A Method for Evaluating the Distribution of the Total Cost of a Random Process over its Lifetime

by

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- The time τ to first exit from A
- The cost (per unit time) f_x of being in state x
- The “path integral”

$$\Gamma = \int_0^\tau f_{X(t)}\, dt,$$

the total cost incurred before leaving A

Examples

- Let $X(t)$ be the water level in a dam at time t. If τ is the time the dam first empties, and $f_x = 1_{\{x>l\}}$, then Γ is the total time that the level is above l:

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- Let $(I(t), S(t))$ be the number of infectives and susceptibles in an epidemic at time t. If τ is the period of infection and $f_{(i,s)} = i$, then Γ is the total amount of infection:

$$\Gamma = \int_0^\tau I(t)\, dt.$$

The problem

Our problem is to determine the *expected total cost*, and the *distribution of the total cost*.

For simplicity, suppose that $X(t)$ takes values in $S = \{0, 1, \ldots\}$. For example, $X(t)$ might be the number in a population at time t, and $A = \{1, 2, \ldots\}$, so the τ is the time to extinction.

A first attempt

Let T_j be the total time that the process spends in state j during the period up to time τ and let N_j be the number of visits to j during that period. Then,

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where X_{jn}, $n = 1, 2, \ldots$, are the successive occupancy times for state j.

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where X_{jn}, $n = 1, 2, \dots$, are the successive occupancy times for state j. If these times are independent and identically distributed, then $E(\Gamma) = \sum_{j \in A} f_j E(N_j) \mu_j$, where μ_j is the mean occupancy time for state j.

Markovian models

We will assume that $(X(t),\, t \geq 0)$ is a *Markov chain* with *transition rates* $Q = (q_{ij},\, i, j \in S)$, so that q_{ij} represents the rate of transition from state i to state j, for $j \neq i$, and $q_{ii} = -q_i$, where $q_i := \sum_{j \neq i} q_{ij}$ ($< \infty$) represents the total rate out of state i.

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$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

The expected value of Γ

Let $e_i = E_i(\Gamma) := E(\Gamma | X(0) = i)$, and condition on the time of the first jump and the state visited at that time, to get

$$E_i(\Gamma) = \int_0^\infty \sum_{k \neq i} \left(\frac{f_i}{q_i} + E_k(\Gamma) \right) \frac{q_{ik}}{q_i} q_i e^{-q_i u} \, du,$$

which leads to

$$q_i e_i = f_i + \sum_{k \neq i} q_{ik} e_k,$$

so that

$$\sum_k q_{ik} e_k + f_i = 0. \qquad (Q\boldsymbol{e} = -\boldsymbol{f})$$

The expected value of Γ

We can do better:

Theorem 1 $\boldsymbol{e} = (e_i,\, i \in A)$, where $e_i = E_i(\Gamma)$, is the *minimal* non-negative solution to

$$\sum_{k \in A} q_{ik} z_k + f_i = 0, \quad i \in A, \qquad (Q\boldsymbol{z} + \boldsymbol{f} = \boldsymbol{0})$$

in the sense that $\boldsymbol{e}$ satisfies these equations, and, if $\boldsymbol{z} = (z_i,\, i \in A)$ is any non-negative solution, then $e_i \leq z_i$ for all $i \in A$.

Birth-death processes

Assume that the birth rates $(\lambda_i,\, i \geq 1)$ and the death rates $(\mu_i,\, i \geq 0)$ are all strictly positive, except that $\lambda_0 = 0$. So, all states in $A = \{1, 2, \dots\}$ intercommunicate, and 0 is an absorbing state (corresponding to population extinction).

Define *potential coefficients* $(\pi_i,\, i \geq 1)$ by $\pi_1 = 1$ and

$$\pi_i = \prod_{j=2}^{i} \frac{\lambda_{j-1}}{\mu_j}, \qquad i \geq 2,$$

and assume that $A := \sum_{i=1}^{\infty} 1/(\mu_i \pi_i)$ diverges, a condition that corresponds to extinction being certain.

Birth-death processes

On applying Theorem 1 we get:

Proposition The expected cost up to the time of extinction is given by

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{\infty} f_k \pi_k,$$

for all $i \geq 1$, this being finite if and only if $\sum_{k=1}^{\infty} f_k \pi_k < \infty$.

The distribution of Γ

- Let $y_i(\theta) = E_i(e^{-\theta\Gamma})$ be the Laplace-Steiltjes transform of the distribution of Γ:

$$y_i(\theta) = \int_0^\infty e^{-\theta x}\, d\,\Pr(\Gamma \le x | X(0) = i).$$

A similar argument leads to:

Theorem 2 For each $\theta > 0$, $\boldsymbol{y}(\theta) = (y_i(\theta), i \in S)$ is the *maximal* solution to

$$\sum_{k \in S} q_{ik} z_k = \theta f_i z_i, \quad i \in A,$$

with $0 \le z_i \le 1$ for $i \in A$ and $z_i = 1$ for $i \notin A$.

A catastrophe process

- Assume that the transition rates have the form

$$q_{ij} = \begin{cases} i\rho a, & i \geq 0,\ j = i+1, \\ -i\rho, & i \geq 0,\ j = i, \\ i\rho d_{i-j}, & i \geq 2,\ 1 \leq j < i, \\ i\rho \sum_{k\geq i} d_k, & i \geq 1,\ j = 0, \end{cases}$$

with all other transition rates equal to 0. Here ρ and a are positive, d_i is positive for at least one i in $A = \{1, 2, \dots\}$ and $a + \sum_{i=1}^{\infty} d_i = 1$.

Clearly 0 is an absorbing state for the process and A is a communicating class.

A catastrophe process

We will consider only the *subcritical case*, where the drift D, given by $D = a - \sum_{i=1}^{\infty} i d_i$, is strictly negative and extinction is certain. Let $b(s) = d(s) - s$, where d is the probability generating function $d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}$, $|s| < 1$. We can evaluate $E_i(e^{-\theta\Gamma})$ for specific choices of f. Take $f_i = i$. We seek the maximal solution to

$$\sum_{j=0}^{\infty} q_{ij} z_j = \theta i z_i, \qquad i \geq 1,$$

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$$\rho a z_{i+1} - \rho z_i + \rho \sum_{j=1}^{i-1} d_{i-j} z_j + \rho z_0 \sum_{j=i}^{\infty} d_j = \theta z_i, \quad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

- Multiplying by s^{i-1} and summing over i gives

$$\sum_{i=1}^{\infty} E_i(e^{-\theta\Gamma})s^{i-1} = \frac{1}{1-s} - \frac{\theta(\gamma_\theta - s)}{(1-\gamma_\theta)(1-s)(\rho b(s) - \theta s)},$$

where γ_θ is the unique solution to $\rho b(s) = \theta s$ on the interval $0 < s < \sigma$. In the case of "geometric catastrophes" ($d_i = d(1-q)q^{i-1}$, $i \geq 1$, where $d > 0$ satisfies $a + d = 1$, and $0 \leq q < 1$), we get

$$E_i(e^{-\theta\Gamma}) = \frac{\beta(\theta) - q}{1-q} \left(\beta(\theta)\right)^{i-1}, \quad i \geq 1,$$

where $\beta(\theta)$ is the smaller of the two zeros of $a\rho s^2 - (\rho(1+qa)+\theta)s + \rho(d+qa) + q\theta$.