Identifying Markov Chains
with a Given Invariant Measure

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AND STATISTICS OF COMPLEX SYSTEMS
Transition functions

State-space. \( S = \{0, 1, \ldots\} \)
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- \( p_{ij}(s + t) = \sum_k p_{ik}(s)p_{kj}(t). \) [Chapman-Kolmogorov]
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It is called standard if

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and honest if

- \( \sum_j p_{ij}(t) = 1, \) for some (and then for all) \( t > 0. \)
The $q$-matrix

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Suppose that $Q$ is given. Assume that $Q$ is stable, that is $q_{ii} < 1$ for all $i$ in $S$. A standard process $P$ will then be called a $q$-process if its $q$-matrix is $Q$. 

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Under this condition, every \( Q \)-process \( P \) satisfies the \textit{backward equations},
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BE_{ij} \quad p_{ij}^t(t) = \sum_k q_{ik} p_{kj}(t), \quad t > 0,
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but might not satisfy the \textit{forward equations},

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FE_{ij} \quad p_{ij}^t(t) = \sum_k p_{ik}(t) q_{kj}, \quad t > 0.
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A collection of positive numbers $\pi = (\pi_j, j \in S)$ is a stationary distribution if $\sum_j \pi_j = 1$ and

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Stationary distributions

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Recipe for finding a stationary distribution!
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**Recipe.** Find a collection of strictly positive numbers $m = (m_j, j \in S)$ such that

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Birth-death processes

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Solve \[ \sum_{i \geq 0} m_i q_{ij} = 0, \ j \geq 0, \] that is, \[ -m_0 \lambda_0 + m_1 \mu_1 = 0, \] and,

\[ m_{j-1} \lambda_{j-1} - m_j (\lambda_j + \mu_j) + m_{j+1} \mu_{j+1} = 0, \quad j \geq 1. \]
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Solution. \( m_0 = 1 \) and

\[ m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1. \]
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So, $m_j = \rho^j$, where $\rho = 1/r$.
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**Transition rates.** Fix \( r > 0 \) and set

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**Solution.** \( m_0 = 1 \) and

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m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.
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So, \( m_j = \rho^j \), where \( \rho = 1/r \), and hence if \( r > 1 \),

\[
\pi_j = (1 - \rho)\rho^j, \quad j \geq 0.
\]
Birth–death process simulation (minimal): $\lambda_i = 2^{2i}$, $\mu_i = 2^{2i-1}$, $x(0) = 1$
What is going wrong?

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In fact, the “process” is *explosive*. ($Q$ is not regular.)
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The relative proportion of births to deaths is \( r \) and so, if \( r > 1 \), the “process” is clearly \textit{transient}.

In fact, the “process” is \textit{explosive}. (\( Q \) is not regular.) R.G. Miller* showed that \( Q \) needs to be regular for the recipe to work.

Motivating question

If $Q$ is regular, then there exists uniquely a $Q$-process, namely the minimal process: the minimal solution $F(\cdot) = (f_{ij}(\cdot), i, j \in S)$ to $BE_{ij}$. 

If $Q$ is not regular, then there are infinitely many $Q$-processes, infinitely many of which are honest.
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Question.
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**Question.** Suppose that there exists a collection of strictly positive numbers \( \pi = (\pi_j, j \in S) \) such that
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\sum_i \pi_i = 1 \quad \text{and} \quad \sum_i \pi_iq_{ij} = 0.
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Does \( \pi \) admit an interpretation as a stationary distribution for any of these processes?
Birth–death simulation (exit): \( \lambda_i = 2^{2i}, \ \mu_i = 2^{2i-1}, \ x(0) = 1 \)
Birth–death simulation (entrance): $\lambda_i = 2^{2i}$, $\mu_i = 2^{2i-1}$, $x(0) = 1$
An invariance result

Let \( m = (m_i, i \in S) \) be a collection of strictly positive numbers.
An invariance result

Let \( m = (m_i, i \in S) \) be a collection of strictly positive numbers. We call \( m \) a subinvariant measure for \( Q \) if
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\sum_i m_i q_{ij} \leq 0
\]
Let $m = (m_i, i \in S)$ be a collection of strictly positive numbers. We call $m$ a subinvariant measure for $Q$ if $\sum_i m_iq_{ij} \leq 0$, and an invariant measure for $Q$ if $\sum_i m_iq_{ij} = 0$. 

Theorem. Let $P$ be an arbitrary $Q$-process. If $m$ is invariant for $P$, then $m$ is subinvariant for $Q$, and invariant for $Q$ if and only if $P$ satisfies the forward equations $FE_{ij}$ over $S$.

Corollary. If $m$ is invariant for the minimal process $F$, then $m$ is invariant for $Q$. 

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An invariance result

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**Theorem.** Let \( P \) be an arbitrary \( Q \)-process. If \( m \) is invariant for \( P \), then \( m \) is subinvariant for \( Q \):

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\sum_i m_i p_{ij}(t) = m_j \quad \Rightarrow \quad \sum_i m_i q_{ij} \leq 0
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Let $m = (m_i, i \in S)$ be a collection of strictly positive numbers. We call $m$ a \textit{subinvariant measure for $Q$} if $\sum_i m_i q_{ij} \leq 0$, and an \textit{invariant measure for $Q$} if $\sum_i m_i q_{ij} = 0$. It is called an \textit{invariant measure for $P$} if $\sum_i m_i p_{ij}(t) = m_j$.

\textbf{Theorem}. Let $P$ be an arbitrary $Q$-process. If $m$ is invariant for $P$, then $m$ is subinvariant for $Q$, and invariant for $Q$ if and only if $P$ satisfies the forward equations $\text{FE}_{ij}$ over $S$:

$$\left(\sum_i m_i p_{ij}(t) = m_j \Rightarrow \sum_i m_i q_{ij} = 0\right) \iff \text{FE}$$
An invariance result

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**Theorem.** Let \( P \) be an arbitrary \( Q \)-process. If \( m \) is invariant for \( P \), then \( m \) is subinvariant for \( Q \), and invariant for \( Q \) if and only if \( P \) satisfies the forward equations \( FE_{ij} \) over \( S \).

**Corollary.** If \( m \) is invariant for the minimal process \( F \), then \( m \) is invariant for \( Q \).
A construction problem

Suppose that $Q$ is a stable and conservative $q$-matrix, and that $m$ is subinvariant for $Q$.

Problem 1. Does there exist a $Q$-process for which $m$ is invariant?

Problem 2. Does there exist an honest $Q$-process for which $m$ is invariant?

Problem 3. When such a $Q$-process exists, is it unique?

Problem 4. In the case of non-uniqueness, can one identify all $Q$-processes (or perhaps all honest $Q$-processes) for which $m$ is invariant?
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The resolvent

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The resolvent

Let $P$ be a transition function. If we write

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\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t) dt, \quad \lambda > 0,
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for the Laplace transform of $p_{ij}(\cdot)$, then

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\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)
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- $\psi_{ij}(\lambda) \geq 0$
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for the Laplace transform of $p_{ij}(\cdot)$, then $\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$ enjoys the following properties:

- $\psi_{ij}(\lambda) \geq 0$, $\sum_j \lambda \psi_{ij}(\lambda) \leq 1$, and
- $\psi_{ij}(\lambda) - \psi_{ij}(\mu) + (\lambda - \mu) \sum_k \psi_{ik}(\lambda) \psi_{kj}(\mu) = 0$. 

$\psi$ is called the resolvent of $P$. Indeed, if $\psi$ is a given resolvent, in that it satisfies these properties, then there exists a standard (!) process $P$ with $\psi$ as its resolvent $\psi$. 

The resolvent

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for the Laplace transform of $p_{ij}(\cdot)$, then

$\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$ enjoys the following properties:

- $\psi_{ij}(\lambda) \geq 0$, $\sum_j \lambda \psi_{ij}(\lambda) \leq 1$, and
- $\psi_{ij}(\lambda) - \psi_{ij}(\mu) + (\lambda - \mu) \sum_k \psi_{ik}(\lambda) \psi_{kj}(\mu) = 0$.

$\Psi$ is called the resolvent of $P$. 

Indeed, if $\Psi$ is a given resolvent, in that it satisfies these properties, then there exists a standard (!) process $P$ with $\Psi$ as its resolvent $\Psi$. 

The resolvent

Let \( P \) be a transition function. If we write

\[
\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t) dt, \quad \lambda > 0,
\]

for the Laplace transform of \( p_{ij}(\cdot) \), then

\( \Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S) \) enjoys the following properties:

1. \( \psi_{ij}(\lambda) \geq 0, \sum_j \lambda \psi_{ij}(\lambda) \leq 1 \), and
2. \( \psi_{ij}(\lambda) - \psi_{ij}(\mu) + (\lambda - \mu) \sum_k \psi_{ik}(\lambda) \psi_{kj}(\mu) = 0. \)

\( \Psi \) is called the \textit{resolvent} of \( P \). Indeed, if \( \Psi \) is a given resolvent, in that it satisfies these properties, then there exists a \textit{standard} (!) process \( P \) with \( \Psi \) as its resolvent*.

Identifying $\mathcal{Q}$-processes

Now, if one is given a stable and conservative $q$-matrix $\mathcal{Q}$, and a resolvent $\Psi$ satisfying the \textit{backward equations},

$$\lambda \psi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik} \psi_{kj}(\lambda), \quad \lambda > 0,$$

then $\Psi$ determines a standard $\mathcal{Q}$-process:
Identifying $Q$-processes

Now, if one is given a stable and conservative $q$-matrix $Q$, and a resolvent $\Psi$ satisfying the *backward equations*,

$$\lambda \psi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik} \psi_{kj}(\lambda), \quad \lambda > 0,$$

then $\Psi$ determines a standard $Q$-process: as $\lambda \to \infty$,

- $\lambda \psi_{ij}(\lambda) \to \delta_{ij}$, and
- $\lambda(\lambda \psi_{ij}(\lambda) - \delta_{ij}) \to q_{ij}$. 

One can also use the resolvent to determine whether or not the $Q$-process is *honest*. This happens if and only if

$$P_j \lambda \psi_{ij}(\lambda) = 1, \quad i \in S, \quad \lambda > 0.$$
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Theorem. Let $P$ be an arbitrary process and let $\Psi$ be its resolvent. Then, $m$ is invariant for $P$ if and only if it is invariant for $\Psi$. 
Identifying invariant measures

**Theorem.** Let $P$ be an arbitrary process and let $\Psi$ be its resolvent. Then, $m$ is invariant for $P$ if and only if it is invariant for $\Psi$, that is,

$$\sum_i m_i p_{ij}(t) = m_j$$

if and only if

$$\sum_i m_i \lambda \psi_{ij}(\lambda) = m_j.$$
Existence of a \( Q \)-process

**Theorem.** Let \( Q \) be a stable and conservative \( q \)-matrix, and suppose that \( m \) is a subinvariant measure for \( Q \).
Existence of a $Q$-process

**Theorem.** Let $Q$ be a stable and conservative $q$-matrix, and suppose that $m$ is a subinvariant measure for $Q$. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$ be the resolvent of the minimal $Q$-process.
**Existence of a \( Q \)-process**

**Theorem.** Let \( Q \) be a stable and conservative \( q \)-matrix, and suppose that \( m \) is a subinvariant measure for \( Q \). Let \( \Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S) \) be the resolvent of the minimal \( Q \)-process and define \( z(\cdot) = (z_i(\cdot), i \in S) \) and \( d(\cdot) = (d_i(\cdot), i \in S) \) by

\[
z_i(\lambda) = 1 - \sum_j \lambda \phi_{ij}(\lambda),
\]
Existence of a $Q$-process

**Theorem.** Let $Q$ be a stable and conservative $q$-matrix, and suppose that $m$ is a subinvariant measure for $Q$. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$ be the resolvent of the minimal $Q$-process and define $z(\cdot) = (z_i(\cdot), i \in S)$ and $d(\cdot) = (d_i(\cdot), i \in S)$ by

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**Existence of a $Q$-process**

**Theorem.** Let $Q$ be a stable and conservative $q$-matrix, and suppose that $m$ is a subinvariant measure for $Q$. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$ be the resolvent of the minimal $Q$-process and define $z(\cdot) = (z_i(\cdot), i \in S)$ and $d(\cdot) = (d_i(\cdot), i \in S)$ by

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and

$$d_i(\lambda) = m_i - \sum_j m_j \lambda \phi_{ji}(\lambda).$$

Then, if $d = 0$, $m$ is invariant for the minimal $Q$-process.
Existence of a $Q$-process

**Theorem.** Let $Q$ be a stable and conservative $q$-matrix, and suppose that $m$ is a subinvariant measure for $Q$. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S')$ be the resolvent of the minimal $Q$-process and define $z(\cdot) = (z_i(\cdot), i \in S)$ and $d(\cdot) = (d_i(\cdot), i \in S)$ by

$$z_i(\lambda) = 1 - \sum_j \lambda \phi_{ij}(\lambda),$$

and

$$d_i(\lambda) = m_i - \sum_j m_j \lambda \phi_{ji}(\lambda).$$

Then, if $d = 0$, $m$ is invariant for the minimal $Q$-process. Otherwise, if $\sum_i d_i(\lambda) \leq \sum_i m_i z_i(\lambda) < \infty$, for all $\lambda > 0$, there exists a $Q$-process $P$ for which $m$ is invariant.
Existence of a $Q$-process

**Theorem continued.** The resolvent of one such process is given by

$$
\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)},
$$

(2)
Existence of a $\mathcal{Q}$-process

Theorem continued. The resolvent of one such process is given by

$$
\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)},
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and this is honest if and only if $\sum_i d_i(\lambda) = \sum_i m_i z_i(\lambda)$, for all $\lambda > 0$. 
Existence of a $Q$-process

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and this is honest if and only if $\sum_i d_i(\lambda) = \sum_i m_i z_i(\lambda)$, for all $\lambda > 0$. A sufficient condition for there to exist an honest $Q$-process for which $m$ is invariant is that $m$ satisfies $\sum_j m_j(1 - \lambda \phi_{jj}(\lambda)) < \infty$, for all $\lambda > 0$. 
Existence of a $Q$-process

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\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)},
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and this is honest if and only if $\sum_i d_i(\lambda) = \sum_i m_i z_i(\lambda)$, for all $\lambda > 0$. A sufficient condition for there to exist an honest $Q$-process for which $m$ is invariant is that $m$ satisfies $\sum_j m_j (1 - \lambda \phi_{jj}(\lambda)) < \infty$, for all $\lambda > 0$.

**Corollary.** If $m$ is a subinvariant *probability distribution* for $Q$, then there exists an honest $Q$-process with stationary distribution $m$. The resolvent of one such process is given by (2).
The single-exit case

Suppose that $Q$ is a single-exit $q$-matrix
The single-exit case

Suppose that $Q$ is a single-exit $q$-matrix, that is, the space of bounded, non-negative vectors $\xi = (\xi_i, i \in S)$ which satisfy

$$\sum_j q_{ij} \xi_j = \alpha \xi_i, \quad \alpha > 0,$$

has dimension 1.
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has dimension 1. (The minimal process has only one available “escape route” to infinity.)
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Suppose that $Q$ is a \textit{single-exit} $q$-matrix, that is, the space of bounded, non-negative vectors $\xi = (\xi_i, i \in S)$ which satisfy

$$\sum_j q_{ij} \xi_j = \alpha \xi_i, \quad \alpha > 0,$$

has dimension 1. (The minimal process has only one available “escape route” to infinity.) Then, the condition

$$\sum_i d_i(\lambda) \leq \sum_i m_i z_i(\lambda) < \infty$$

is \textit{necessary} for the existence of a $Q$-process for which the specified measure is invariant;
The single-exit case

Suppose that $Q$ is a single-exit $q$-matrix, that is, the space of bounded, non-negative vectors $\xi = (\xi_i, i \in S)$ which satisfy

$$\sum_j q_{ij} \xi_j = \alpha \xi_i, \quad \alpha > 0,$$

has dimension 1. (The minimal process has only one available “escape route” to infinity.) Then, the condition

$$\sum_i d_i(\lambda) \leq \sum_i m_i z_i(\lambda) < \infty$$

is necessary for the existence of a $Q$-process for which the specified measure is invariant; the $Q$-process is then determined uniquely by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)}.$$
Non-uniqueness

Consider a pure-birth process with strictly positive birth rates \((q_i, i \geq 0)\), but imagine that we have two distinct sets of birth rates, \((q^{(0)}_i, i \geq 0)\) and \((q^{(1)}_i, i \geq 0)\), which satisfy

\[
\sum_{i=0}^{\infty} \frac{1}{q^{(r)}_i} < \infty, \quad r = 0, 1.
\]
Non-uniqueness

Consider a pure-birth process with strictly positive birth rates \((q_i, i \geq 0)\), but imagine that we have two distinct sets of birth rates, \((q^{(0)}_i, i \geq 0)\) and \((q^{(1)}_i, i \geq 0)\), which satisfy \(\sum_{i=0}^{\infty} 1/q^{(r)}_i < \infty\), \(r = 0, 1\). Let \(S = \{0, 1\} \times \{0, 1, \ldots\}\) and define \(Q = (q_{xy}, x, y \in S)\) by

\[
q(r, i)(s, j) = \begin{cases} 
q^{(r)}_i, & \text{if } j = i + 1 \text{ and } s = r, \\
-q^{(r)}_i, & \text{if } j = i \text{ and } s = r, \\
0, & \text{otherwise},
\end{cases}
\]

for \(r = 0, 1\) and \(i \geq 0\).
Non-uniqueness

Consider a pure-birth process with strictly positive birth rates \((q_i, i \geq 0)\), but imagine that we have two distinct sets of birth rates, \((q^{(0)}_i, i \geq 0)\) and \((q^{(1)}_i, i \geq 0)\), which satisfy
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\sum_{i=0}^{\infty} 1/q^{(r)}_i < \infty, \quad r = 0, 1.
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-q^{(r)}_i, & \text{if } j = i \text{ and } s = r, \\
0, & \text{otherwise},
\end{cases}
\]
for \(r = 0, 1\) and \(i \geq 0\). The measure \(m = (m_x, x \in S)\), given by \(m(r,i) = 1/q^{(r)}_i\), \(r = 0, 1, i \geq 0\), is subinvariant for \(Q\).
Non-uniqueness
Non-uniqueness

The resolvents of two distinct $Q$-processes for which $m$ is invariant are given by
Non-uniqueness

The resolvents of \textit{two distinct} \(Q\)-processes for which \(m\) is invariant are given by

\[
\psi_{(r,i)(s,j)}(\lambda) = \delta_{rs} \phi_{ij}^{(r)}(\lambda) + \frac{z_{i}^{(r)}(\lambda)\phi_{0j}^{(s)}(\lambda)}{2 - \{z_{0}^{(0)}(\lambda) + z_{0}^{(1)}(\lambda)\}}
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The resolvents of two distinct $Q$-processes for which $m$ is invariant are given by

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$$

and

$$
\psi_{(r,i)(s,j)}(\lambda) = \begin{cases} 
\phi_{ij}^{(r)}(\lambda) + \frac{z_i^{(r)}(\lambda) z_0^{(1-r)}(\lambda) \phi_{0j}^{(r)}(\lambda)}{1 - z_0^{(0)}(\lambda) z_0^{(1)}(\lambda)}, & s = r \\
\frac{z_i^{(r)}(\lambda) \phi_{0j}^{(1-r)}(\lambda)}{1 - z_0^{(0)}(\lambda) z_0^{(1)}(\lambda)}, & s \neq r.
\end{cases}
$$
Non-uniqueness

Interpretation.
Non-uniqueness

*Interpretation.*

The first process chooses between $(0, 0)$ and $(1, 0)$ with equal probability as the starting point following an explosion, no matter which was the most recently traversed path.
Non-uniqueness

*Interpretation.*

The first process chooses between \((0, 0)\) and \((1, 0)\) with equal probability as the starting point following an explosion, no matter which was the most recently traversed path.

The second process traverses *alternate paths* following successive explosions.
The reversible case

Suppose that \( Q \) is \textit{symmetrically reversible} with respect to \( m \).
The reversible case

Suppose that $Q$ is symmetrically reversible with respect to $m$, that is, $m_{ij}q_{ij} = m_{ji}q_{ji}$, $i, j \in S$. 

\[ d_i(\cdot) = m_{i}z_i(\cdot), \]

and so we arrive at the following corollary.

Corollary. If $Q$ is reversible with respect to $m$, then there exists uniquely a $Q$-function $P$ for which $m$ is invariant if and only if $P_jm_jz_j(\cdot) < 1$, for all $\cdot > 0$. It is honest and its resolvent is given by

\[ \Lambda_{ij}(\cdot) = \Lambda_{ij}(\cdot) + z_i(\cdot)m_jz_j(\cdot), \]

\[ \cdot P_k2 \in S m_kz_k(\cdot). \]

Moreover, $P$ is reversible with respect to $m$ in that

\[ m_{i}p_{ij}(t) = m_{j}p_{ji}(t) \quad \text{(or, equivalently,} \quad m_{i}q_{ij}(\cdot) = m_{j}q_{ji}(\cdot)). \]

\[ \text{Hou Chen-Ting and Chen Mufa (1980) Markov processes and field theory.} \]

\[ \text{Tongbao} 25, 807–811. \]
The reversible case

Suppose that $Q$ is \textit{symmetrically reversible} with respect to $m$, that is, $m_i q_{ij} = m_j q_{ji}$, $i, j \in S$. Then, $d_i(\lambda) = m_i z_i(\lambda)$, and so we arrive at the following corollary*. 

The reversible case

Suppose that $Q$ is *symmetrically reversible* with respect to $m$, that is, $m_i q_{ij} = m_j q_{ji}$, $i, j \in S$. Then, $d_i(\lambda) = m_i z_i(\lambda)$, and so we arrive at the following corollary*.

**Corollary.** If $Q$ is reversible with respect to $m$
The reversible case

Suppose that \( Q \) is \textit{symmetrically reversible} with respect to \( m \), that is, \( m_i q_{ij} = m_j q_{ji}, \ i, j \in S \). Then, \( d_i(\lambda) = m_i z_i(\lambda) \), and so we arrive at the following corollary.*

\textbf{Corollary}. If \( Q \) is reversible with respect to \( m \), then there exists uniquely a \( Q \)-function \( P \) for which \( m \) is invariant \textit{if and only if} \( \sum_j m_j z_j(\lambda) < \infty \), for all \( \lambda > 0 \).
The reversible case

Suppose that $Q$ is \textit{symmetrically reversible} with respect to $m$, that is, $m_{i}q_{ij} = m_{j}q_{ji}$, $i, j \in S$. Then, $d_{i}(\lambda) = m_{i}z_{i}(\lambda)$, and so we arrive at the following corollary*

\textbf{Corollary}. If $Q$ is reversible with respect to $m$, then there exists uniquely a $Q$-function $P$ for which $m$ is invariant \textit{if and only if} $\sum_{j} m_{j}z_{j}(\lambda) < \infty$, for all $\lambda > 0$. It is honest and its resolvent is given by

$$
\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_{i}(\lambda)m_{j}z_{j}(\lambda)}{\lambda \sum_{k \in S} m_{k}z_{k}(\lambda)}.
$$
The reversible case

Suppose that $Q$ is \textit{symmetrically reversible} with respect to $m$, that is, $m_i q_{ij} = m_j q_{ji}$, $i, j \in S$. Then, $d_i(\lambda) = m_i z_i(\lambda)$, and so we arrive at the following corollary*.

\textbf{Corollary}. If $Q$ is reversible with respect to $m$, then there exists uniquely a $Q$-function $P$ for which $m$ is invariant \textit{if and only if} $\sum_j m_j z_j(\lambda) < \infty$, for all $\lambda > 0$. It is honest and its resolvent is given by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) m_j z_j(\lambda)}{\lambda \sum_{k \in S} m_k z_k(\lambda)}.$$ 

Moreover, $P$ is reversible with respect to $m$ in that $m_i p_{ij}(t) = m_j p_{ji}(t)$ (or, equivalently, $m_i \psi_{ij}(\lambda) = m_j \psi_{ji}(\lambda)$).

Birth-death processes

Suppose that the birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 1)\) are strictly positive.
Suppose that the birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 1)\) are strictly positive. \(Q\) is then regular if and only if

\[
\sum_{i=0}^{\infty} \frac{1}{\lambda_i m_i} \sum_{j=0}^{i} m_j = \infty. \tag{3}
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**Proposition.** Let \(m = (m_i, i \in S)\) be the essentially unique invariant measure for \(Q\).
Birth-death processes

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Proposition. Let \( m = (m_i, \ i \in S) \) be the essentially unique invariant measure for \( Q \).

- \( m \) is invariant for the minimal \( Q \)-process if and only if (3) holds.
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Suppose that the birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 1)\) are strictly positive. \(Q\) is then regular if and only if

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**Proposition.** Let \(m = (m_i, i \in S)\) be the essentially unique invariant measure for \(Q\).

- \(m\) is invariant for the minimal \(Q\)-process if and only if (3) holds.
- When (3) fails, there exists uniquely a \(Q\)-process \(P\) for which \(m\) is invariant if and only if \(m\) is finite.
Birth-death processes

Suppose that the birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 1)\) are strictly positive. \(Q\) is then regular if and only if

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**Proposition.** Let \(m = (m_i, i \in S)\) be the essentially unique invariant measure for \(Q\).

- \(m\) is invariant for the minimal \(Q\)-process if and only if (3) holds.
- When (3) fails, there exists uniquely a \(Q\)-process \(P\) for which \(m\) is invariant if and only if \(m\) is finite, in which case \(P\) is the unique, honest \(Q\)-process which satisfies \(FE_{ij}\).
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Suppose that the birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 1)\) are strictly positive. \(Q\) is then regular if and only if

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**Proposition.** Let \(m = (m_i, i \in S)\) be the essentially unique invariant measure for \(Q\).

- \(m\) is invariant for the minimal \(Q\)-process if and only if (3) holds.

- When (3) fails, there exists uniquely a \(Q\)-process \(P\) for which \(m\) is invariant if and only if \(m\) is finite, in which case \(P\) is the unique, honest \(Q\)-process which satisfies \(\text{FE}_{ij}\); \(P\) is positive recurrent and its stationary distribution is obtained by normalizing \(m\).
μ-Invariance

Suppose that $S = \{0\} \cup C$, where 0 is an absorbing state and $C$ is irreducible (for $F$).
Suppose that $S = \{0\} \cup C$, where 0 is an absorbing state and $C$ is irreducible (for $F$). Let $\mu \geq 0$. 
**μ-Invariance**

Suppose that $S = \{0\} \cup C$, where 0 is an absorbing state and $C$ is irreducible (for $F$). Let $\mu \geq 0$. A collection $m = (m_i, i \in C)$ of strictly positive numbers is called a $\mu$-subinvariant measure for $Q$ if

$$\sum_{i \in C} m_i q_{ij} \leq -\mu m_j, \quad j \in C,$$
Suppose that $S = \{0\} \cup C$, where $0$ is an absorbing state and $C$ is irreducible (for $F$). Let $\mu \geq 0$. A collection $m = (m_i, i \in C)$ of strictly positive numbers is called a $\mu$-subinvariant measure for $Q$ if

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and a $\mu$-invariant measure for $Q$ if

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Suppose that $S = \{0\} \cup C$, where 0 is an absorbing state and $C$ is irreducible (for $F$). Let $\mu \geq 0$. A collection $m = (m_i, i \in C)$ of strictly positive numbers is called a $\mu$-subinvariant measure for $Q$ if

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and a $\mu$-invariant measure for $Q$ if

$$\sum_{i \in C} m_i q_{ij} = -\mu m_j, \quad j \in C.$$

It is called a $\mu$-invariant measure for $P$, where $P$ is any transition function, if

$$\sum_{i \in C} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in C.$$
Quasi-stationary distributions

**Proposition.** A probability distribution $\pi = (\pi_i, i \in C)$ is a $\mu$-invariant measure for some $\mu > 0$, that is,

$$\sum_{i \in C} \pi_i p_{ij}(t) = e^{-\mu t} \pi_j, \quad j \in C,$$

if and only if it is a *quasi-stationary distribution*
**Quasi-stationary distributions**

**Proposition.** A probability distribution $\pi = (\pi_i, i \in C)$ is a $\mu$-invariant measure for some $\mu > 0$, that is,

$$\sum_{i \in C} \pi_i p_{ij}(t) = e^{-\mu t} \pi_j, \quad j \in C,$$

if and only if it is a **quasi-stationary distribution**: for $j \in C$,

$$p_j(t) = \sum_{i \in C} m_i p_{ij}(t) \Rightarrow \frac{p_j(t)}{\sum_{k \in C} p_k(t)} = m_j.$$
\( \mu \)-invariance for \( F \)

**Theorem.** If \( m \) is \( \mu \)-invariant for \( P \), then \( m \) is \( \mu \)-subinvariant for \( Q \).
$\mu$-invariance for $F$

**Theorem.** If $m$ is $\mu$-invariant for $P$, then $m$ is $\mu$-subinvariant for $Q$, and $\mu$-invariant for $Q$ *if and only if* $P$ satisfies the forward equations *over* $C$. 
**\( \mu \)-invariance for \( F \)**

**Theorem.** If \( m \) is \( \mu \)-invariant for \( P \), then \( m \) is \( \mu \)-subinvariant for \( Q \), and \( \mu \)-invariant for \( Q \) if and only if \( P \) satisfies the forward equations over \( C \). For example, if \( m \) is \( \mu \)-invariant for the minimal process, then it is \( \mu \)-invariant for \( Q \).
Theorem. If $m$ is $\mu$-invariant for $P$, then $m$ is $\mu$-subinvariant for $Q$, and $\mu$-invariant for $Q$ if and only if $P$ satisfies the forward equations over $C$. For example, if $m$ is $\mu$-invariant for the minimal process, then it is $\mu$-invariant for $Q$.

Theorem. If $m$ is $\mu$-invariant for $Q$, then it is $\mu$-invariant for $F$ if and only if the equations $\sum_{i \in C} y_i q_{ij} = -\nu y_j$, $0 \leq y_i \leq m_i$, $i \in C$, have no non-trivial solution for some (and then all) $\nu < \mu$. 

$\mu$-invariance for $F$
**μ-invariance for \( F \)**

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**Theorem.** If \( m \) is \( \mu \)-invariant for \( Q \), then it is \( \mu \)-invariant for \( F \) if and only if the equations \( \sum_{i \in C} y_i q_{ij} = -\nu y_j \), \( 0 \leq y_i \leq m_i, \ i \in C \), have no non-trivial solution for some (and then all) \( \nu < \mu \).

**Theorem.** If \( m \) is a finite \( \mu \)-invariant measure for \( Q \), then

\[
\mu \sum_{i \in C} m_i a_i^F \leq \sum_{i \in C} m_i q_{i0}, \tag{4}
\]

where \( a_i^F = \lim_{t \to \infty} f_{i0}(t) \), and \( m \) is \( \mu \)-invariant for \( F \) if and only if equality holds in (4).
\( \mathcal{Q} \)-processes with a given \( m \)

**Theorem.** Suppose that \( \mathcal{Q} \) is single-exit and that \( m \) is a finite \( \mu \)-subinvariant measure for \( \mathcal{Q} \).
Theorem. Suppose that $Q$ is single-exit and that $m$ is a finite $\mu$-subinvariant measure for $Q$. Then, there exists a $Q$-process for which $m$ is $\mu$-invariant if and only if

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$Q$-processes with a given $m$

**Theorem.** Suppose that $Q$ is single-exit and that $m$ is a finite $\mu$-subinvariant measure for $Q$. Then, there exists a $Q$-process for which $m$ is $\mu$-invariant if and only if

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The resolvent $\Psi$ of any $Q$-process for which $m$ is $\mu$-invariant must be of the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{(\lambda + \mu) \sum_{k \in C} m_k z_k(\lambda)}, \quad i, j \in S,$$
**Q-processes with a given m**

**Theorem.** Suppose that $Q$ is single-exit and that $m$ is a *finite* $\mu$-subinvariant measure for $Q$. Then, there exists a $Q$-process for which $m$ is $\mu$-invariant if and only if

$$\sum_{i \in C} mi q_{i0} \leq \mu \sum_{i \in C} mi.$$ 

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where

$$d_j(\lambda) = m_j - \sum_{i \in C} mi(\lambda + \mu)\phi_{ij}(\lambda), \quad j \in C,$$

and

$$d_0(\lambda) = e/\lambda - \sum_{i \in C} mi(\lambda + \mu)\phi_{i0}(\lambda),$$
Q-processes with a given \( m \)

**Theorem.** Suppose that \( Q \) is single-exit and that \( m \) is a finite \( \mu \)-subinvariant measure for \( Q \). Then, there exists a \( Q \)-process for which \( m \) is \( \mu \)-invariant if and only if

\[
\sum_{i \in C} m_i q_{i0} \leq \mu \sum_{i \in C} m_i.
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The resolvent \( \Psi \) of any \( Q \)-process for which \( m \) is \( \mu \)-invariant must be of the form

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\[
d_0(\lambda) = e / \lambda - \sum_{i \in C} m_i (\lambda + \mu) \phi_{i0}(\lambda),
\]

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$Q$-processes with a given $m$

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\[ \sum_{i \in C} m_i q_{i0} \leq \mu \sum_{i \in C} m_i, \]

then all \( Q \)-processes for which \( m \) is \( \mu \)-invariant can be constructed in this way by varying \( e \) in the range

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\(Q\)-processes with a given \(m\)

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Exactly one of these is honest; this is obtained by setting \(e = \mu \sum_{i \in C} m_i\). And, exactly one satisfies the forward equations \(FE_{i0}\) over \(i \in C\); this is obtained by setting \(e = \sum_{i \in C} m_i q_{i0}\).