Infinite-patch metapopulation models: branching, convergence and chaos

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Centre of Excellence for Mathematics
and Statistics of Complex Systems

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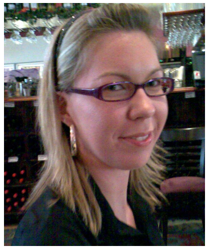
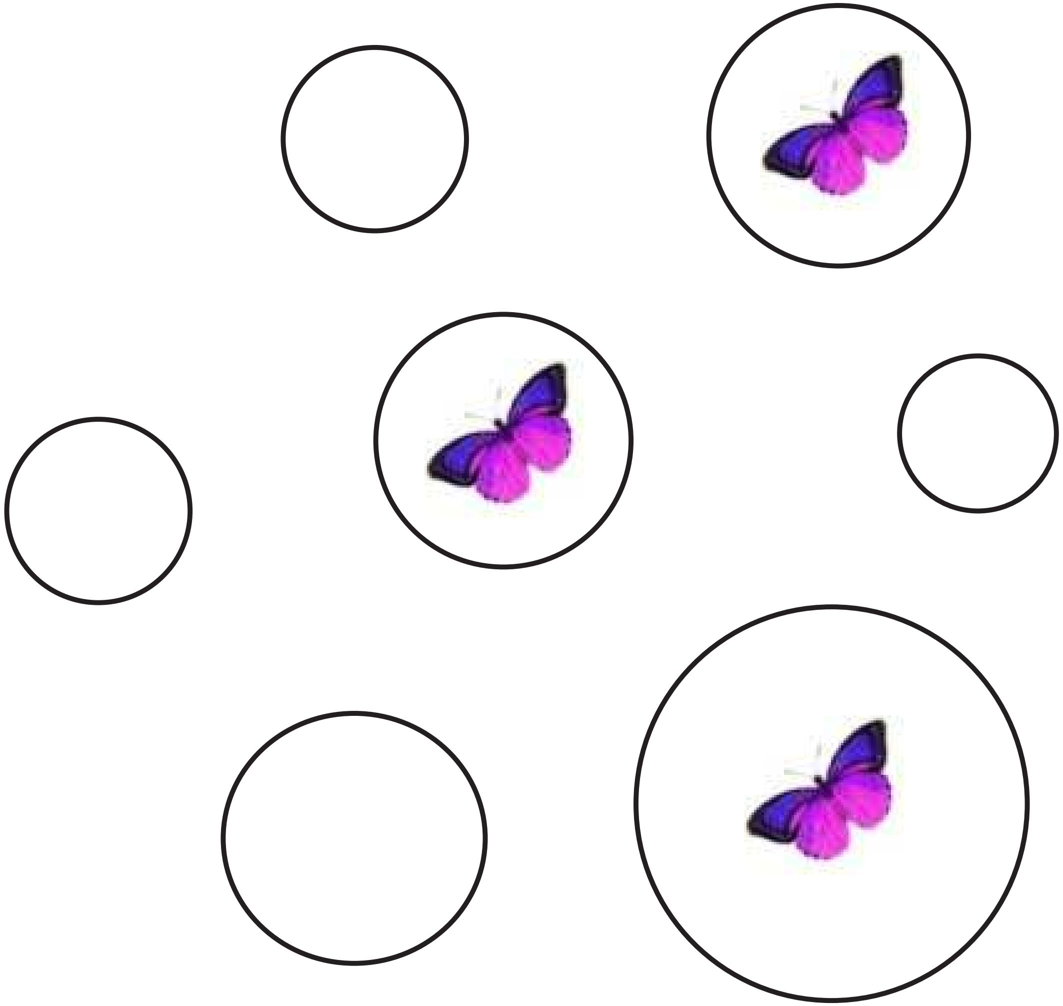
MASCOS PhD Scholar
University of Queensland



*Buckley, F.M. and Pollett, P.K. (2010) Limit theorems for discrete-time metapopulation models. Probability Surveys 7, 53-83.

Metapopulations



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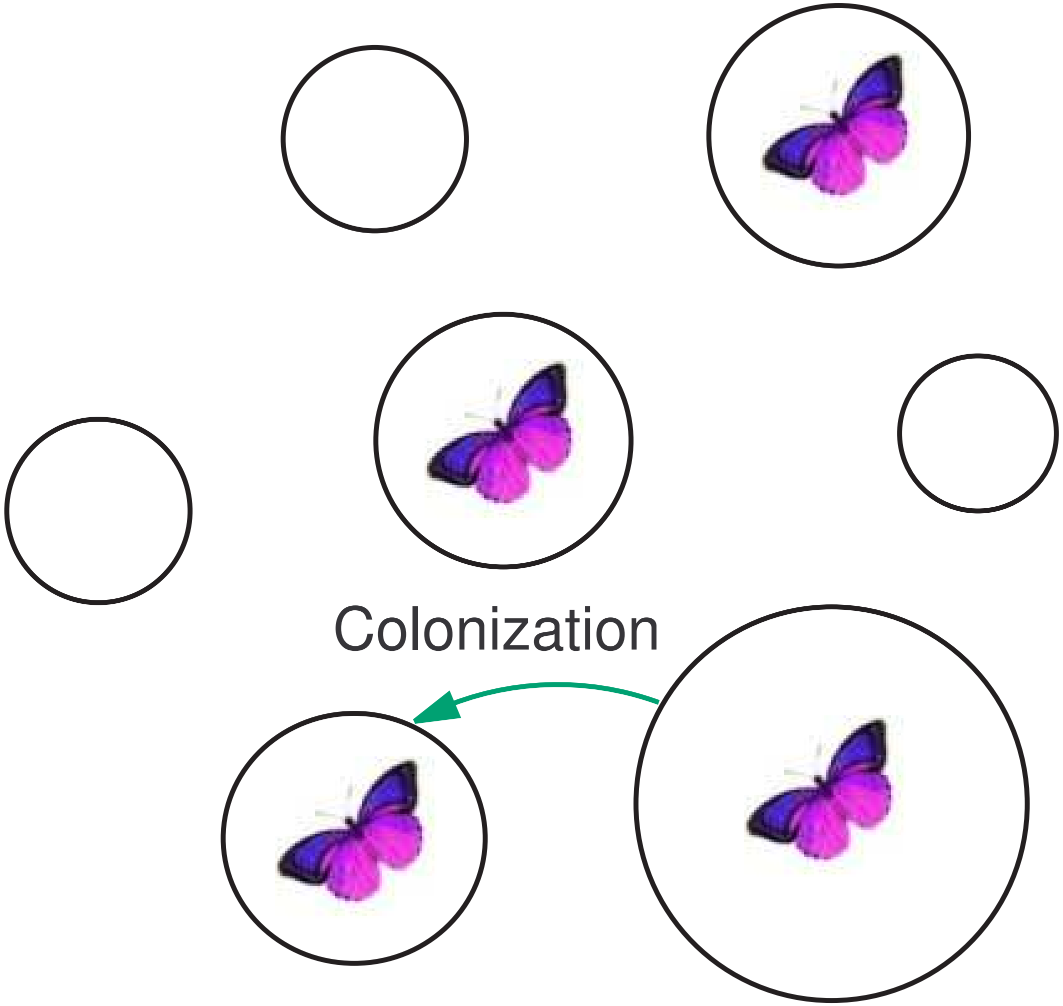

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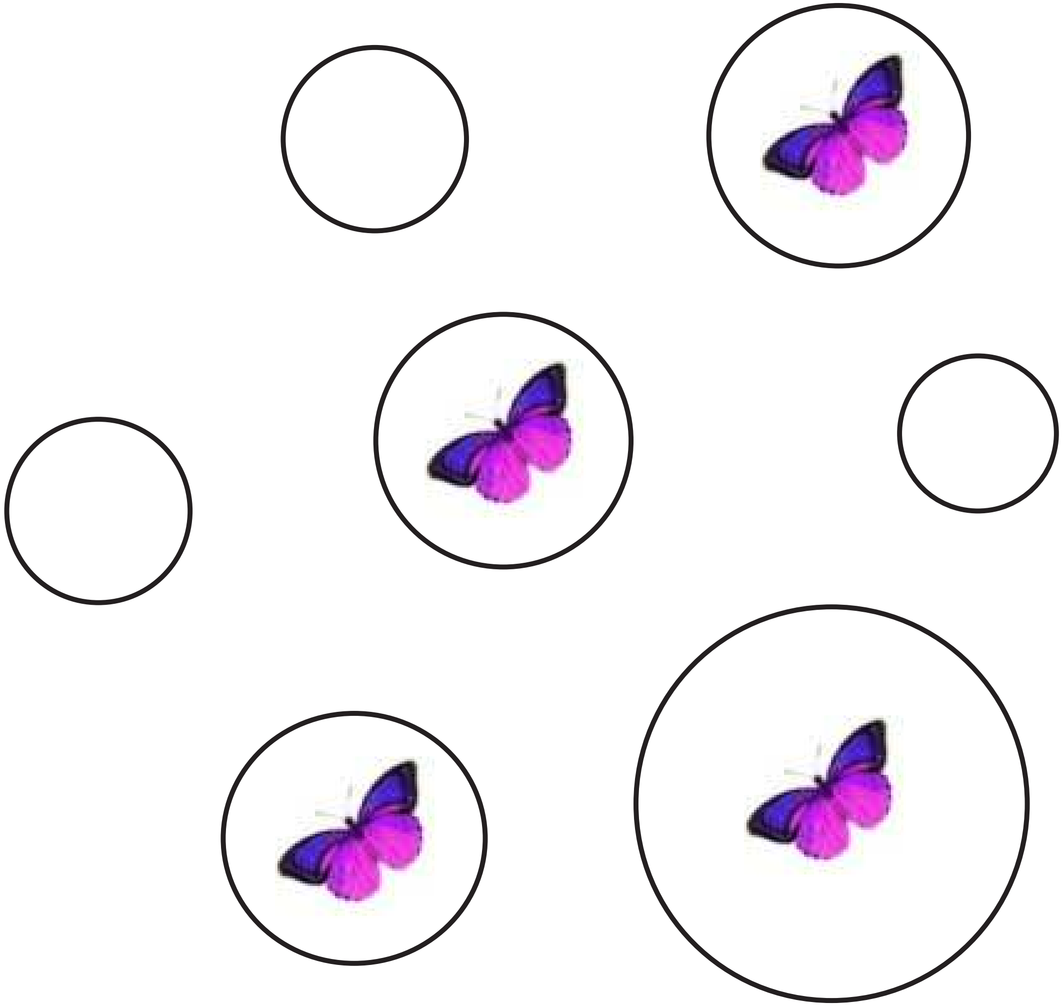

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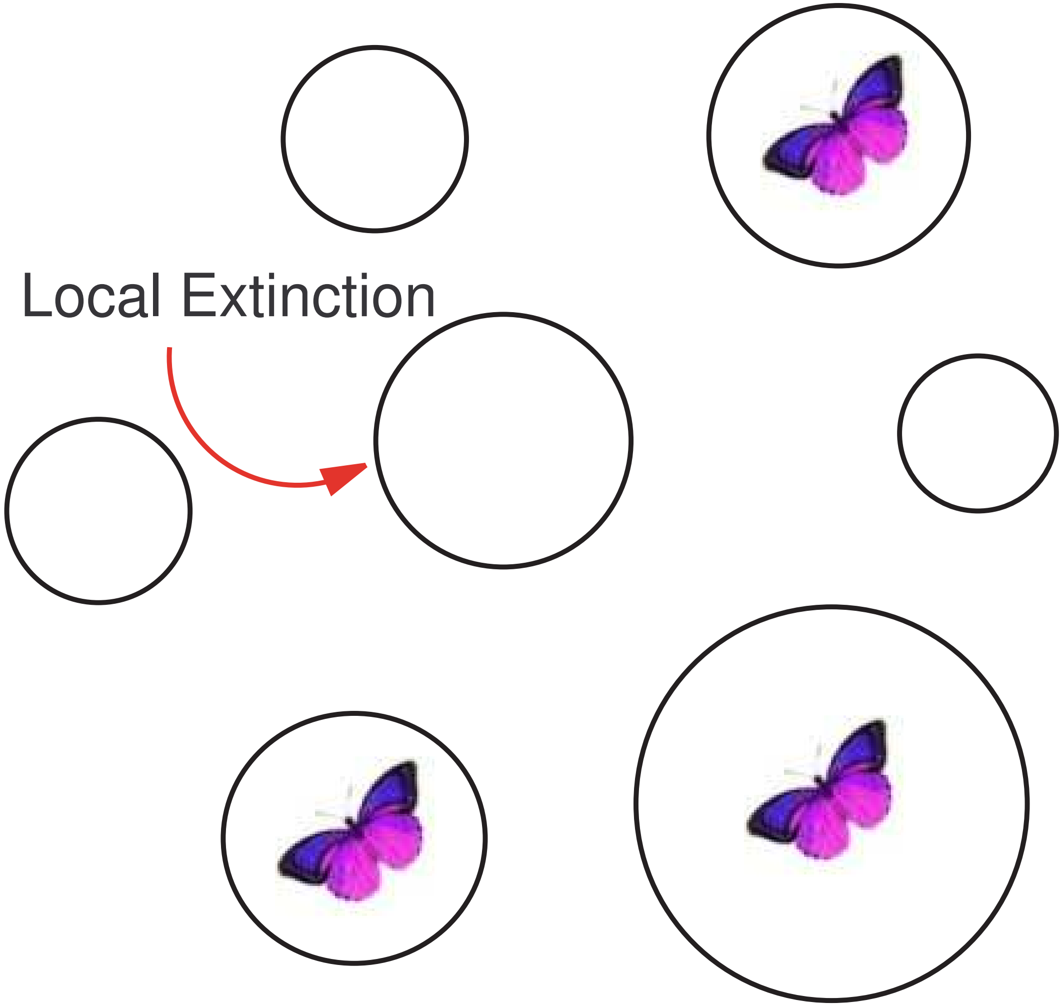

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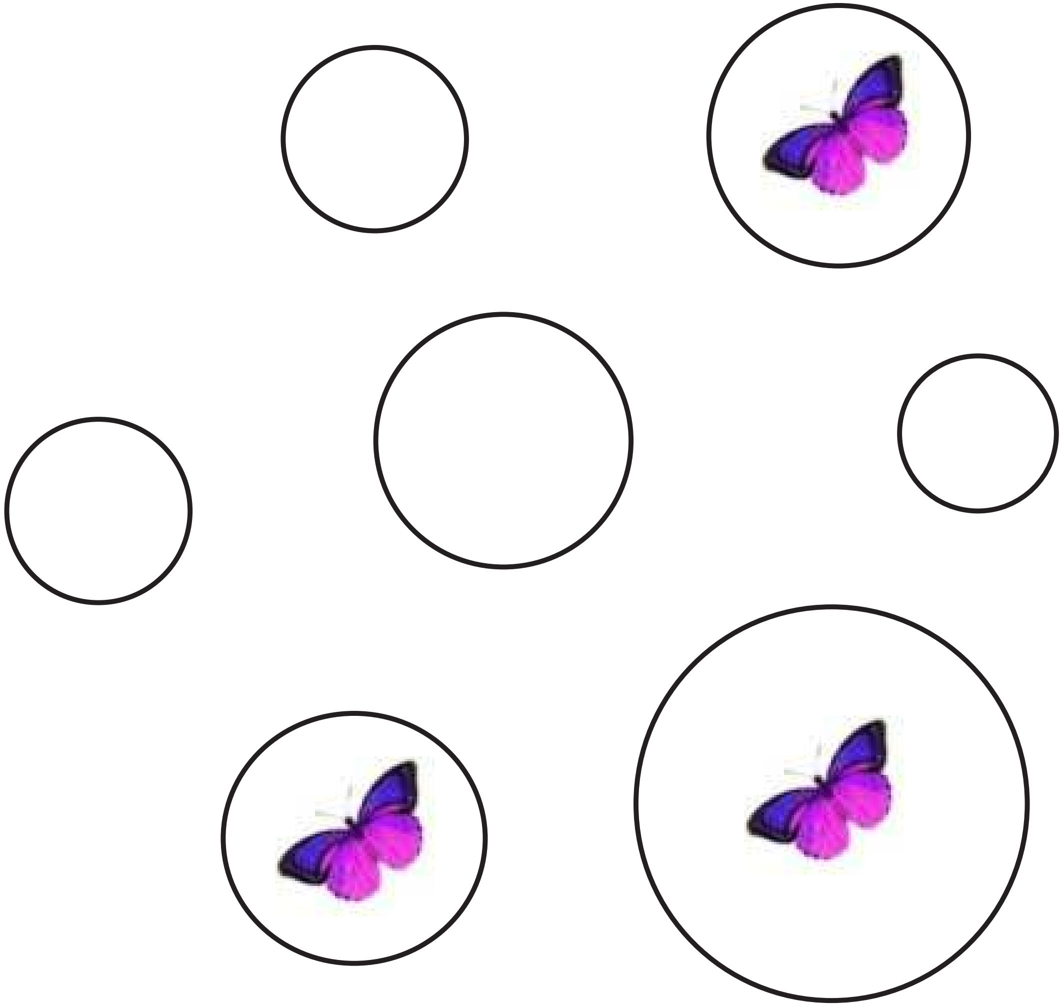

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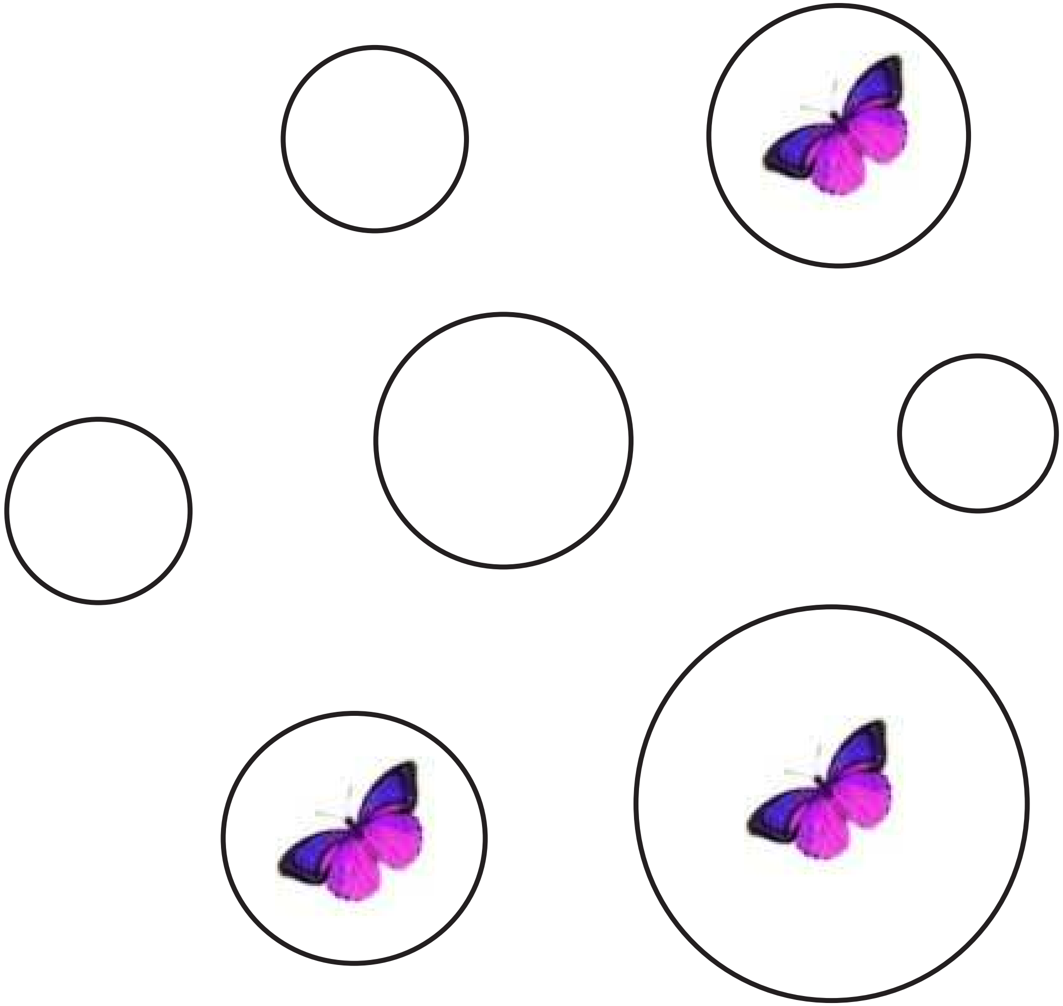

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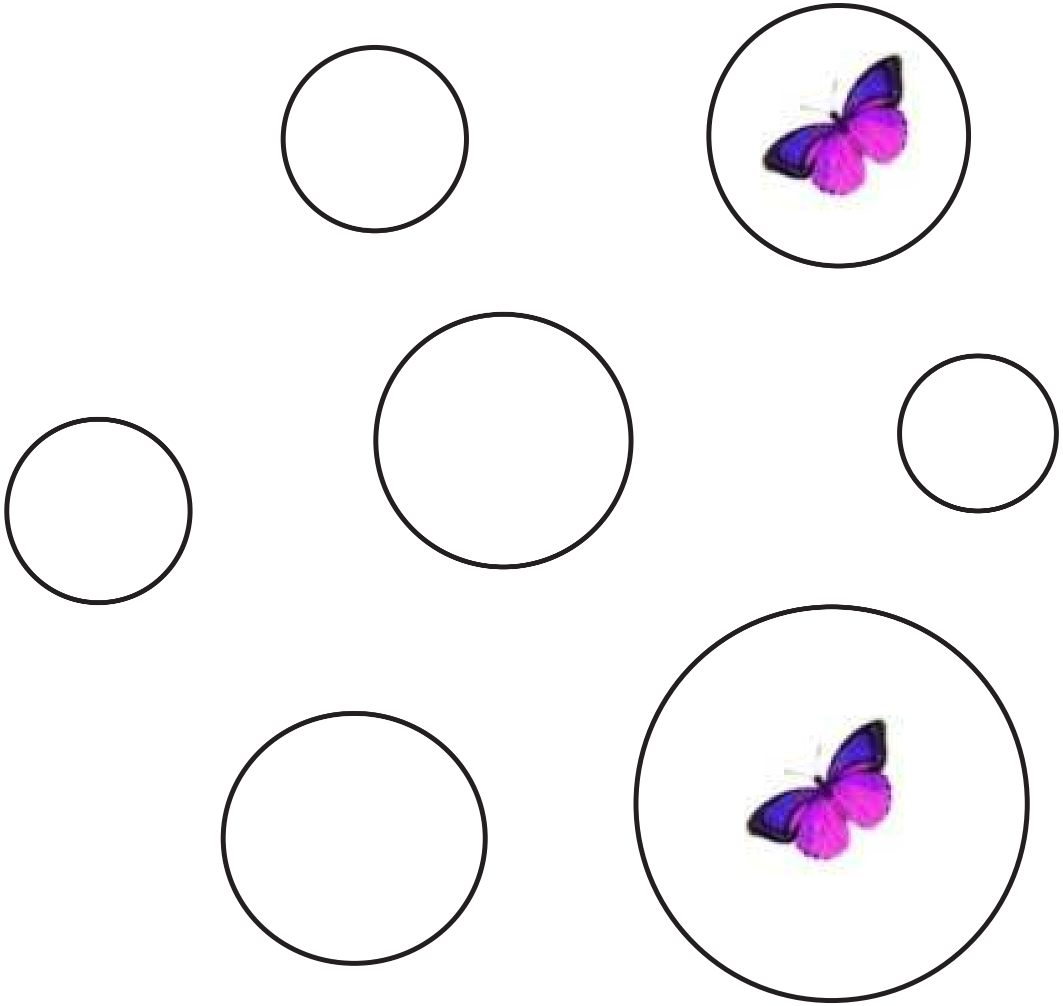

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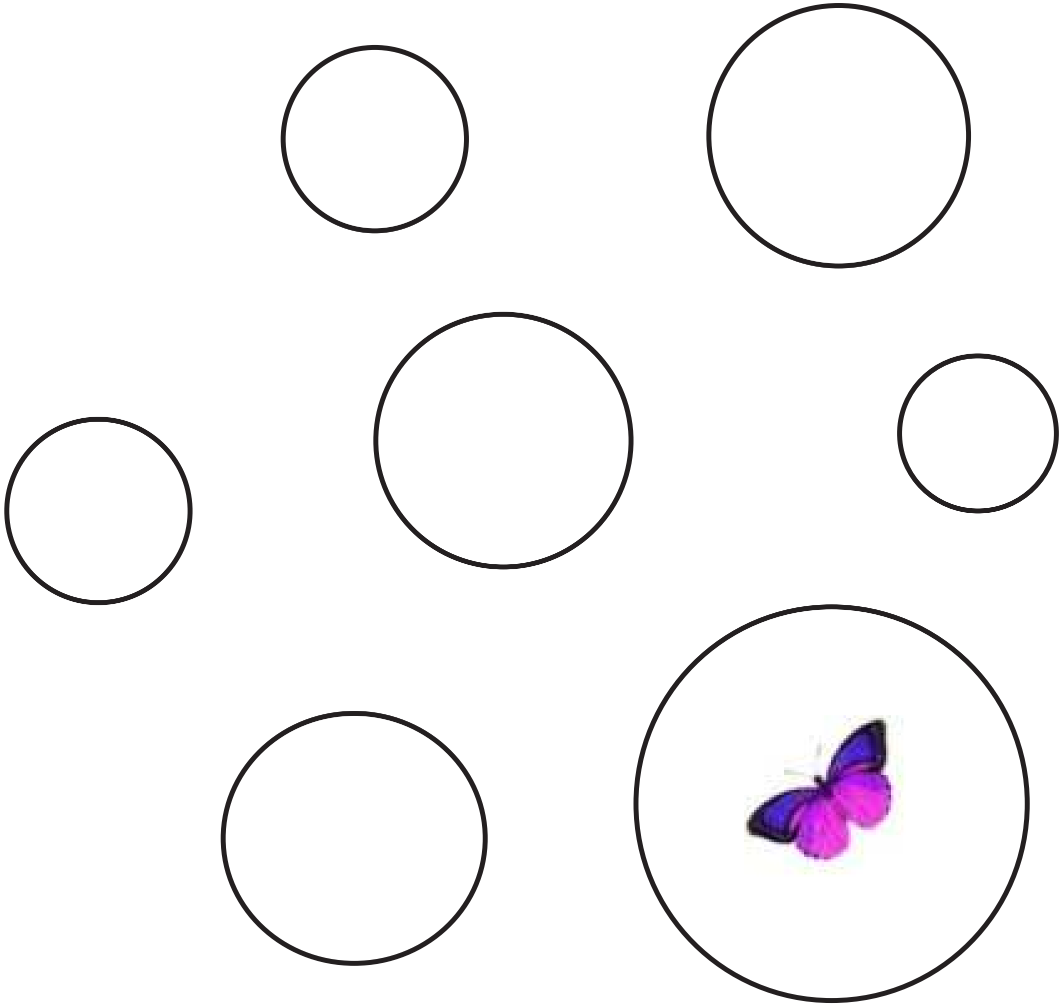

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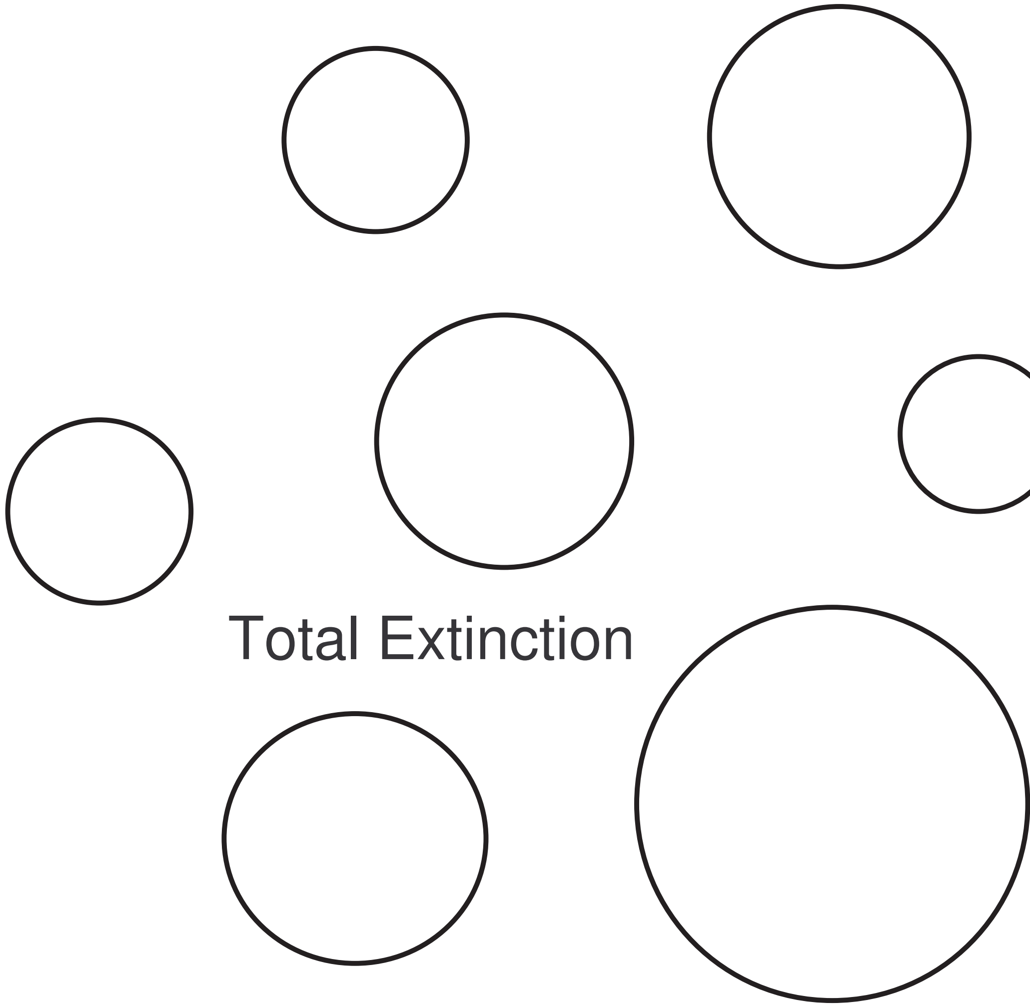

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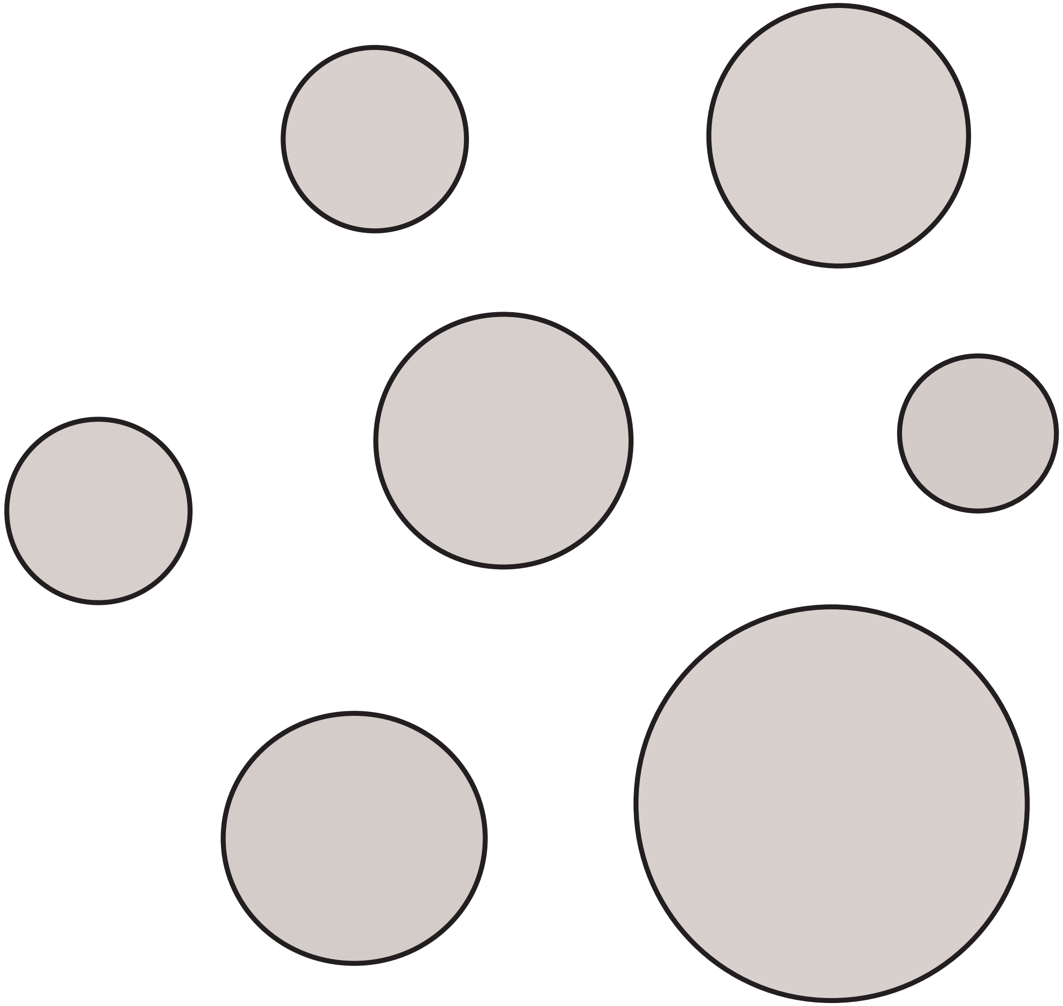

A Stochastic Patch Occupancy Model (SPOM)

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Colonization and extinction happen in distinct, successive phases.

SPOM - Phase structure

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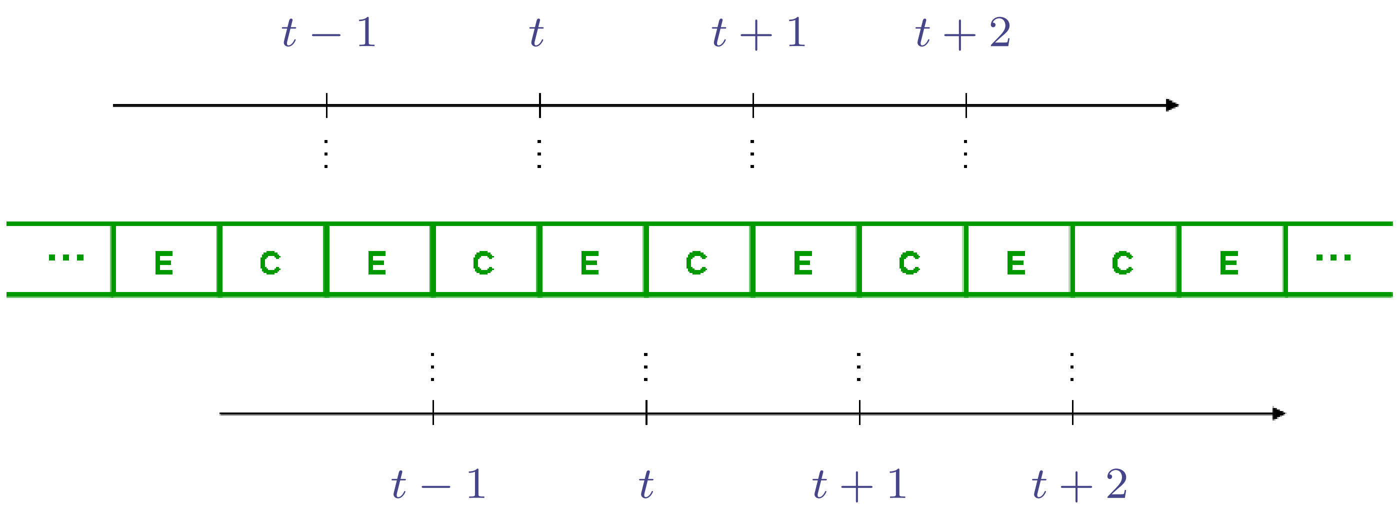

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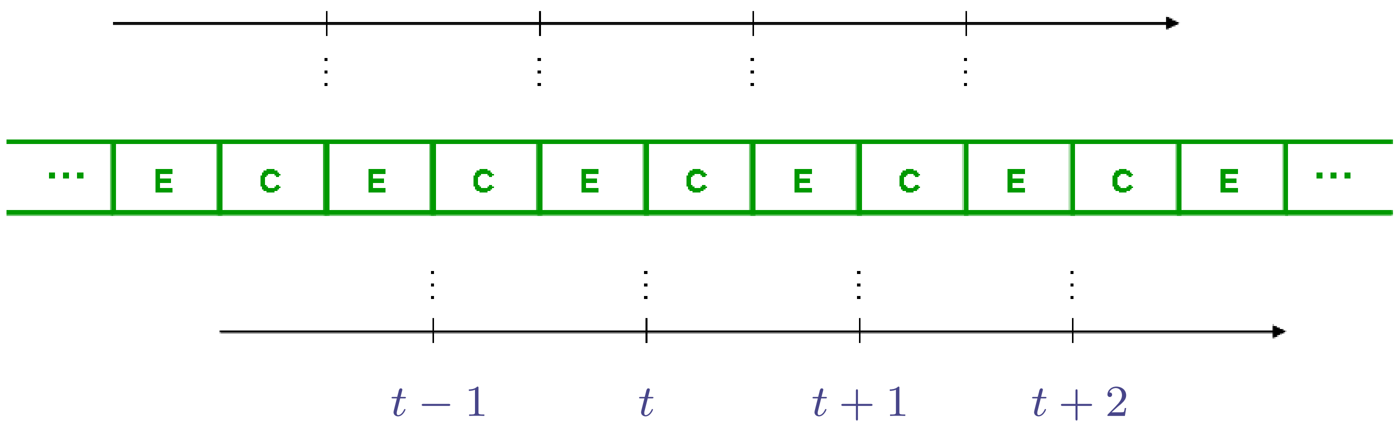

We will we assume that the population is *observed after successive extinction phases* (CE Model).

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Colonization: unoccupied patches become occupied independently with probability $c(n_t/N)$, where $c : [0, 1] \to [0, 1]$ is continuous, increasing and concave, and $c'(0) > 0$.

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Extinction: occupied patches remain occupied independently with probability s.

N-patch SPOM

We have the following *Chain Binomial* structure:

$$n_{t+1} \stackrel{\mathsf{D}}{=} \mathsf{Bin}\Big(n_t + \mathsf{Bin}\Big(N - n_t, c(n_t/N)\Big), s\Big)$$

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Notation: $\mathsf{Bin}(m, p)$ is a binomial random variable with m trials and success probability p.

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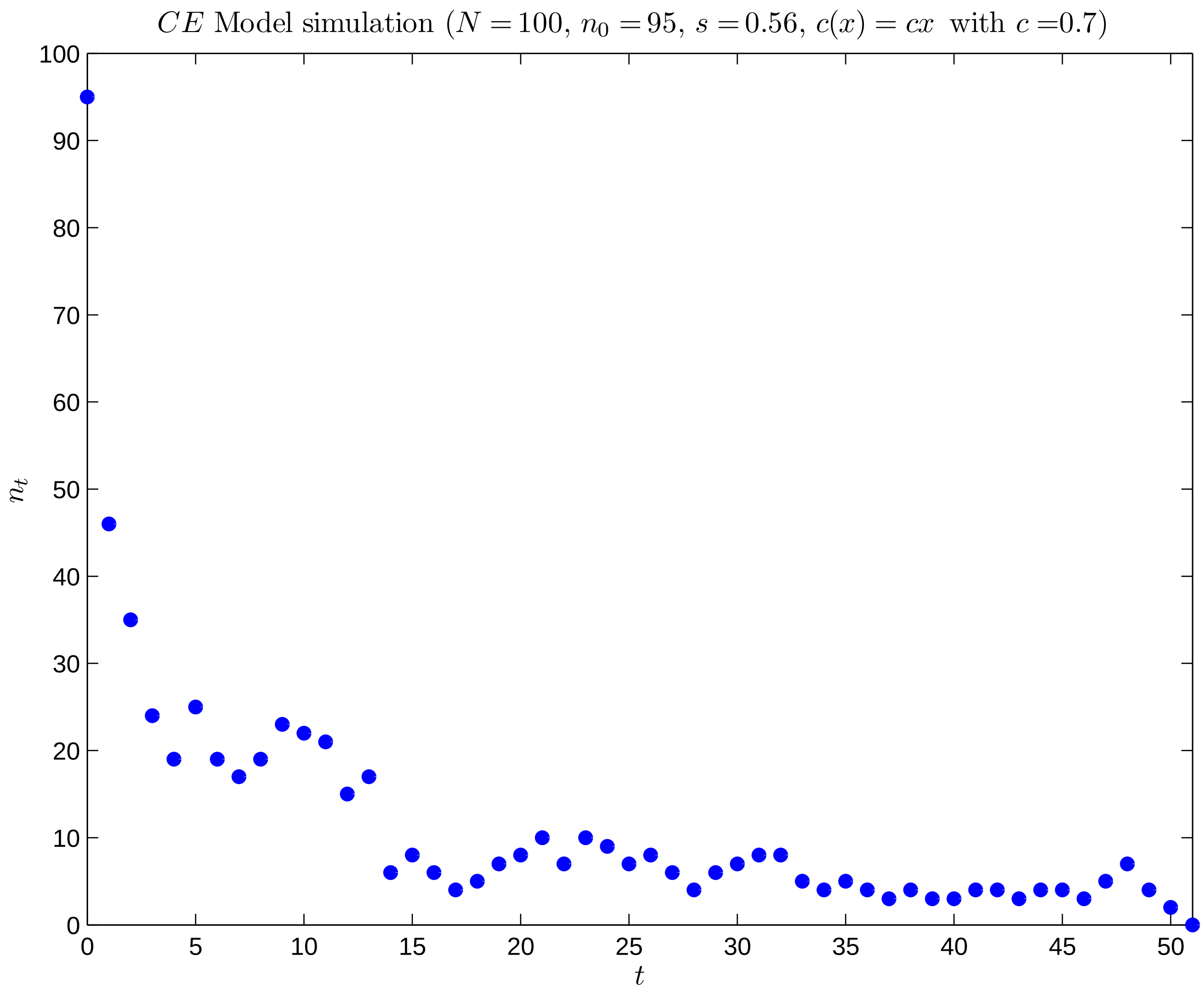
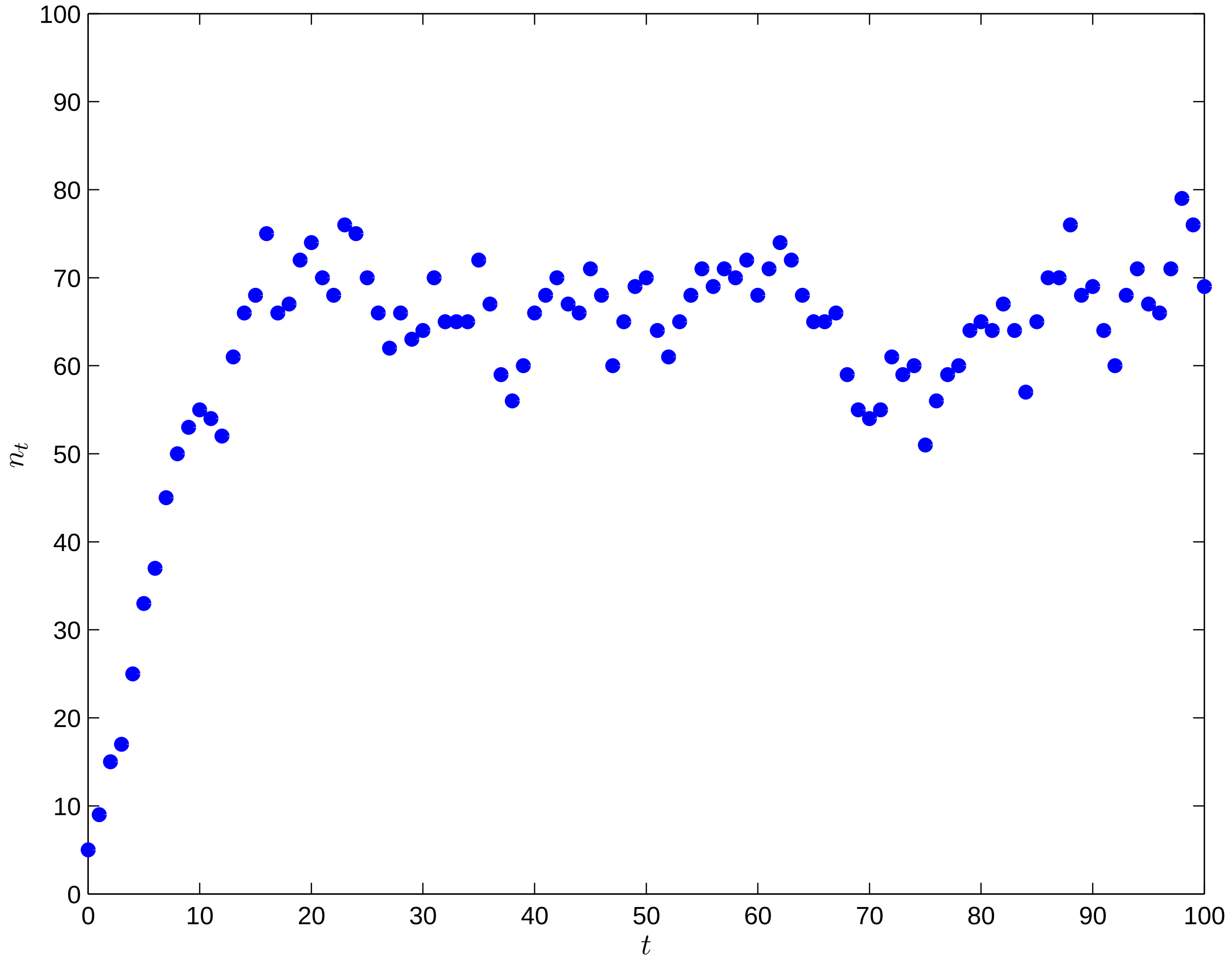
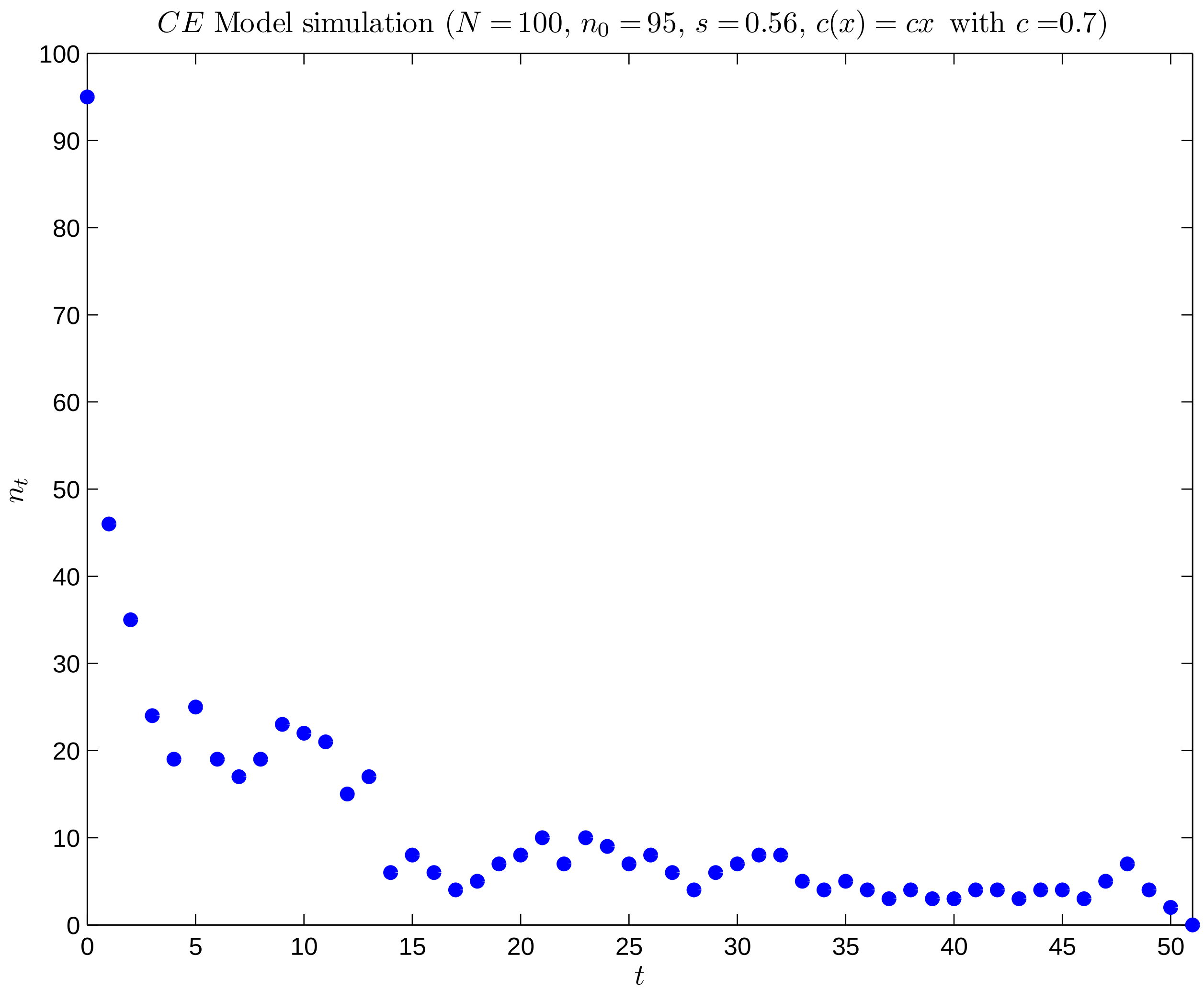

CE Model simulation ($N = 100$, $n_0 = 95$, $s = 0.56$, $c(x) = cx$ with $c = 0.7$)
n_t
t
100
90
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
40
45
50

CE Model

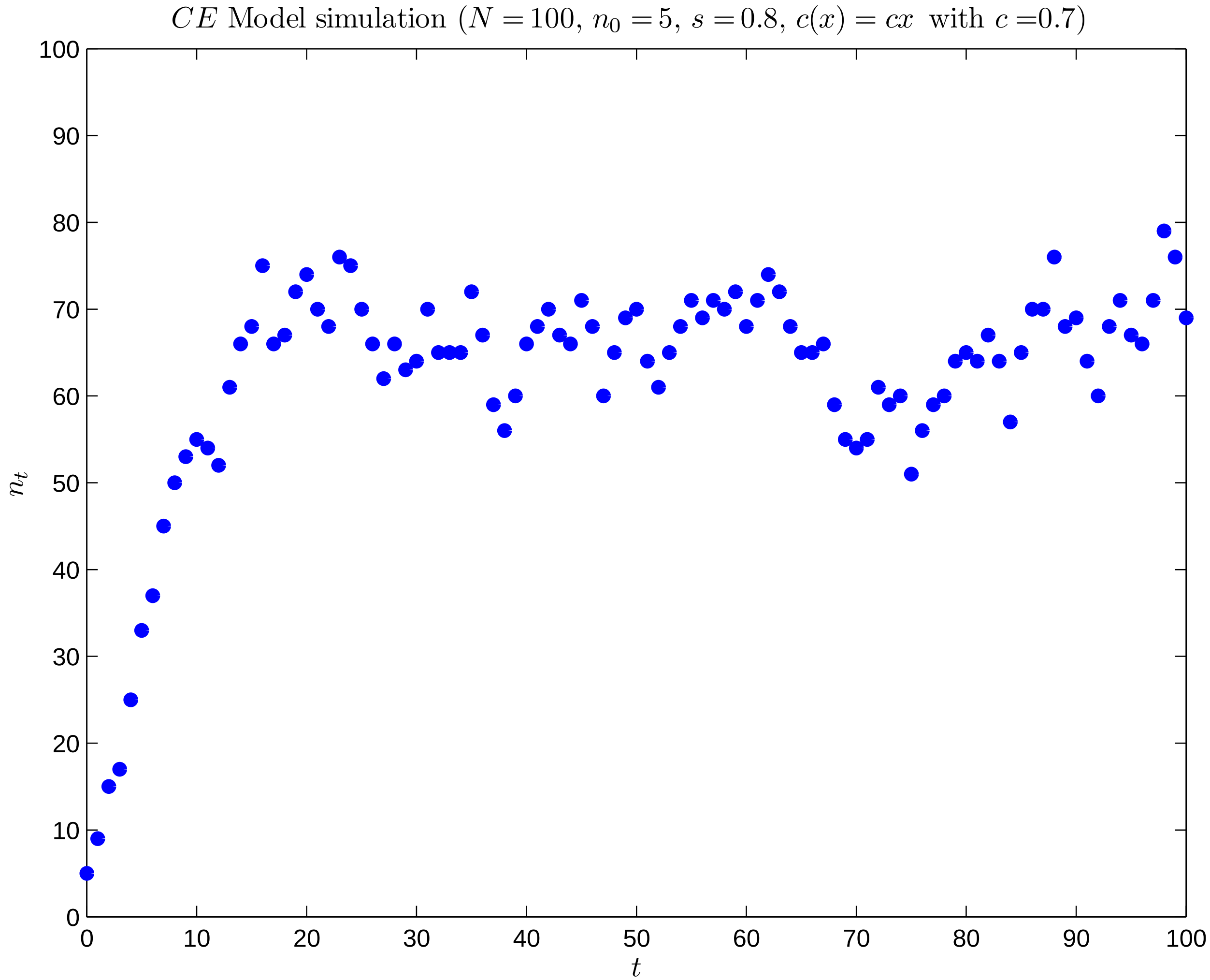
CE Model simulation ($N = 100$, $n_0 = 5$, $s = 0.8$, $c(x) = cx$ with $c = 0.7$)



CE Model - Evanescence



CE Model - Quasi stationarity



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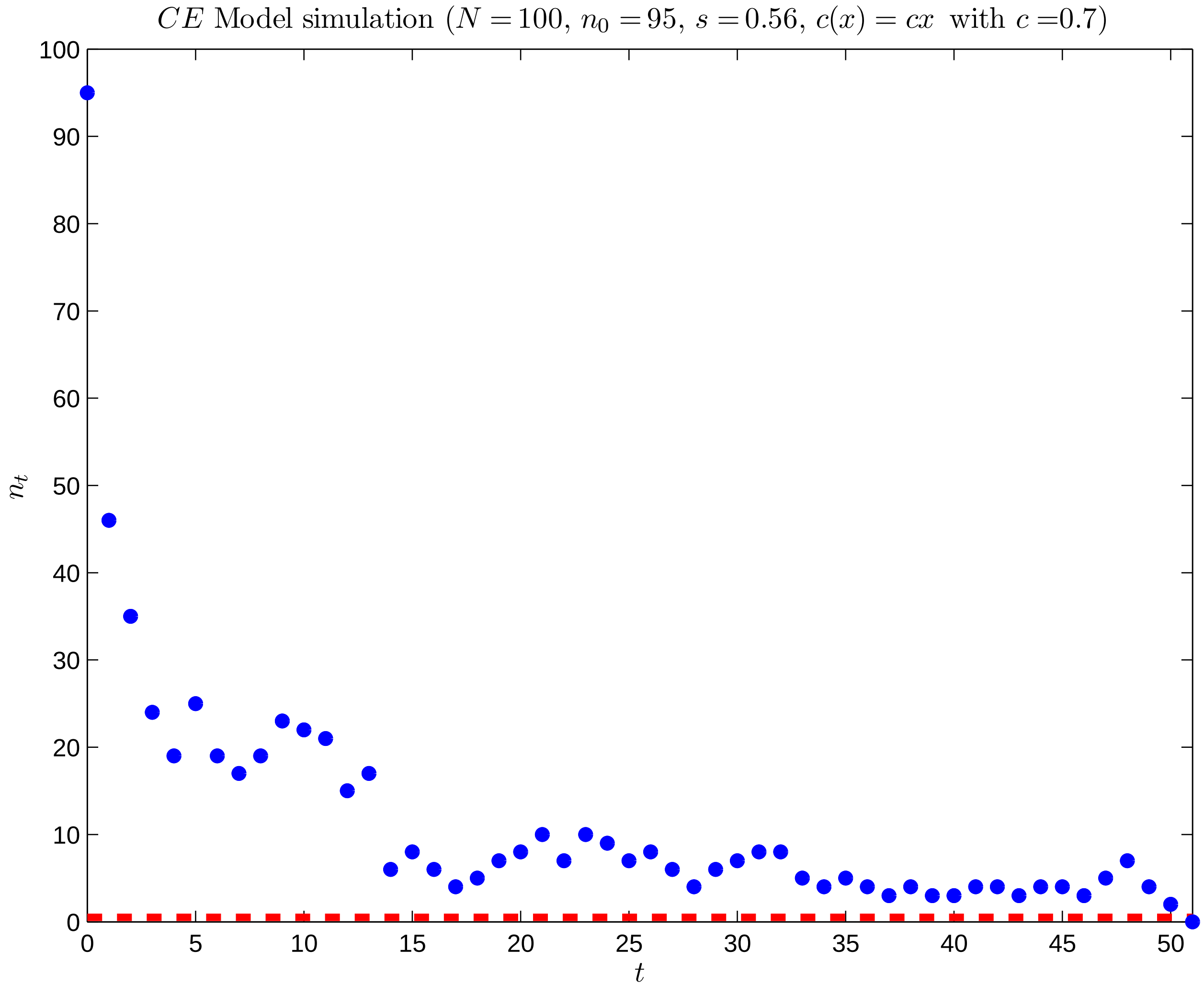

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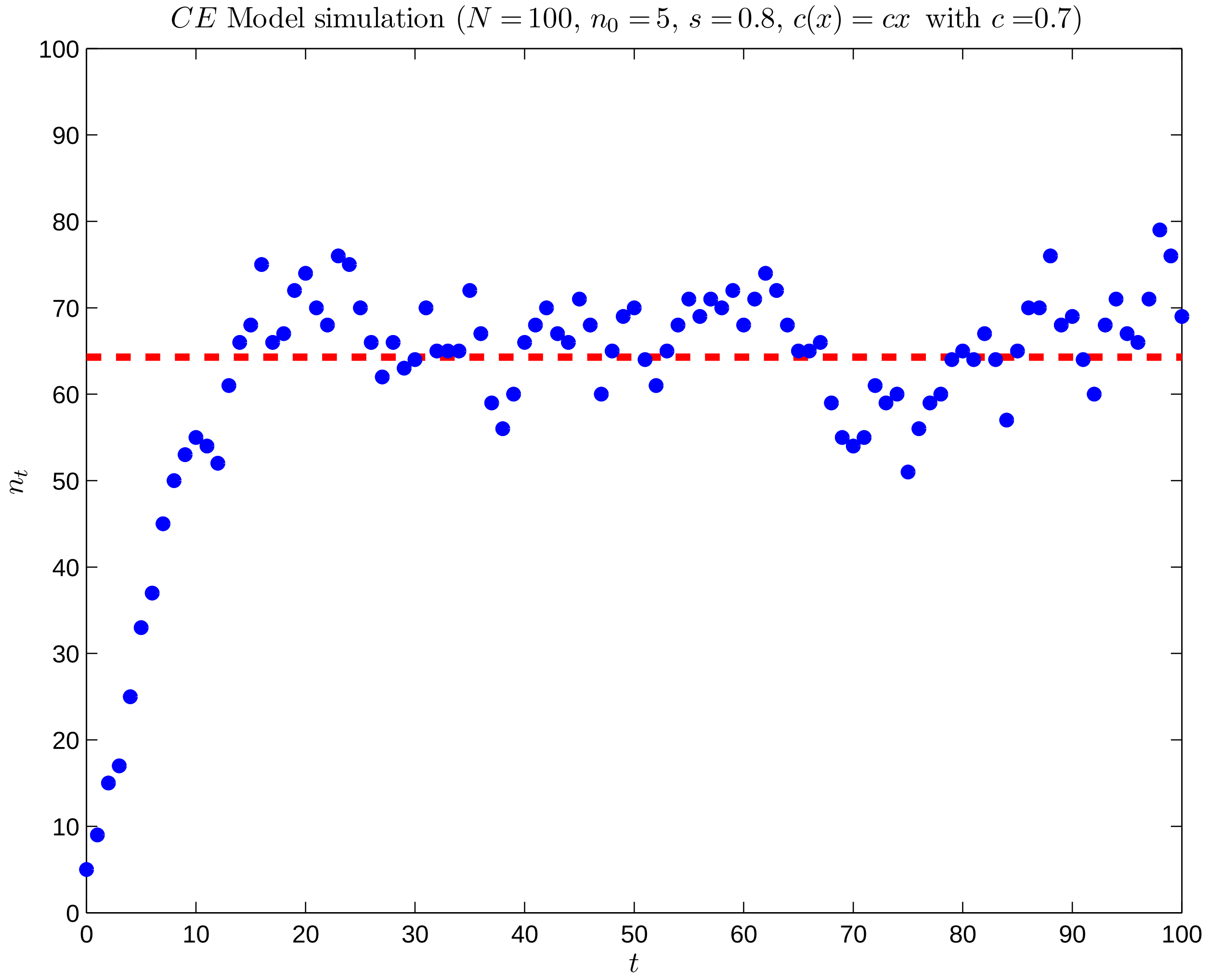

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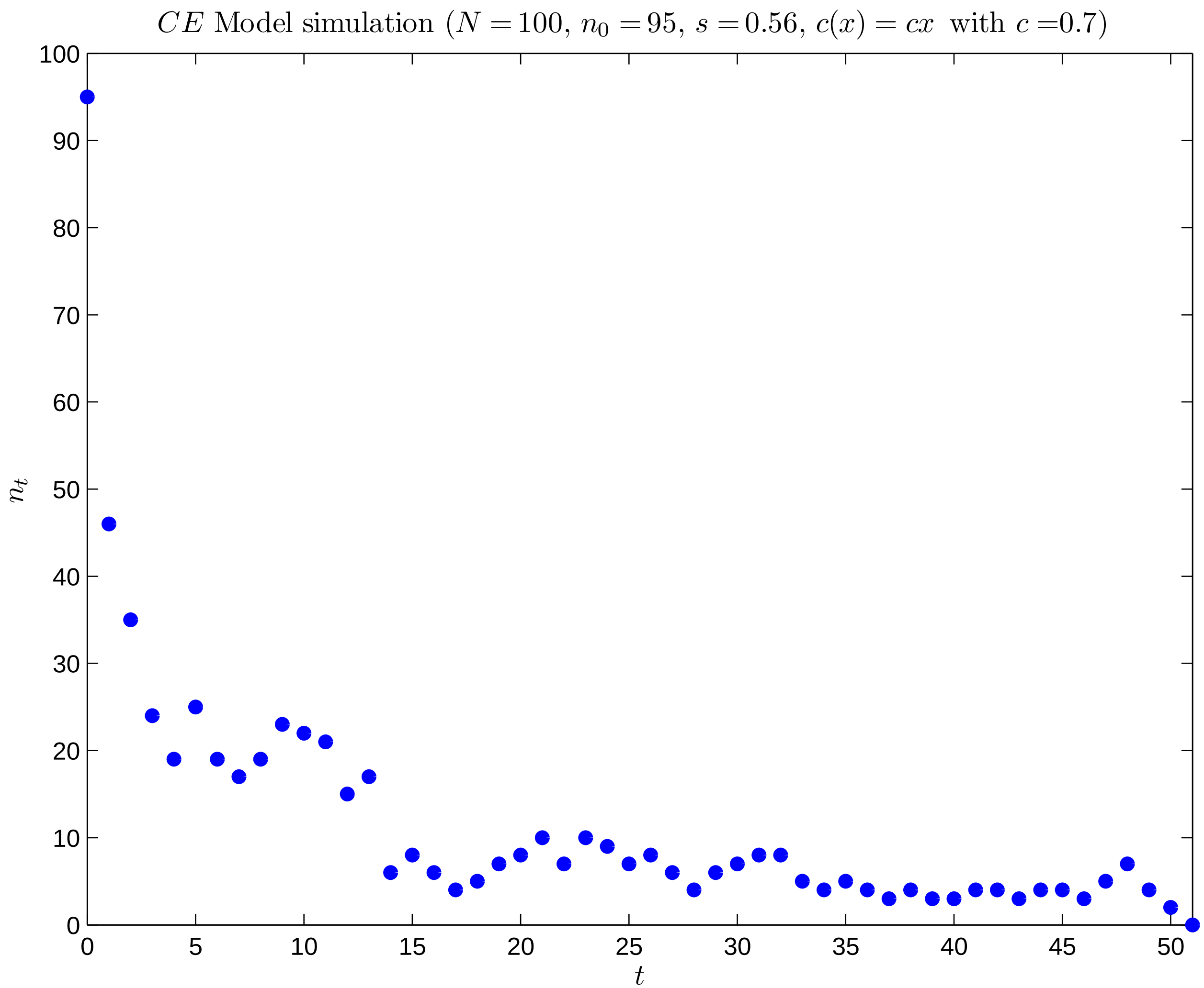

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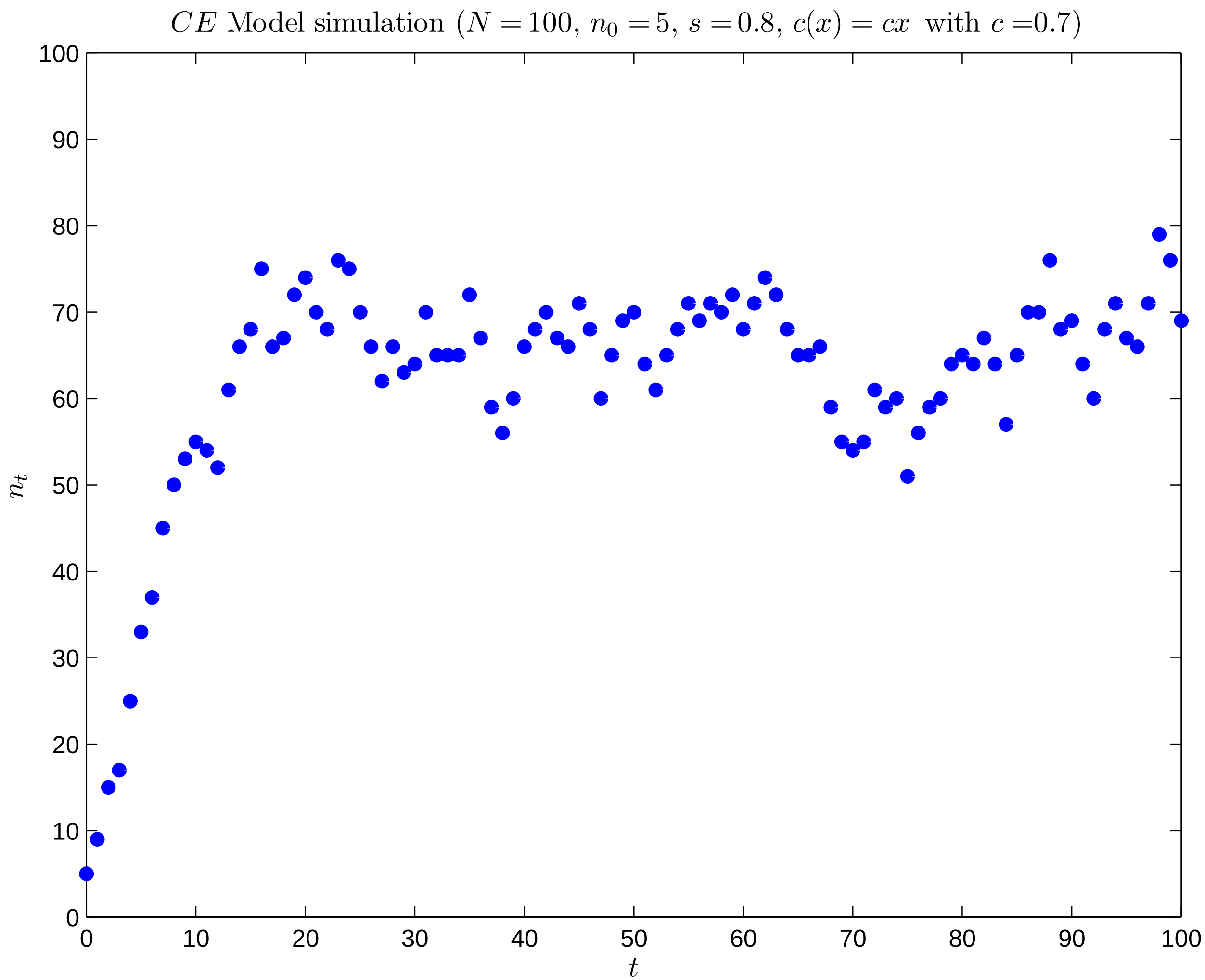
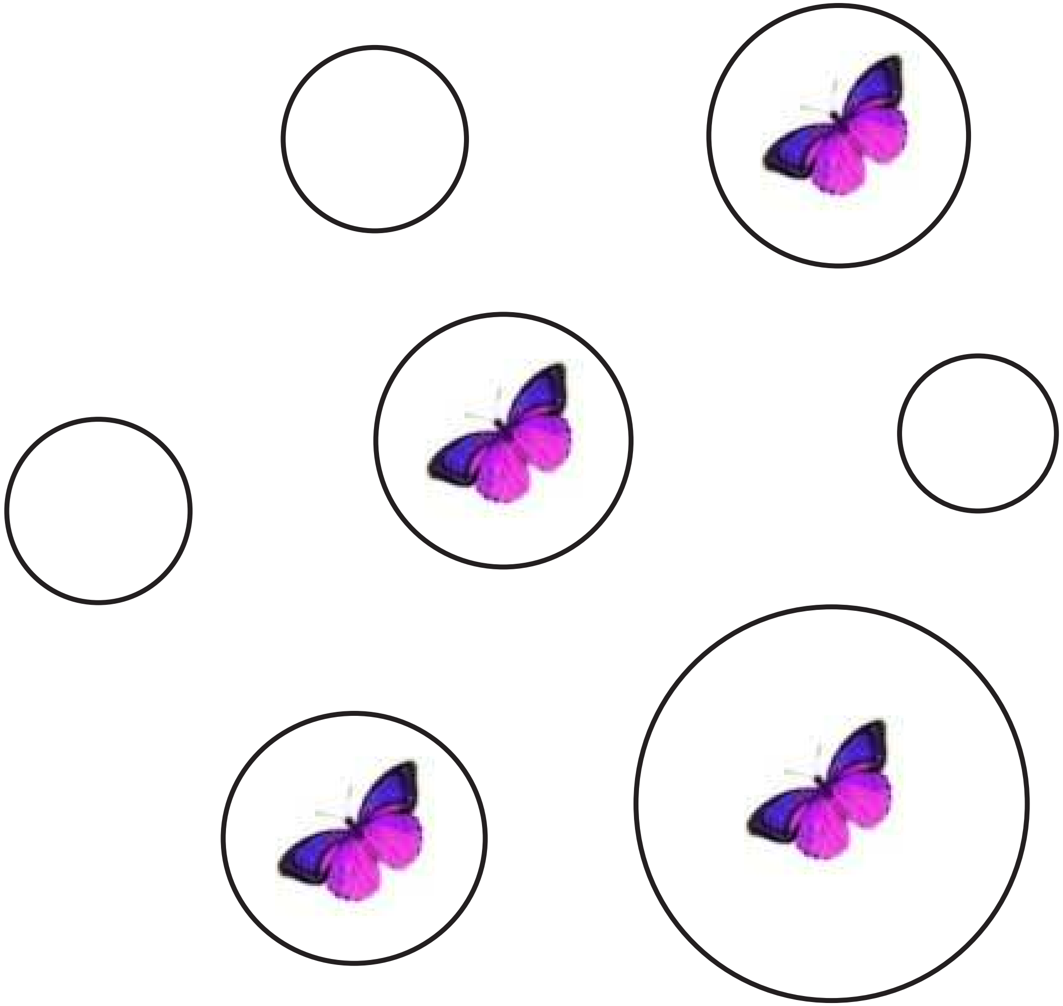


Metapopulations



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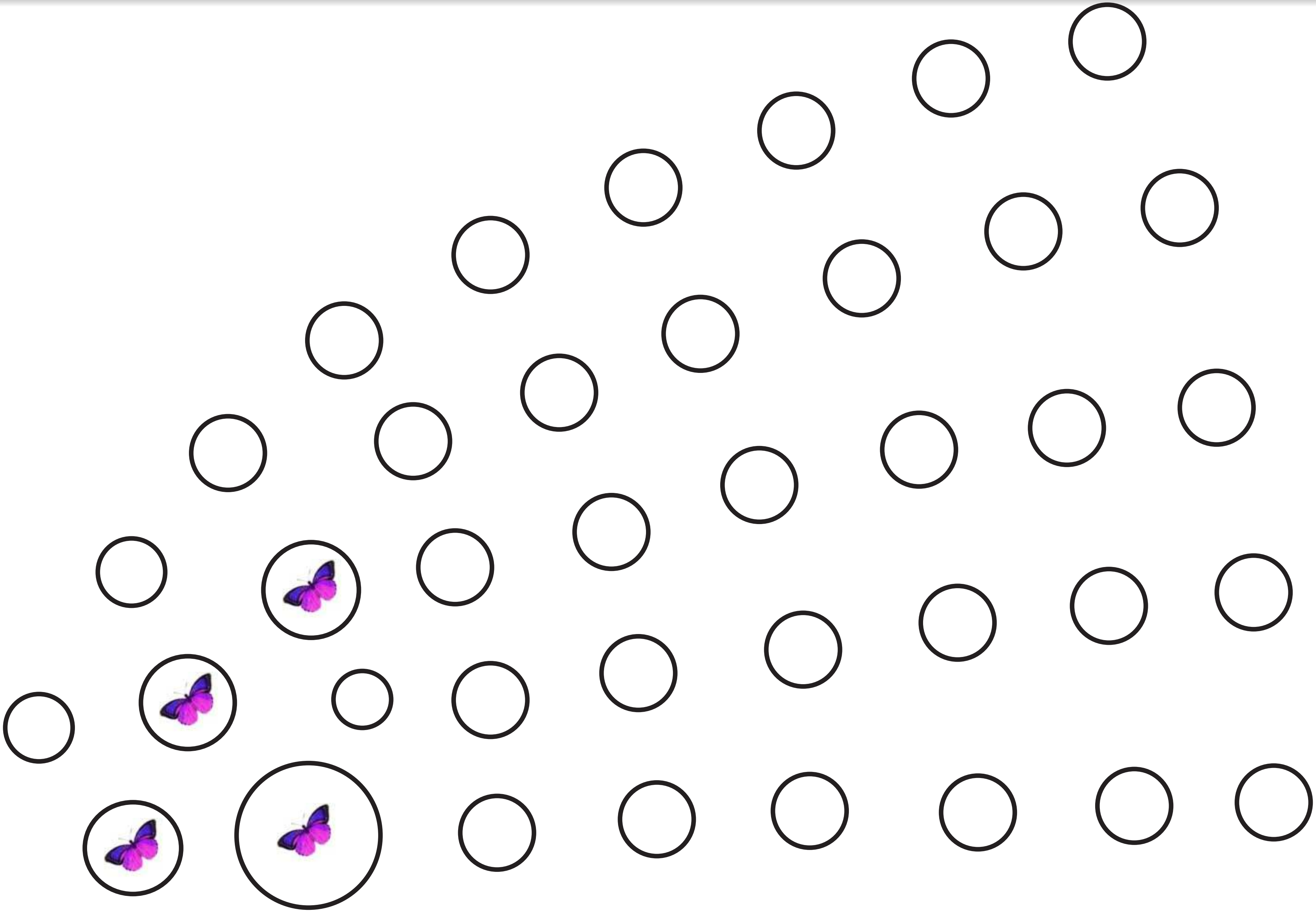

Infinite-patch SPOM

Prelude If $c(0) = 0$ and c has a continuous second derivative near 0, then, for fixed n,

$$\text{Bin}(N - n, c(n/N)) \xrightarrow{\text{D}} \text{Poi}(mn), \quad \text{as } N \to \infty,$$

where $m = c'(0)$.

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We have the following structure:

$$n_{t+1} \stackrel{\mathsf{D}}{=} \mathsf{Bin}\big(n_t + \mathsf{Poi}\big(mn_t\big), s\big)$$

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Claim The process $(n_t,\, t = 0, 1, \ldots)$ is a *branching process* (Galton-Watson process) whose offspring distribution has pgf $G(z) = (1 - s + sz)\mathrm{e}^{-ms(1-z)}$.

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(We think of the census times as marking the ‘generations’, the ‘particles’ being the occupied patches, and the ‘offspring’ being the occupied patches that they notionally replace in the succeeding generation.)

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The mean number of offspring is $\mu = (1 + m)s$.

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So, for example, $\mathsf{E}(n_t|n_0) = n_0\mu^t \; (t \geq 1)$.

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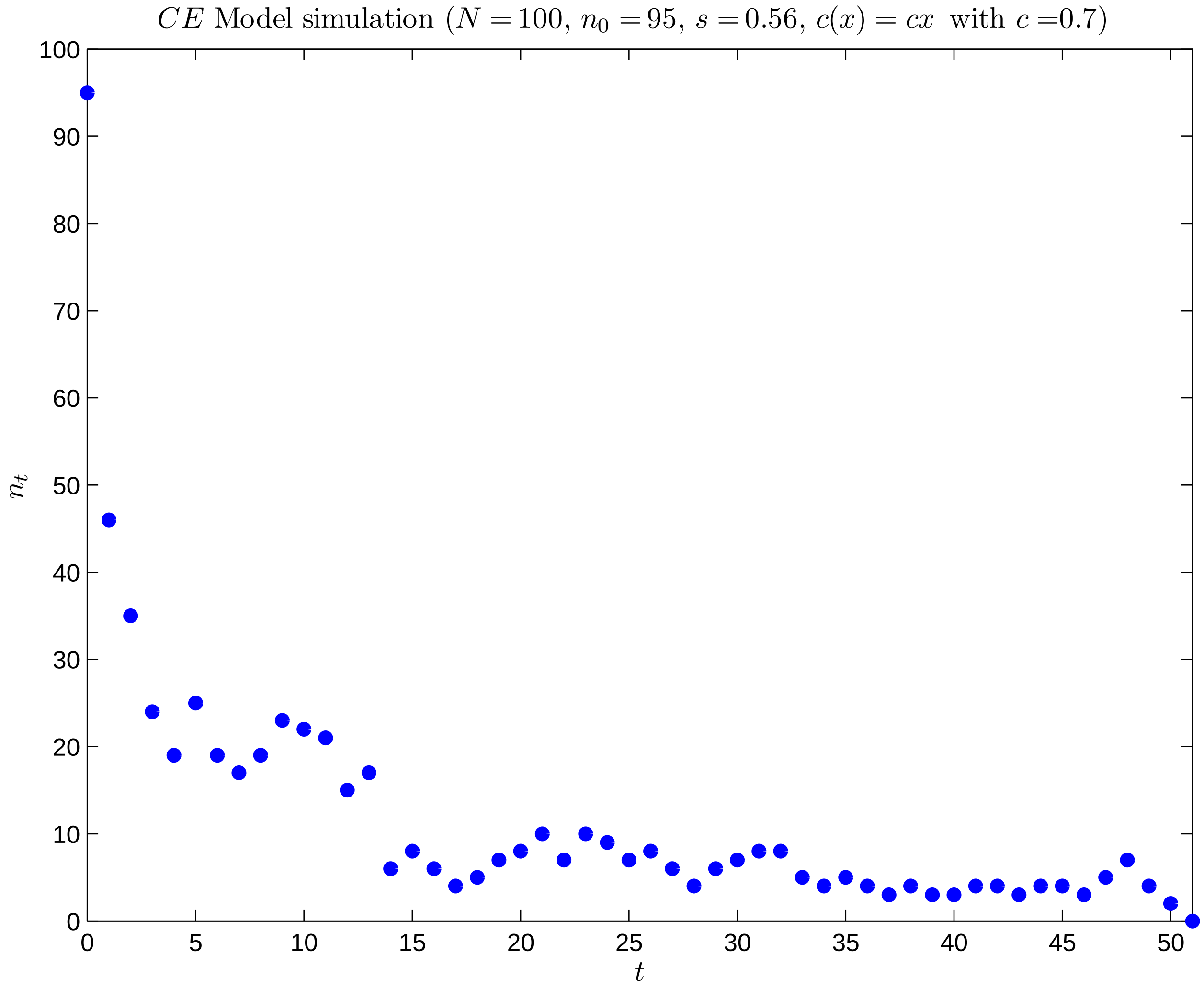
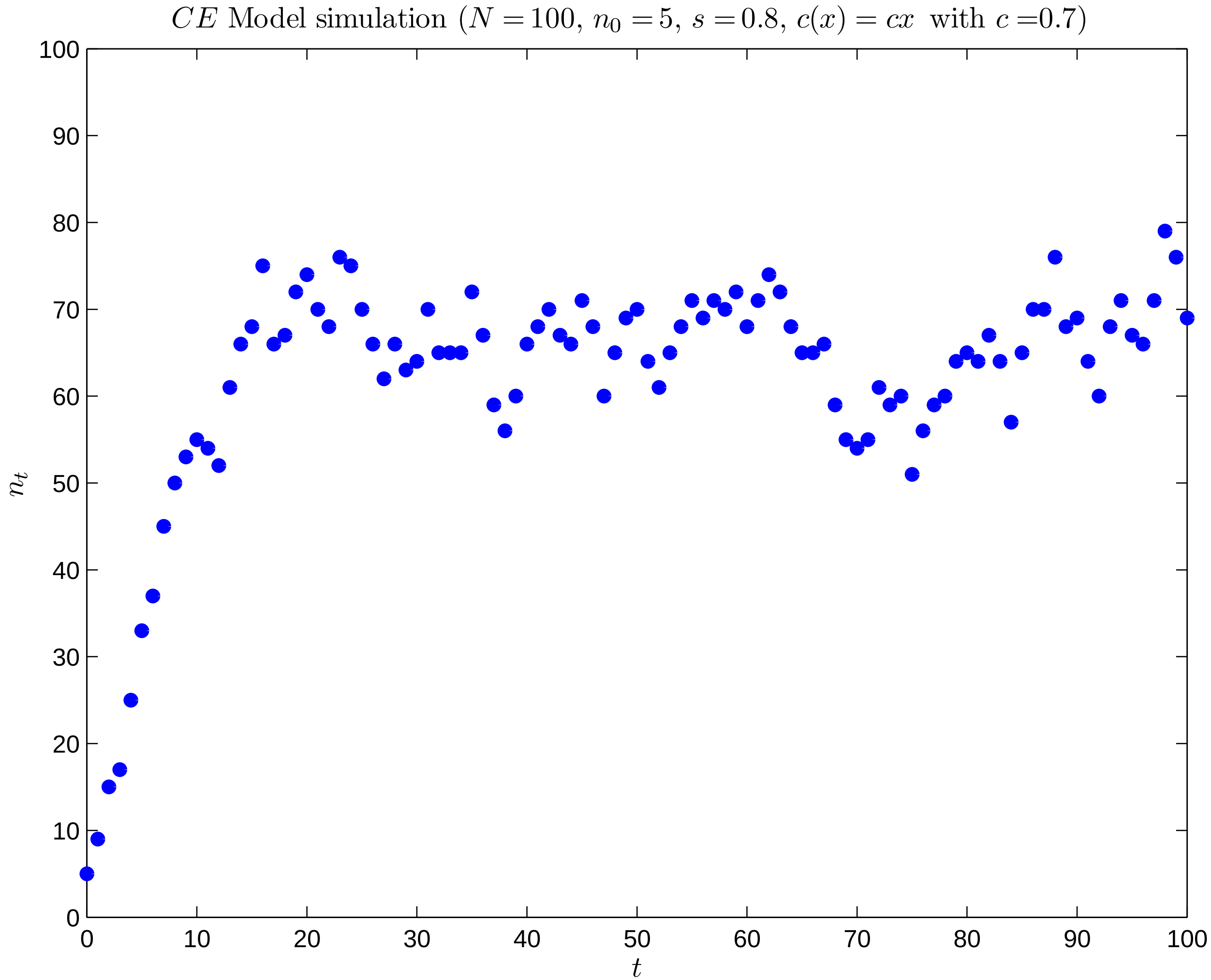
Theorem Extinction occurs with probability 1 if and only if $m \leq (1-s)/s$; otherwise total extinction occurs with probability η^{n_0}, where η is the unique fixed point of G on the interval $(0, 1)$.

CE Model $c'(0) < (1-s)/s \quad (\eta = 1)$



CE Model $c'(0) > (1-s)/s \ \ (\eta^{n_0} = 0.0020837)$



Infinite-patch SPOM with regulation

Assume the following structure:

$$n_{t+1} \stackrel{\mathsf{D}}{=} \mathsf{Bin}\big(n_t + \mathsf{Poi}\big(m(n_t)\big), s\big)$$

where $m(n) \geq 0$.

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where $m(n) \geq 0$. A moment ago we had $m(n) = mn$.

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For some index N write $m(n) = N\mu(n/N)$, and assume μ is continuous with bounded first derivative.

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We may take N to be simply n_0 or, more generally, following Klebaner*, we may interpret N as being a 'threshold' with the property that $n_0/N \to x_0$ as $N \to \infty$.

*Klebaner (1993) Population-dependent branching processes with a threshold. Stochastic Process. Appl. 46, 115–127.

Infinite-patch SPOM with regulation

By choosing μ appropriately, we may allow for a degree of regulation in the colonisation process.

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For example, $\mu(x)$ might be of the form

- $\mu(x) = rx(a-x)$ $(0 \leq x \leq a)$ (logistic growth);
- $\mu(x) = x\mathrm{e}^{r(1-x)}$ $(x \geq 0)$ (Ricker dynamics);
- $\mu(x) = \lambda x/(1+ax)^b$ $(x \geq 0)$ (Hassell dynamics).

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We can establish a *law of large numbers* for $X_t^{(N)} = n_t/N$, the number of occupied patches at census t measured *relative to* the threshold.

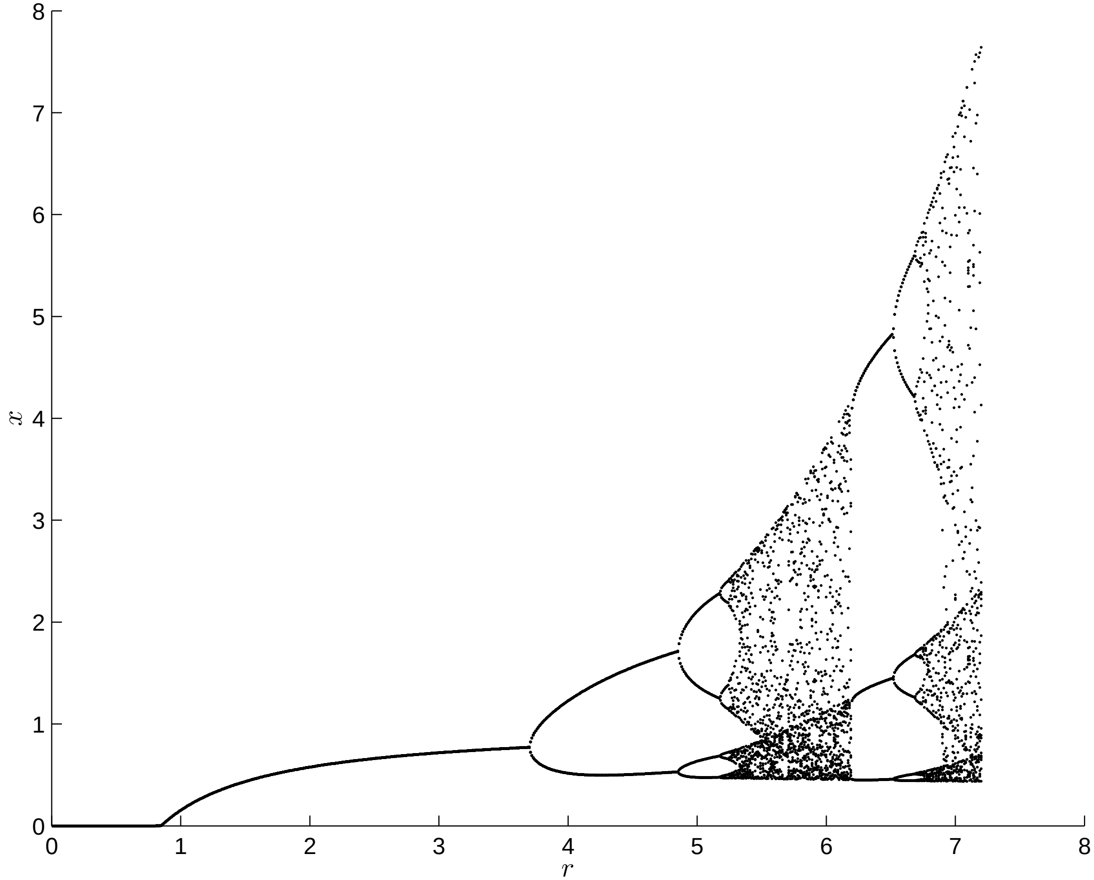
Infinite-patch SPOM - Convergence

Theorem For the infinite-patch CE model with parameters s and $\mu(x)$, let $X_t^{(N)} = n_t/N$ be the number of occupied patches at census t relative to the threshold N.

Suppose that μ is continuous with bounded first derivative.

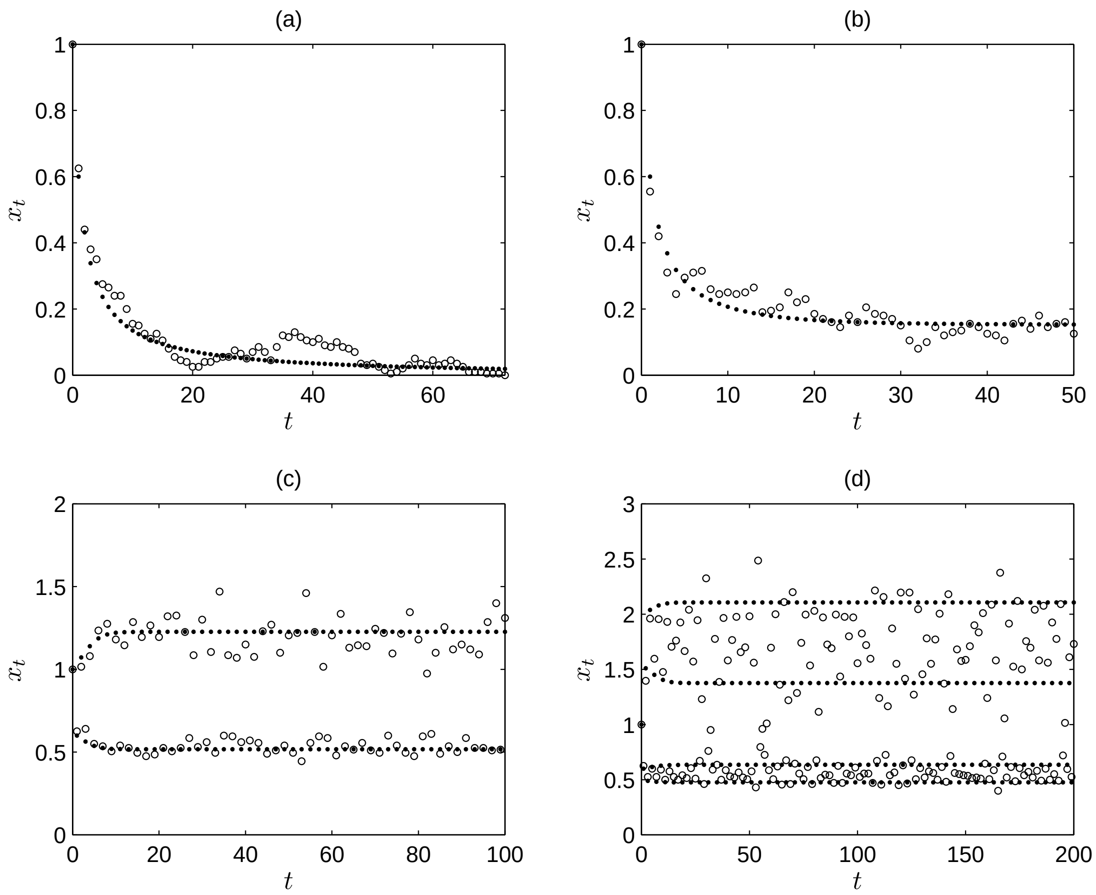
If $X_0^{(N)} \xrightarrow{2} x_0$ as $N \to \infty$, then $X_t^{(N)} \xrightarrow{2} x_t$ for all $t \geq 1$, where (x_t) is determined by $x_{t+1} = f(x_t)$ $(t \geq 0)$, where $f(x) = s(x + \mu(x))$.

Infinite-patch SPOM - Ricker dynamics



Bifurcation diagram for the infinite-patch deterministic CE model with Ricker growth dynamics: $x_{n+1} = 0.3\, x_n(1 + \mathrm{e}^{r(1-x_n)})$ (r ranges from 0 to 7.2).

Infinite-patch SPOM - Ricker dynamics



Simulation (open circles) of the infinite-patch CE model with Ricker growth dynamics, together with the corresponding limiting deterministic trajectories (solid circles). Here $s = 0.3$, $N = 200$ and (a) $r = 0.84$, (b) $r = 1$ (c) $r = 4$, (d) $r = 5$.