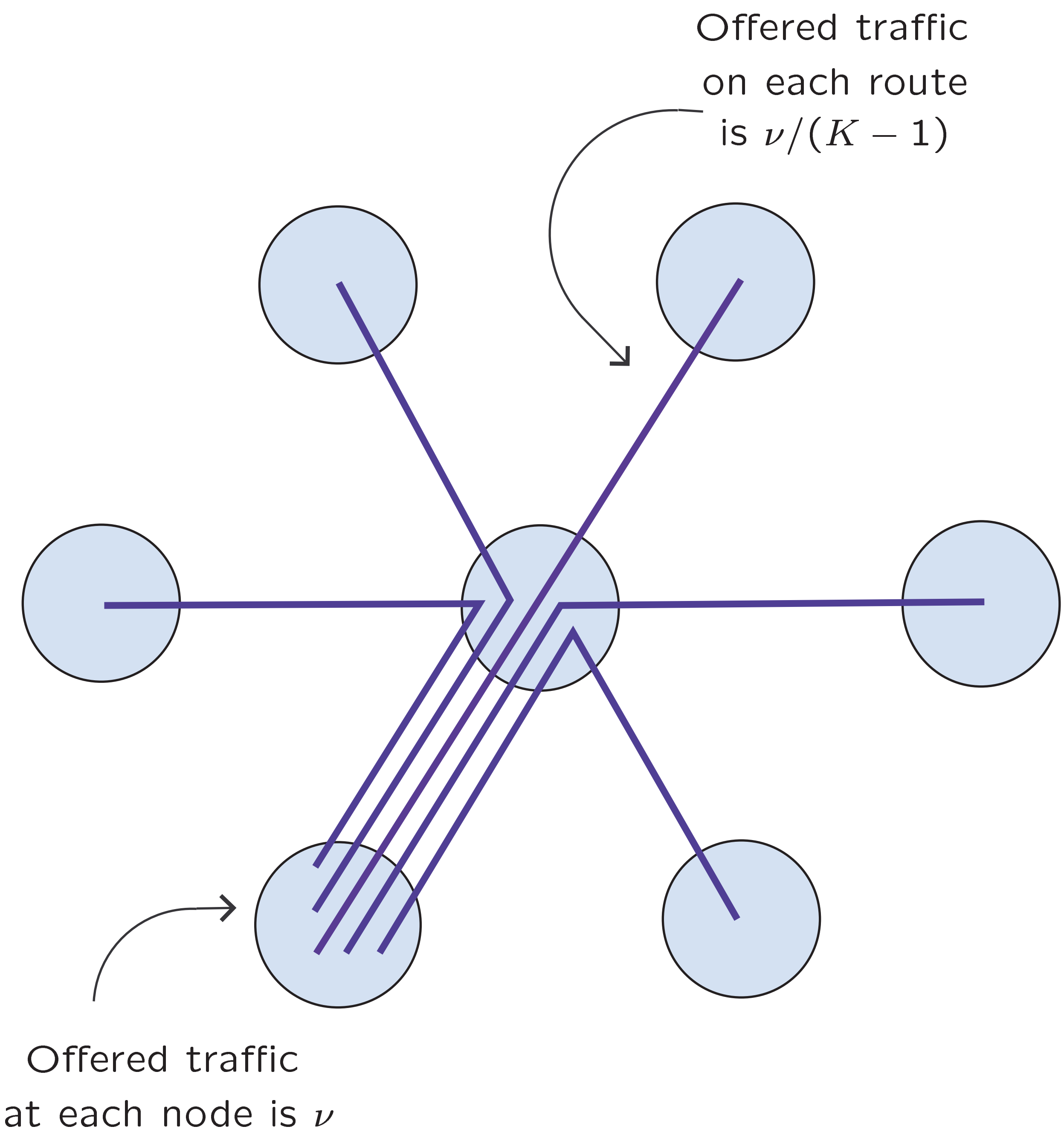


Fig 2. A typical symmetric star network

$\left(K = 6\right.$ outer nodes and $\binom{K}{2} = 15$ routes$\left.\right)$

## The symmetric star network

This consists of a collection of $K$ outer nodes, which communicate via a single central node. Take $\mathcal{R}$ to be all those routes consisting of a *pair* of links (there are $\binom{K}{2}$ of them). Fix $\nu > 0$ and take $\nu_r = \nu/(K-1)$ for all $r \in R$, and fix $C_j = C$ for all links $j = 1, \dots, K$.

**Theorem** *If $L^{(K)}$ is the common route loss probability, then $L^{(K)} \to L$, given by*

$$L = 1 - (1 - B)^2,$$

*and $B$, the Erlang Fixed Point, is the unique solution to*

$$B = E(\nu(1 - B), C),$$

where $E(\cdot, \cdot)$ is Erlang's Formula.

## When can the EFP be expected to perform badly?

The EFP can perform badly if the network exhibits any of the following features:

- highly linear structure (lines and rings)
- low capacities
- priority controls (eg, trunk reservation)

In these cases, the independent blocking assumption may not be valid.

Can we model the link dependencies, and thus obtain improved fixed point methods?

## A symmetric ring network with two types of traffic

We shall examine a network consisting of $K$ nodes with $K$ links which form a loop (see Figure 3). $\mathcal{R}$ consist of all 1-link routes (type-1 traffic), as well as all 2-link routes (type-2 traffic) comprising pairs of adjacent links ($2K$ routes in all). Type-$t$ traffic is offered at rate $\nu_t$ on all type-$t$ routes, for $t \in \{1, 2\}$, and $C_j = C$ for each link $j$.

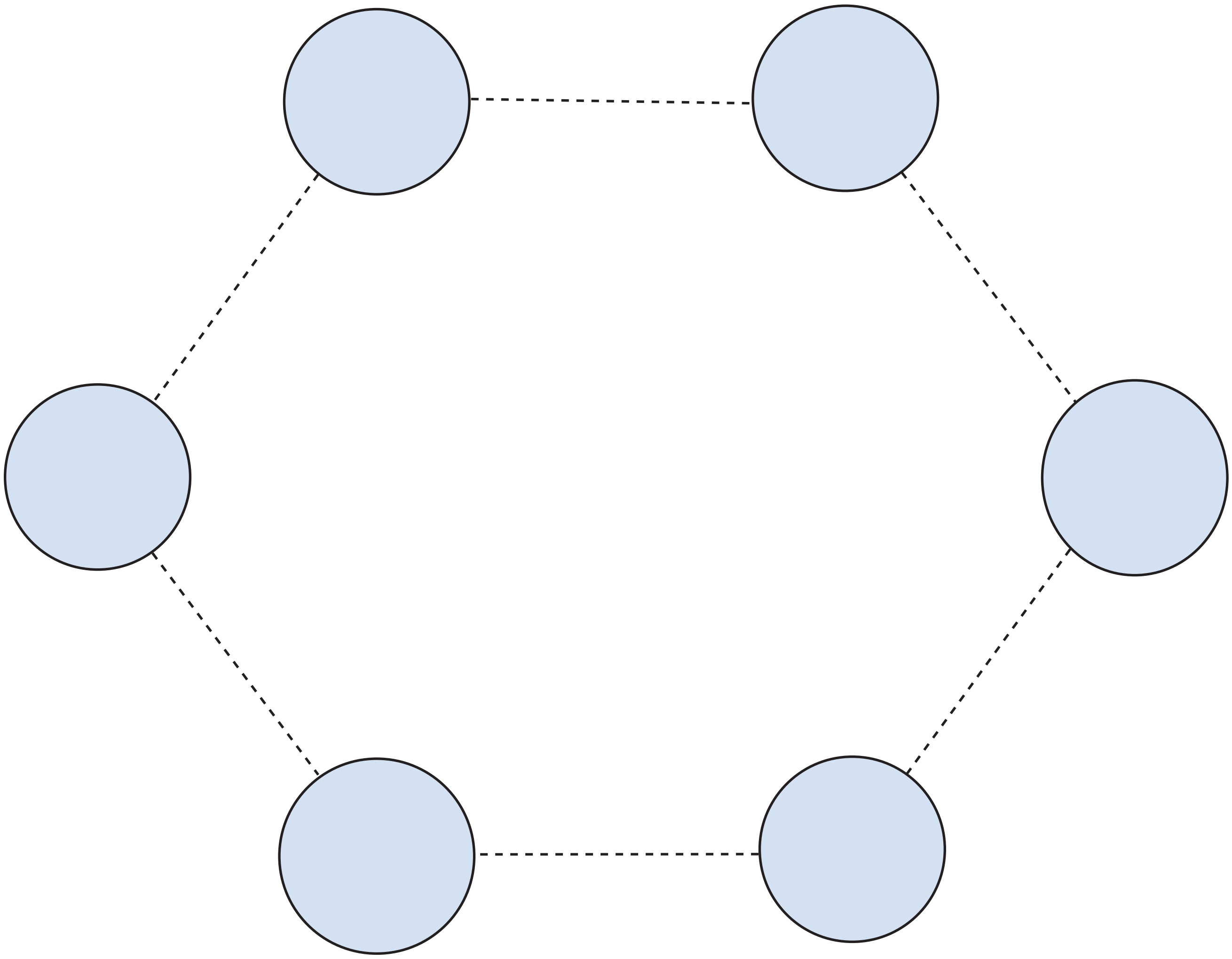

Fig 3. A Ring Network (6 nodes)

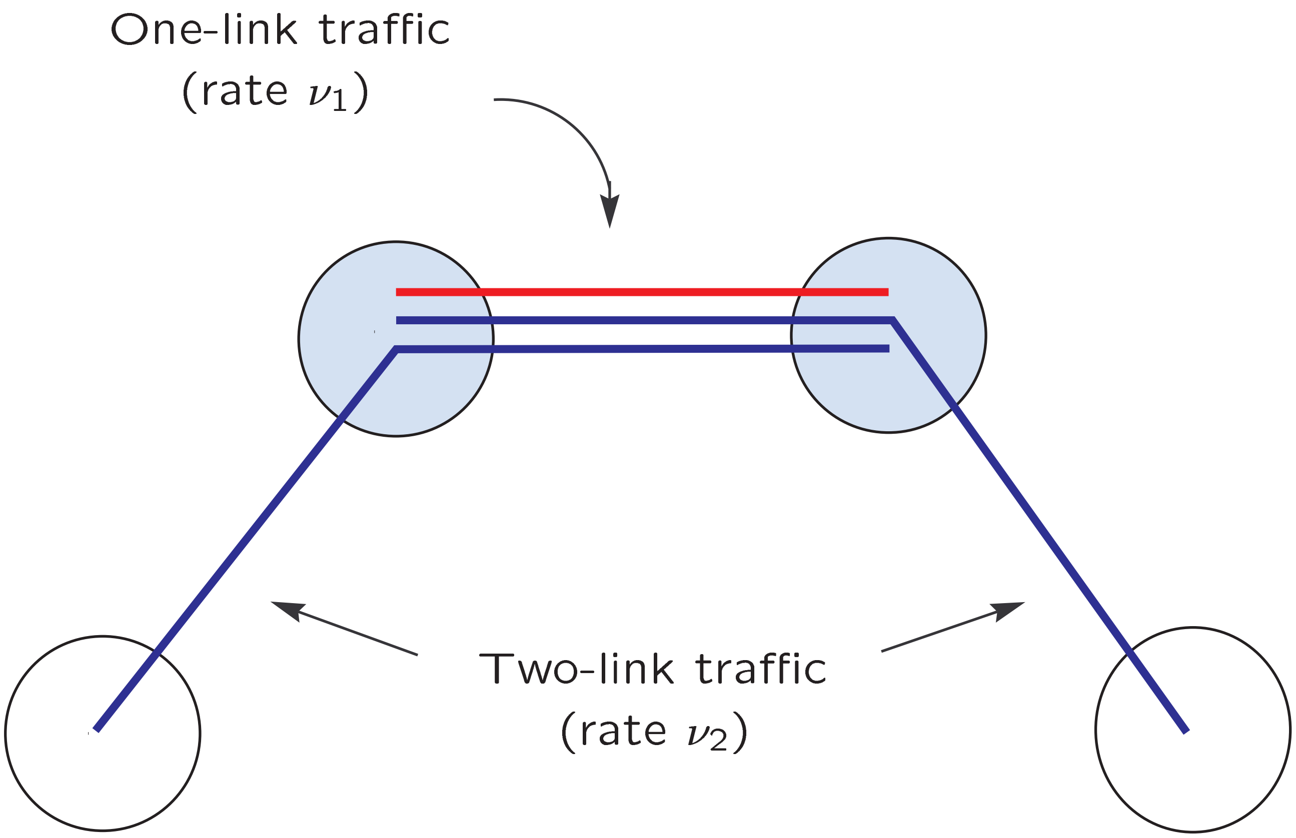


Fig 4. One- and two-link traffic
incorporating a given link

Two-link subnetwork

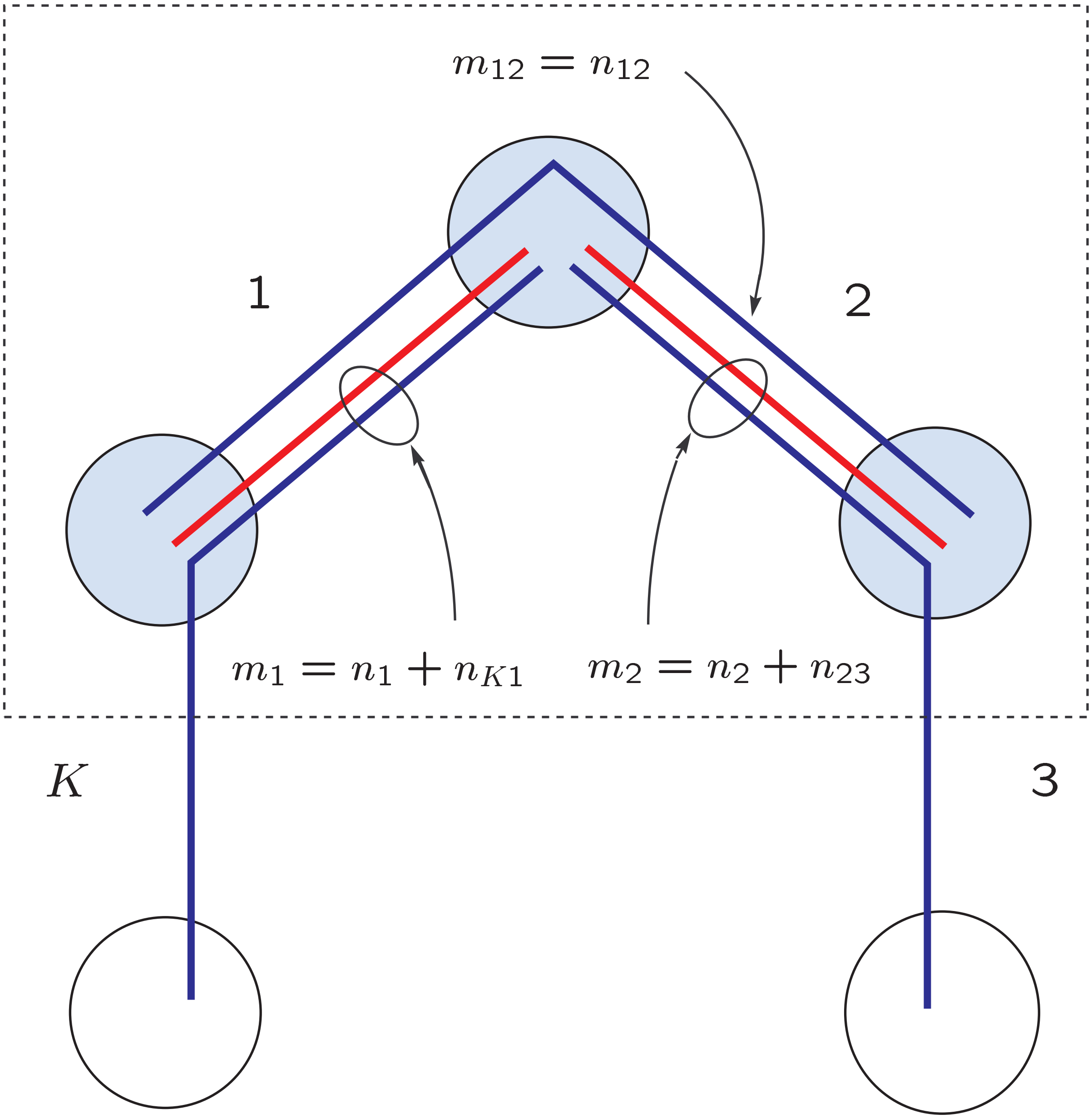


Fig 5. Definition of $m_1$, $m_2$ and $m_{12}$ for the symmetric ring network

## A two-link subnetwork

Take links 1 and 2 as "reference" links. The subnetwork consisting of just these two links has three routes: $\mathcal{R}_{12} = \{\{1\},\{2\},\{1,2\}\}$. Let $m_r$ denote the number of calls on route $r$, for $r \in \mathcal{R}_{12}$. Then, $m_1 = n_1 + n_{K1}$ is the number of calls occupying capacity on link 1 *but not* on link 2; $m_2 = n_2 + n_{23}$ is the number occupying capacity on link 2 *but not* on link 1; and $m_{12} = n_{12}$ is the number of calls occupying capacity on both links (see Figure 5).

First examine the correlation between links 1 and 2, in particular,

$$\mathrm{Corr}\left(I_{\{m_1+m_{12}<C\}}, I_{\{m_2+m_{12}<C\}}\right) .$$

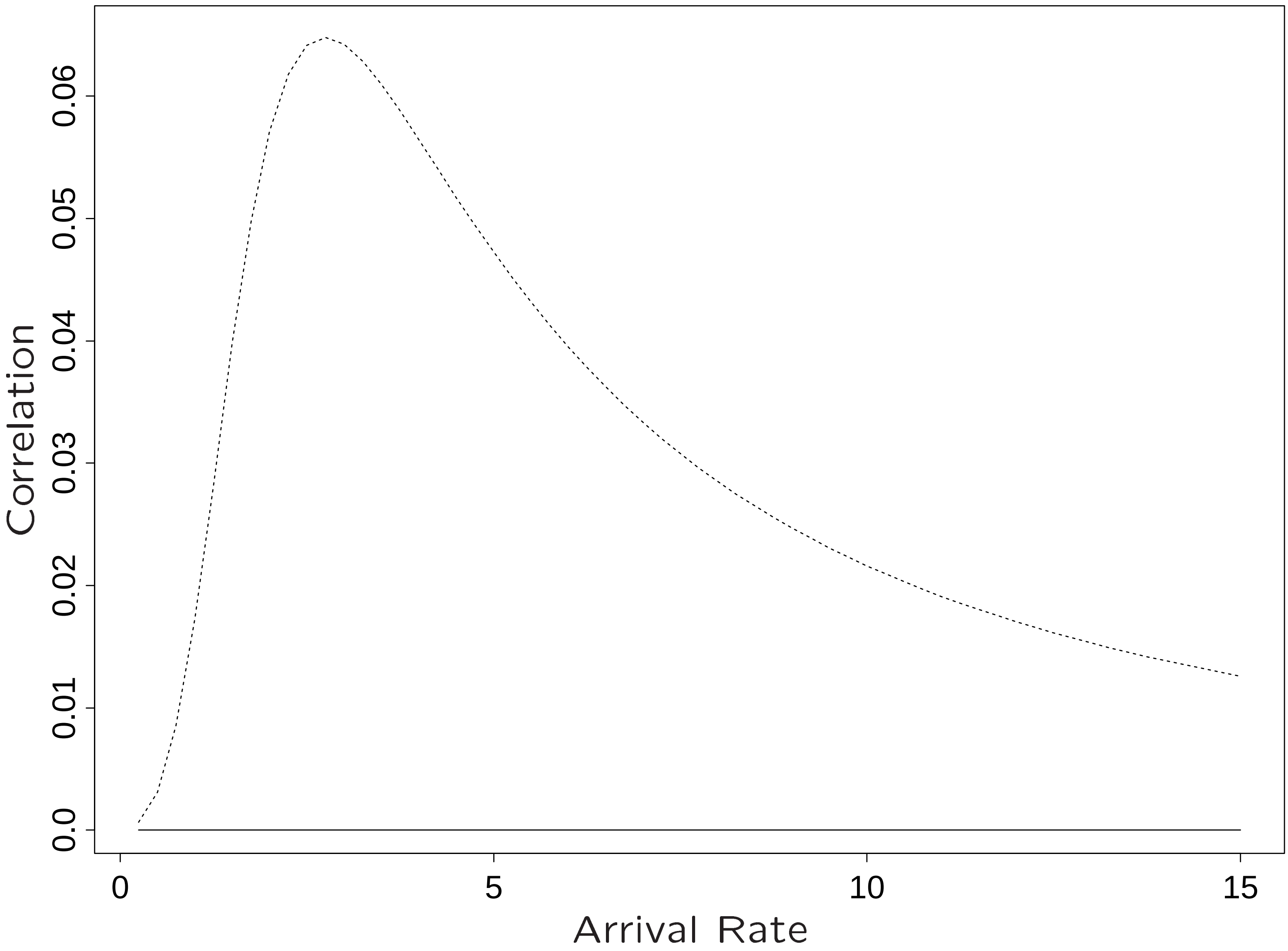


Fig 6. Correlation between adjacent links
($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

## The EFP approximation

If $L_t$ is the common loss probability for type-$t$ calls, for $t \in \{1, 2\}$, then the EFP approximation is

$$L_1 \simeq B, \qquad L_2 \simeq 1 - (1 - B)^2,$$

where the Erlang Fixed Point $B$ is the unique solution to

$$B = E(\nu_1 + 2\nu_2(1 - B), C),$$

where $E(\cdot, \cdot)$ is Erlang's Formula.

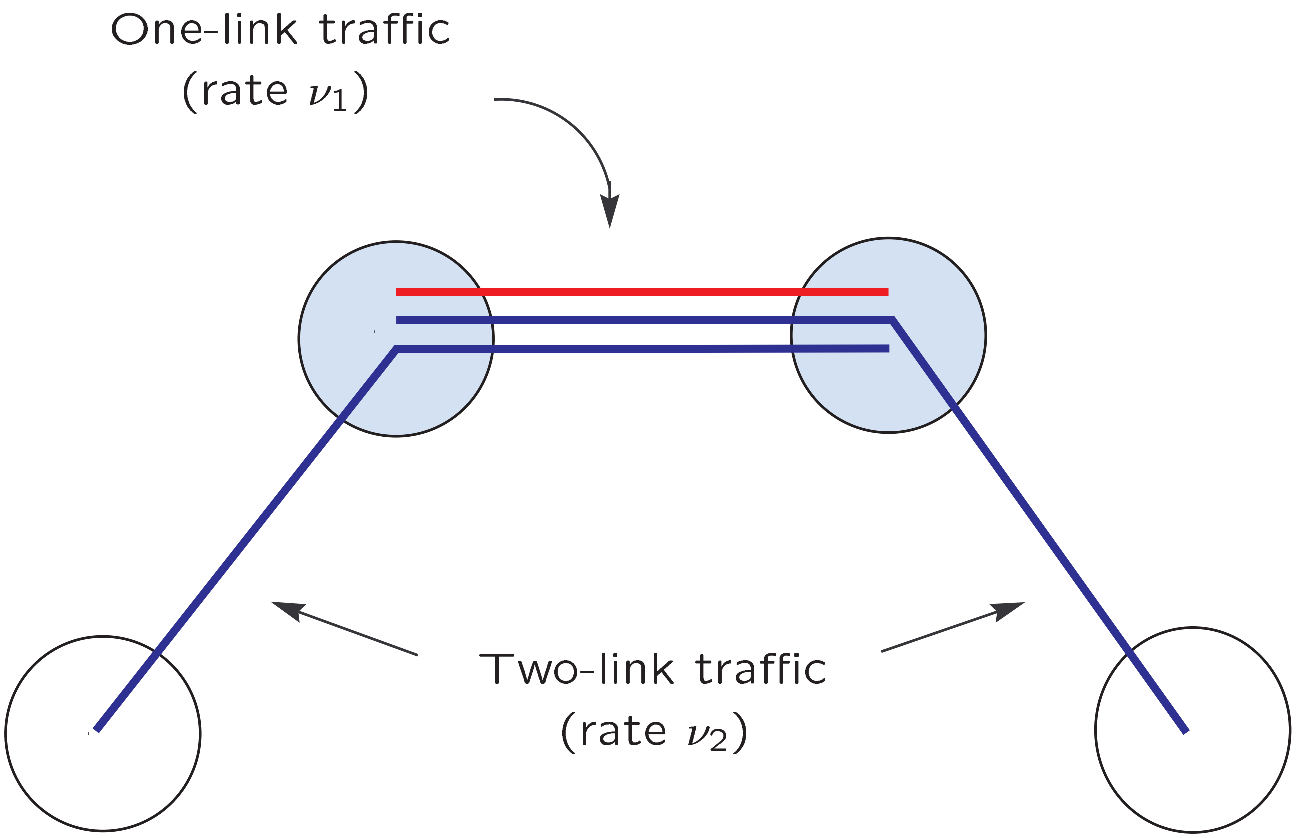


Fig 4a. One- and two-link traffic incorporating a given link

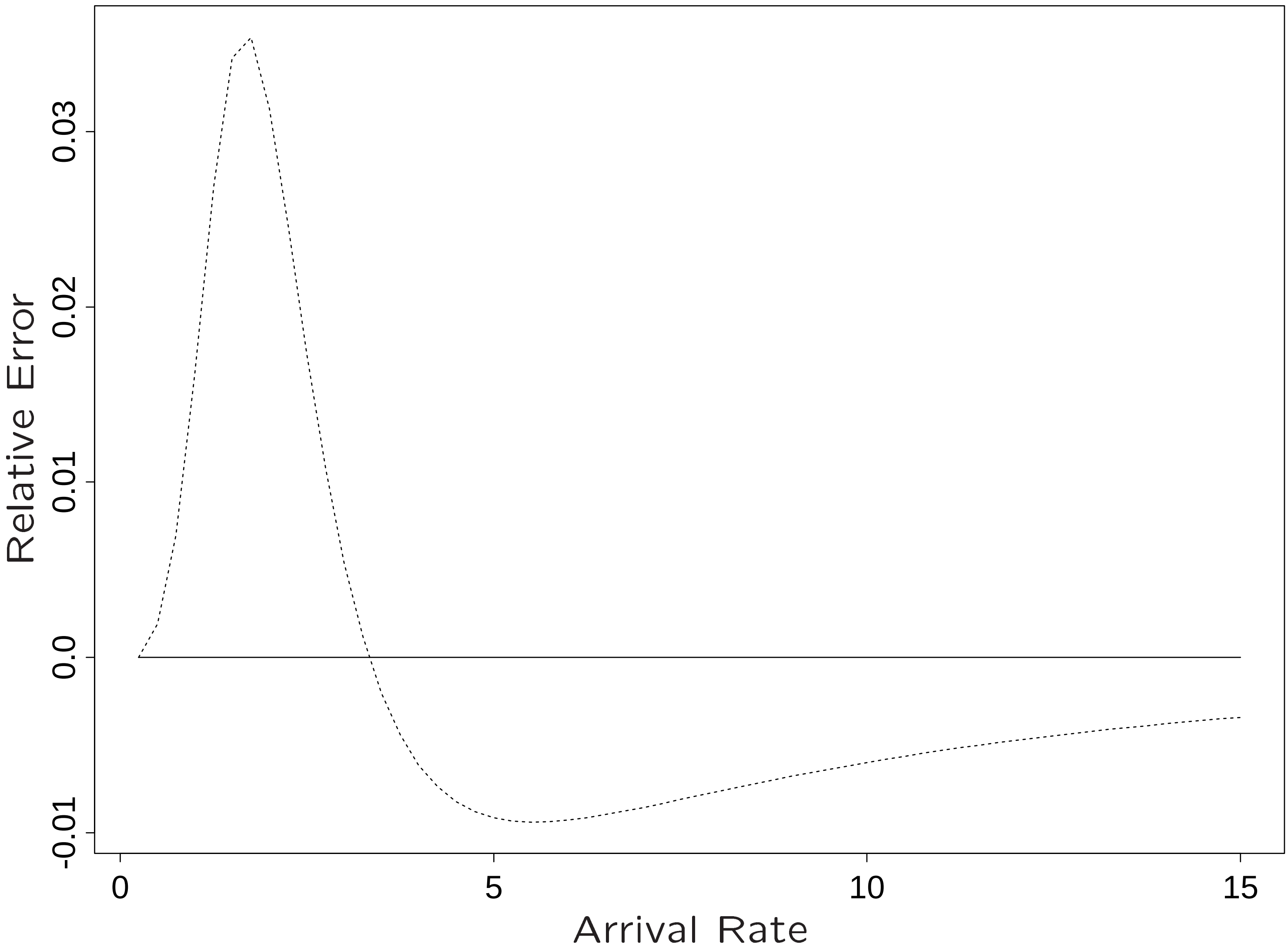


Fig 7. Relative error in EFP for the blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate).

# Approximation I

This follows the work of Pallant (1992). The network is decomposed into independent subnetworks and the stationary distribution for each is evaluated.

The state space for the subnetwork is

$$S_{12} = \{(m_1, m_2, m_{12}) : m_i + m_{12} \le C, i = 1, 2\}$$

and the stationary distribution is

$$\pi(\boldsymbol{m}) = G^{-1} \frac{(\nu_1 + \nu_2(1-B))^{m_1+m_2} \nu_2^{m_{12}}}{m_1! m_2! m_{12}!}.$$

We estimate $B$, the probability that a link adjacent to the two-link subnetwork is fully occupied, by

$$\begin{aligned} B &= \sum_{\boldsymbol{m} : m_1 + m_{12} = C} \pi(m_1, m_2, m_{12}) \\ &= \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}), \end{aligned}$$

and use these expressions iteratively to determine the "correct" value of $\pi$, and hence $B$.

Two-link subnetwork

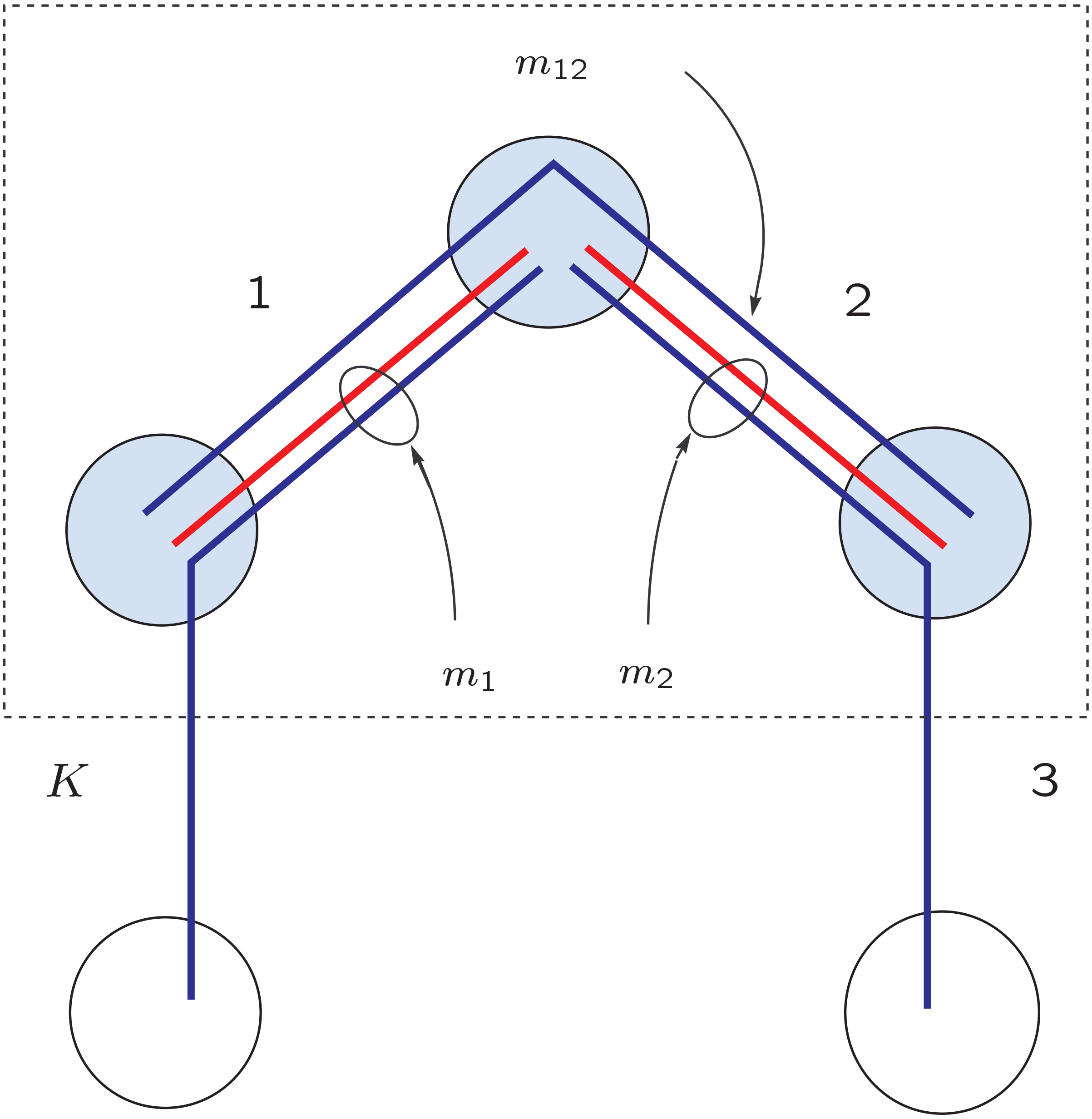


Fig 5a. Recall the definition of $m_1$, $m_2$ and $m_{12}$

# Approximation II

Our second approximation uses additional knowledge of the state of a given link in estimating the probability that the adjacent link is full. We use state-dependent arrival rates, $\rho_j = \nu_1 + \nu_2(1 - b_j)$, $j \in \{1, 2\}$, where $b_j$ is the probability that link $K$ is fully occupied, conditional on $m_1 = j$ ($b_j$ is also the probability that link 3 is fully occupied, conditional on $m_2 = j$), so that

$$\pi(\boldsymbol{m}) = G^{-1} \frac{\nu_2^{m_{12}} \left(\prod_{s=0}^{m_1-1} \rho_s\right) \left(\prod_{s=0}^{m_2-1} \rho_s\right)}{m_1! m_2! m_{12}!}.$$

Once $b_j$ is estimated and $\pi$ determined, we set

$$L_1 (= B) = \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}).$$

Similarly, $L_2$ can be expressed in terms of $\pi$.

An estimate of $b_j$ can be found by assuming (incorrectly) that $b_j$ does not depend on $m_{12}$. For $j = 0, \ldots, C-1$, set

$$b_j = \frac{\sum_{m=0}^{j} p(j-m, m, C-m)}{\sum_{m=0}^{j} \sum_{r=0}^{C-m} p(j-m, m, r)},$$

where

$$p(n_1, n_{K1}, m_K) = \frac{\nu_1^{n_1} \nu_2^{n_{K1}} \left( \prod_{s=0}^{m_K-1} \rho_s \right)}{n_1! n_{K1}! m_K!}.$$

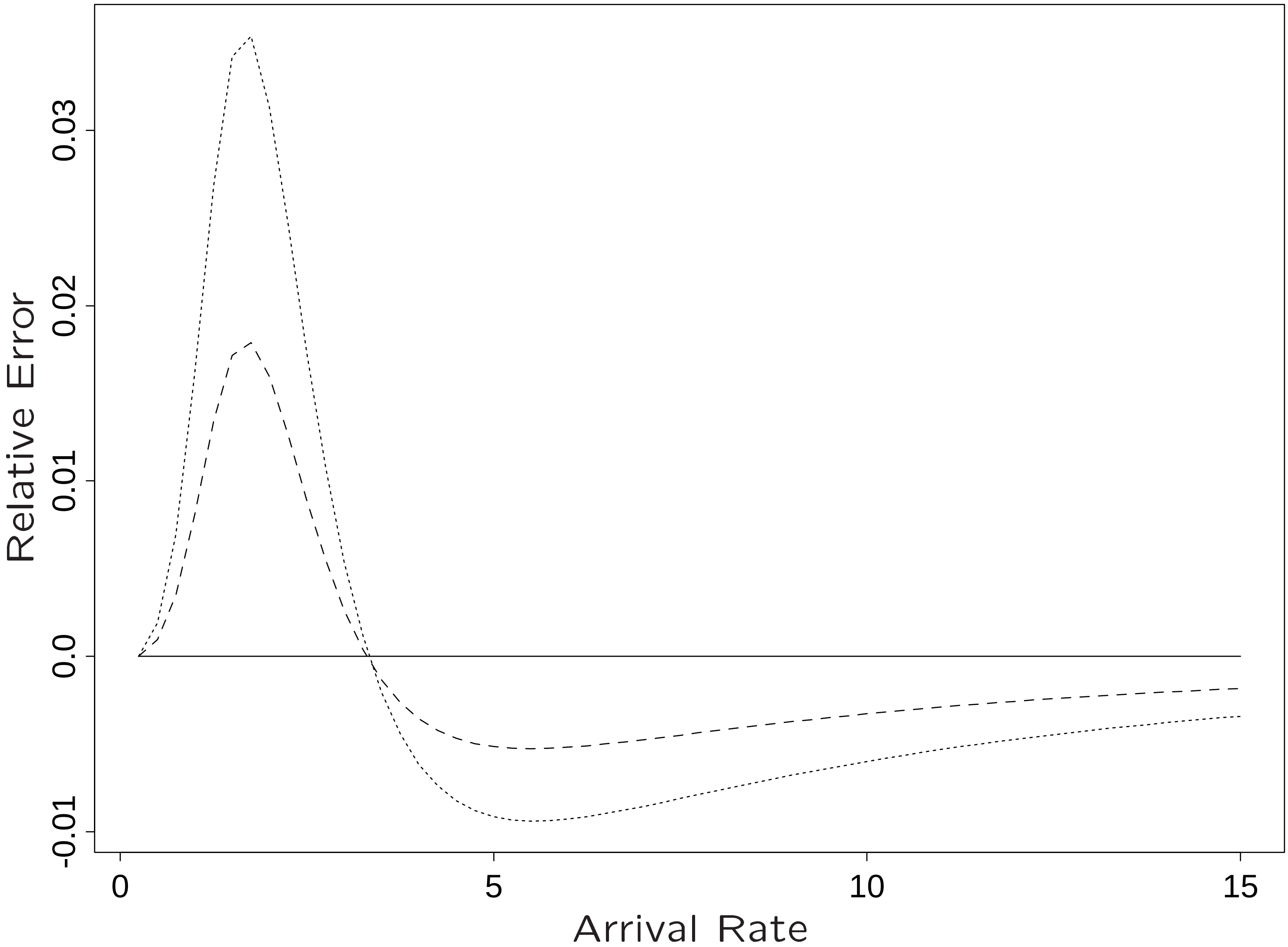


Fig 8. Relative error in the estimated blocking probability for type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate).

$\cdots\cdots\cdots\cdots\cdots$ EFP - - - - - Approx. I

__________ Approx. II

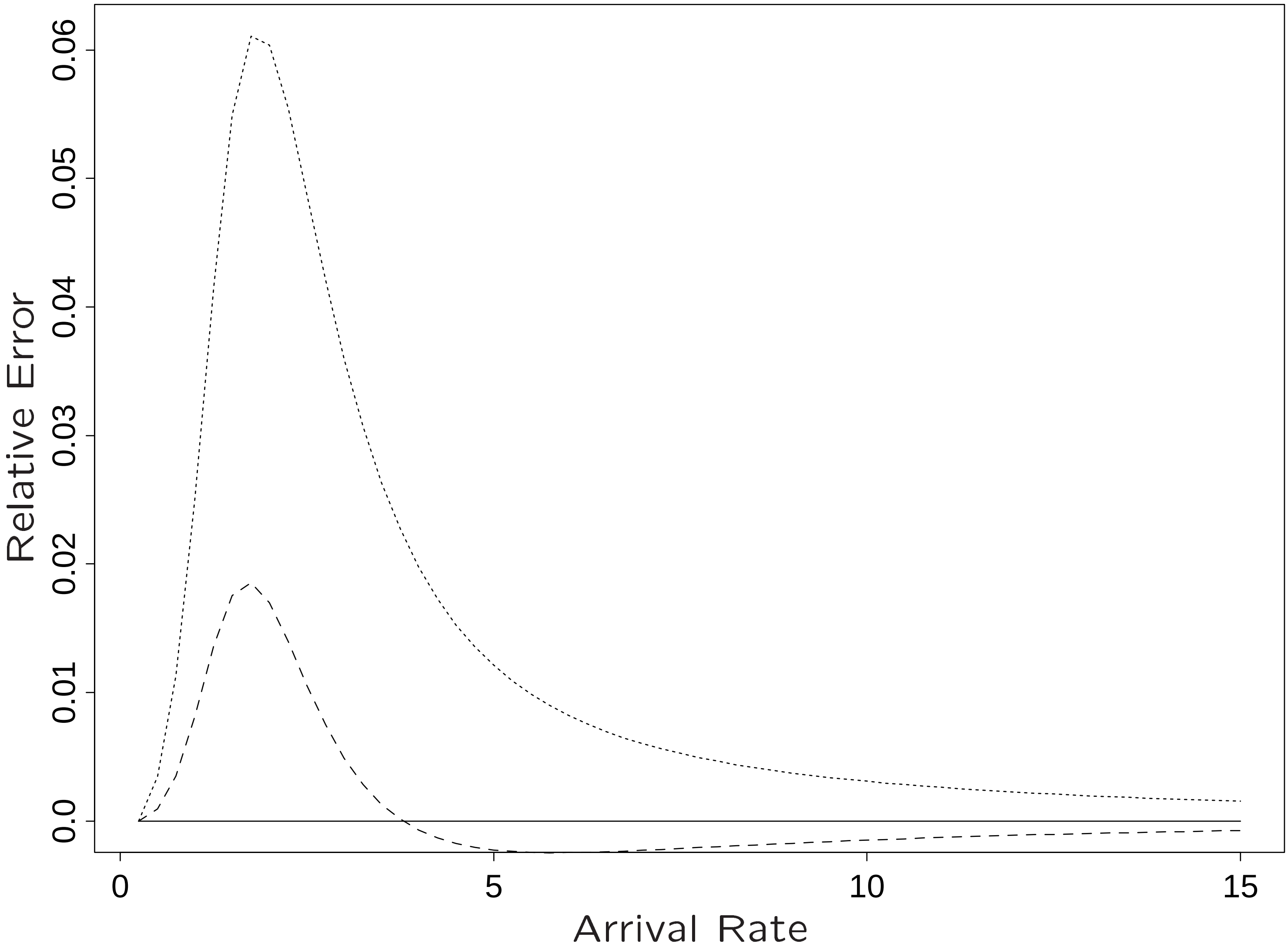


Fig 9. Relative error in the estimated blocking probability for type-2 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate).

·············· EFP - - - - - - Approx. I

________ Approx. II

## Some remarks on Approximation II

The dependence of $a_j$ on $m_{12}(= n_{12})$ is due to the cyclic nature of the network, but is expected to be slight for large networks. The approximation is *exact* for the infinite line network, as Zachary (1985) shows for a network which is equivalent to the one considered here, with $\nu_1 = 0$ (no single-link traffic). The expression for $b_j$ given above is equivalent to that obtained in his paper for the infinite line network, although written in a different form.

State-dependent arrival rates such as we have here are also discussed by various authors (for example, Pallant and Taylor (1994)). Kelly (1985) gives an exact expression in a line network (including some asymmetric cases).