Identifying Markov Chains with a Given Invariant Measure

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It is called *standard* if

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and *honest* if

- $\sum_j p_{ij}(t) = 1$, for some (and then for all) $t > 0$.

The q-matrix

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Suppose that Q is given. Assume that Q is *stable*, that is $q_i < \infty$ for all i in S. A standard process P will then be called a *Q-process* if its q-matrix is Q.

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but might not satisfy the *forward equations*,

$$\mathrm{FE}_{ij} \qquad p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj}, \quad t > 0.$$

Stationary distributions

A collection of positive numbers $\pi = (\pi_j, j \in S)$ is a *stationary distribution* if $\sum_j \pi_j = 1$ and

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Recipe for finding a stationary distribution!

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$$m_{j-1}\lambda_{j-1} - m_j(\lambda_j + \mu_j) + m_{j+1}\mu_{j+1} = 0, \qquad j \geq 1.$$

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Solution. $m_0 = 1$ and

$$m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.$$

Miller's example

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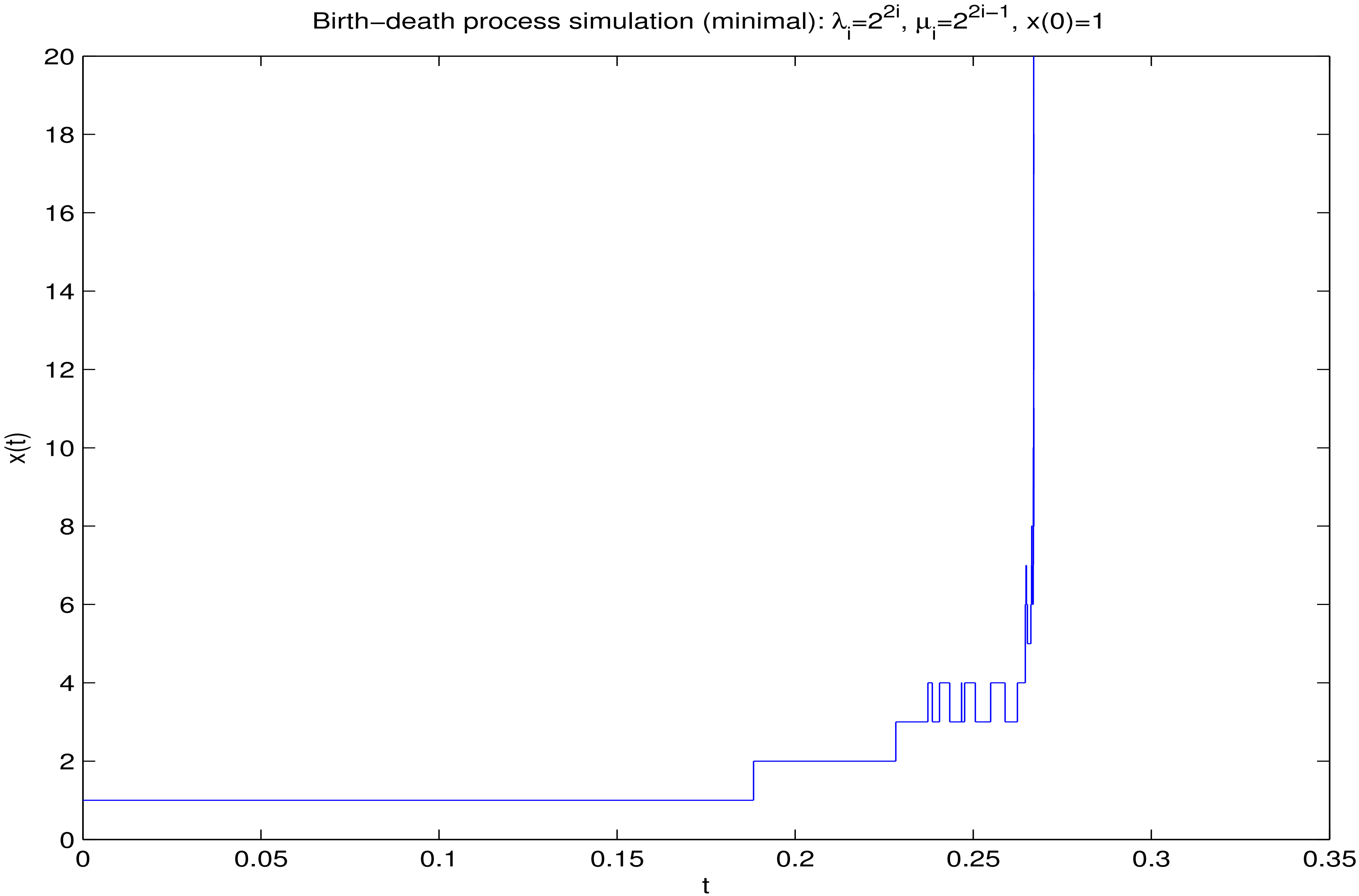
So, $m_j = \rho^j$, where $\rho = 1/r$, and hence if $r > 1$,

$$\pi_j = (1-\rho)\rho^j, \quad j \geq 0.$$

Simulation



Birth−death process simulation (minimal): λi=2^2i, μi=2^2i−1, x(0)=1
x(t)
t
0
2
4
6
8
10
12
14
16
18
20
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

What is going wrong?

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The relative proportion of births to deaths is r and so, if $r > 1$, the "process" is clearly *transient*.

In fact, the "process" is *explosive*. (Q is not regular.) R.G. Miller* showed that Q needs to be regular for the recipe to work.

*Miller, R.G. Jr. (1963) Stationary equations in continuous time Markov chains. *Trans. Amer. Math. Soc.* 109, 35–44.

Motivating question

If Q is regular, then there exists uniquely a Q-process, namely the minimal process: the minimal solution $F(\cdot) = (f_{ij}(\cdot), i, j \in S)$ to BE$_{ij}$.

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Question.

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Question. Suppose that there exists a collection of strictly positive numbers $\pi = (\pi_j, j \in S)$ such that

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Does π admit an interpretation as a stationary distribution for any of these processes?

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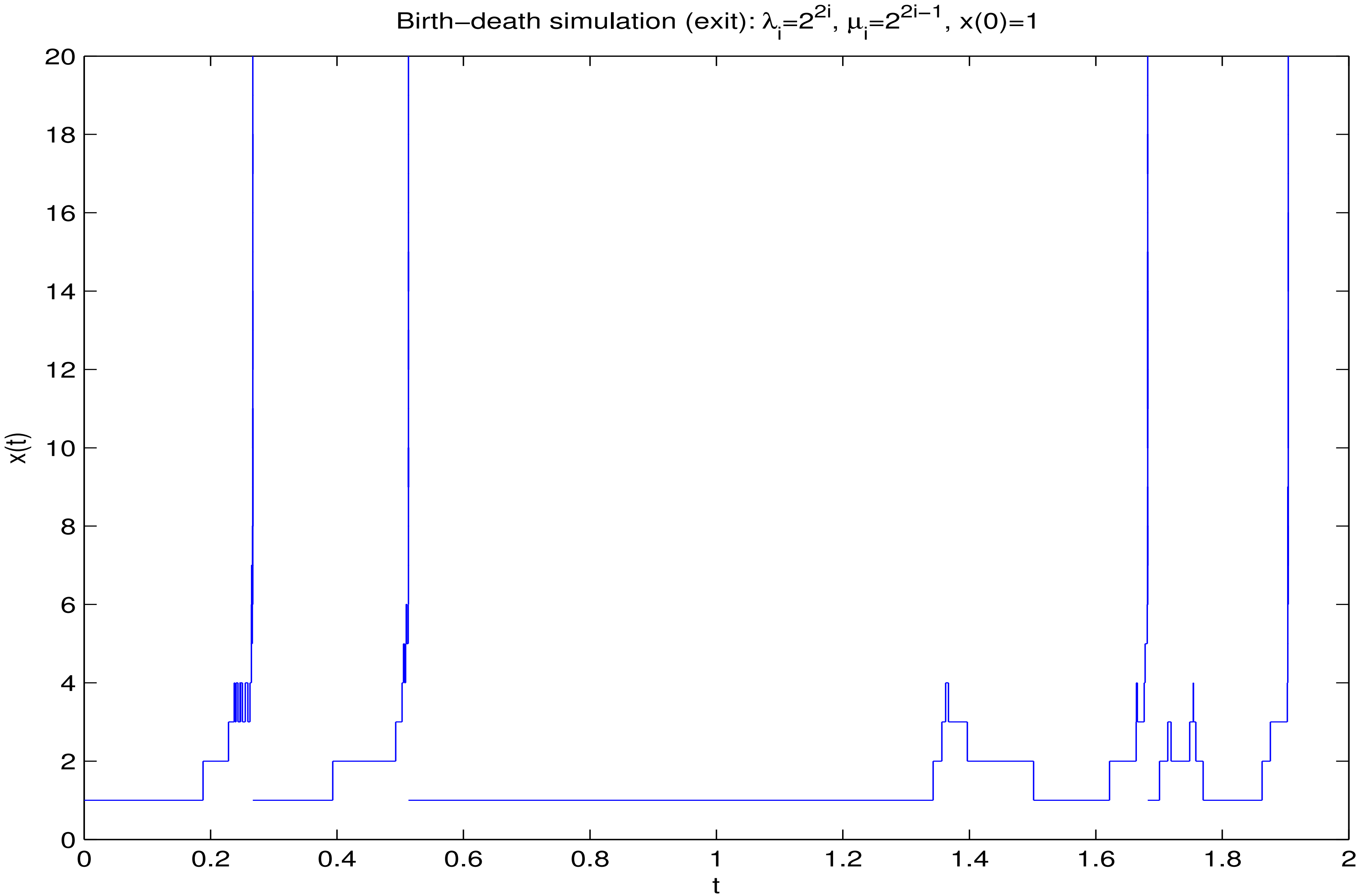

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20
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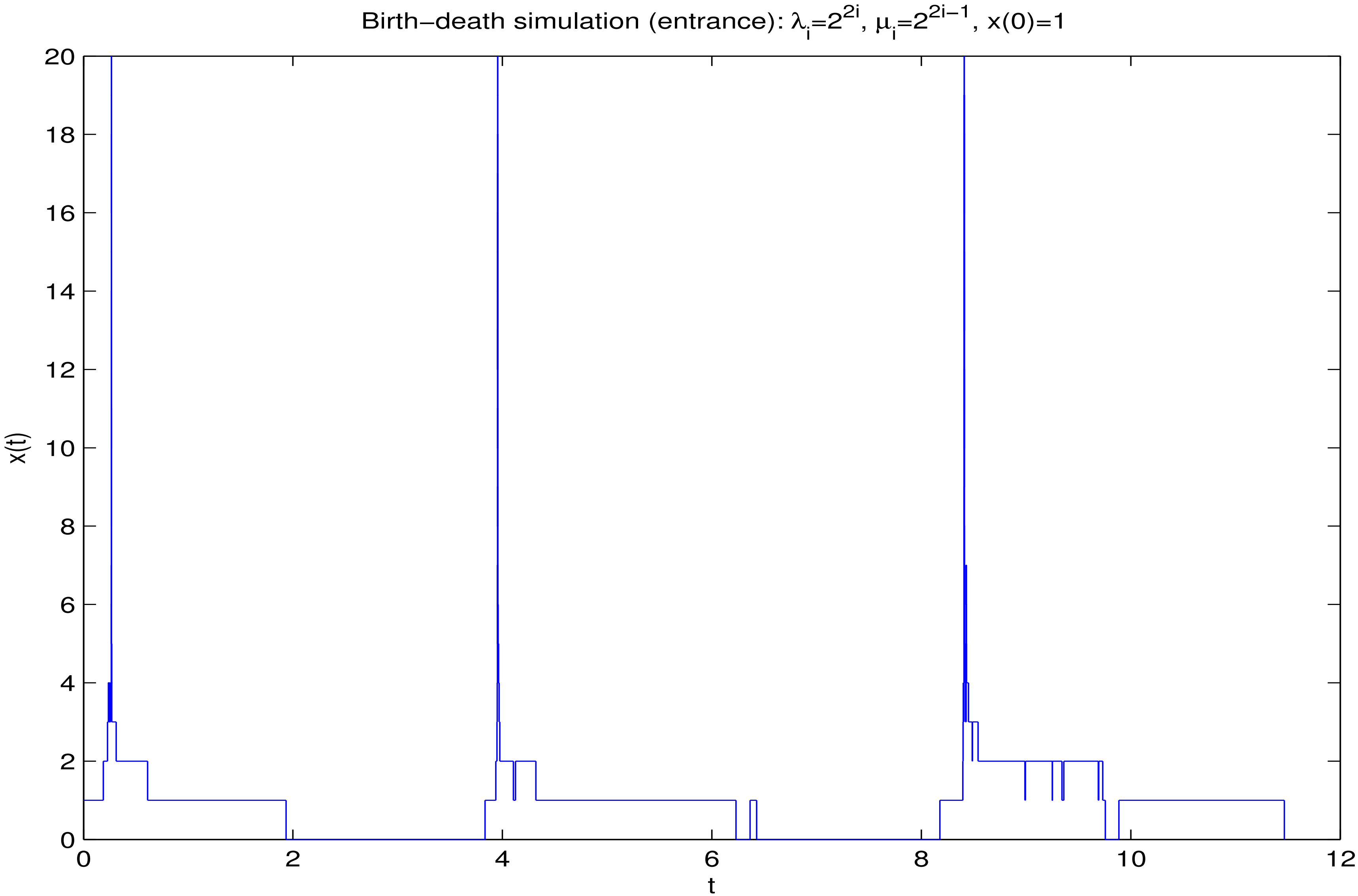
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Simulation



Birth−death simulation (entrance): $\lambda_i=2^{2i}$, $\mu_i=2^{2i-1}$, x(0)=1
x(t)
t
0
2
4
6
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Theorem. Let P be an arbitrary Q-process. If m is invariant for P, then m is subinvariant for Q:

$$\sum_i m_i p_{ij}(t) = m_j \quad \Rightarrow \quad \sum_i m_i q_{ij} \le 0$$

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Theorem. Let P be an arbitrary Q-process. If m is invariant for P, then m is subinvariant for Q, and invariant for Q *if and only if* P satisfies the forward equations FE_{ij} over S:

$$\left(\sum_i m_i p_{ij}(t) = m_j \quad \Rightarrow \quad \sum_i m_i q_{ij} = 0 \right) \quad \Leftrightarrow \quad \mathrm{FE}$$

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Theorem. Let P be an arbitrary Q-process. If m is invariant for P, then m is subinvariant for Q, and invariant for Q *if and only if* P satisfies the forward equations FE_{ij} over S.

Corollary. If m is invariant for the minimal process F, then m is invariant for Q.

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Problem 4. In the case of non-uniqueness, can one identify all Q-processes (or perhaps all honest Q-processes) for which m is invariant?

The resolvent

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for the Laplace transform of $p_{ij}(\cdot)$, then
$\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$ enjoys the following properties:

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- $\psi_{ij}(\lambda) \geq 0$

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- $\psi_{ij}(\lambda) \geq 0$, $\sum_j \lambda \psi_{ij}(\lambda) \leq 1$, and
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- $\psi_{ij}(\lambda) - \psi_{ij}(\mu) + (\lambda - \mu) \sum_k \psi_{ik}(\lambda)\psi_{kj}(\mu) = 0$.

Ψ is called the *resolvent* of P. Indeed, if Ψ is a given resolvent, in that it satisfies these properties, then there exists a *standard* (!) process P with Ψ as its resolvent*.

*Reuter, G.E.H. (1967) Note on resolvents of denumerable submarkovian processes. *Z. Wahrscheinlichkeitstheorie* 9, 16–19.

Identifying Q-processes

Now, if one is given a stable and conservative q-matrix Q, and a resolvent Ψ satisfying the *backward equations*,

$$\lambda\psi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik}\psi_{kj}(\lambda), \qquad \lambda > 0,$$

then Ψ determines a standard *Q-process*:

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$$\lambda\psi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik}\psi_{kj}(\lambda), \qquad \lambda > 0,$$

then Ψ determines a standard *Q-process*: as $\lambda \to \infty$,

- $\lambda\psi_{ij}(\lambda) \to \delta_{ij}$, and
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One can also use the resolvent to determine whether or not the Q-process is *honest*. This happens if and only if

$$\sum_j \lambda\psi_{ij}(\lambda) = 1, \qquad i \in S,\ \lambda > 0.$$

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Then, if $d = 0$, m is invariant for the minimal Q-process. Otherwise, if $\sum_i d_i(\lambda) \leq \sum_i m_i z_i(\lambda) < \infty$, for all $\lambda > 0$, there exists a Q-process P for which m is invariant.

Existence of a Q-process

Theorem continued. The resolvent of one such process is given by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)}, \tag{2}$$

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Corollary. If m is a subinvariant *probability distribution* for Q, then there exists an honest Q-process with stationary distribution m. The resolvent of one such process is given by (2).

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is *necessary* for the existence of a Q-process for which the specified measure is invariant; the Q-process is then determined *uniquely* by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda\sum_k m_k z_k(\lambda)}.$$

Non-uniqueness

Consider a *pure-birth process* with strictly positive birth rates $(q_i, i \geq 0)$, but imagine that we have *two distinct* sets of birth rates, $(q_i^{(0)}, i \geq 0)$ and $(q_i^{(1)}, i \geq 0)$, which satisfy $\sum_{i=0}^{\infty} 1/q_i^{(r)} < \infty$, $r = 0, 1$.

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$$q_{(r,i)(s,j)} = \begin{cases} q_i^{(r)}, & \text{if } j = i + 1 \text{ and } s = r, \\ -q_i^{(r)}, & \text{if } j = i \text{ and } s = r, \\ 0, & \text{otherwise}, \end{cases}$$

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for $r = 0, 1$ and $i \geq 0$. The measure $m = (m_x, x \in S)$, given by $m_{(r,i)} = 1/q_i^{(r)}$, $r = 0, 1, \; i \geq 0$, is subinvariant for Q.

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$$\psi_{(r,i)(s,j)}(\lambda) = \begin{cases} \phi_{ij}^{(r)}(\lambda) + \dfrac{z_i^{(r)}(\lambda)z_0^{(1-r)}(\lambda)\phi_{0j}^{(r)}(\lambda)}{1-z_0^{(0)}(\lambda)z_0^{(1)}(\lambda)}, & s = r \\[2em] \dfrac{z_i^{(r)}(\lambda)\phi_{0j}^{(1-r)}(\lambda)}{1-z_0^{(0)}(\lambda)z_0^{(1)}(\lambda)}, & s \neq r. \end{cases}$$

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Moreover, P is reversible with respect to m in that $m_i p_{ij}(t) = m_j p_{ji}(t)$ (or, equivalently, $m_i \psi_{ij}(\lambda) = m_j \psi_{ji}(\lambda)$).

*Hou Chen-Ting and Chen Mufa (1980) Markov processes and field theory. *Kexue. Tongbao* 25, 807–811.

Birth-death processes

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μ-Invariance

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It is called a *μ-invariant measure for P*, where P is any transition function, if

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Quasi-stationary distributions

Proposition. A probability distribution $\pi = (\pi_i, i \in C)$ is a μ-invariant measure for some $\mu > 0$, that is,

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if and only if it is a *quasi-stationary distribution*: for $j \in C$,

$$p_j(t) = \sum_{i \in C} m_i p_{ij}(t) \Rightarrow \frac{p_j(t)}{\sum_{k \in C} p_k(t)} = m_j.$$

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Theorem. If m is μ-invariant for Q, then it is μ-invariant for F *if and only if* the equations $\sum_{i \in C} y_i q_{ij} = -\nu y_j$, $0 \le y_i \le m_i$, $i \in C$, have no non-trivial solution for some (and then all) $\nu < \mu$.

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Theorem. If m is a *finite* μ-invariant measure for Q, then

$$\mu \sum_{i \in C} m_i a_i^F \leq \sum_{i \in C} m_i q_{i0}, \qquad (4)$$

where $a_i^F = \lim_{t \to \infty} f_{i0}(t)$, and m is μ-invariant for F *if and only if* equality holds in (4).

Q-processes with a given m

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The resolvent Ψ of any Q-process for which m is μ-invariant must be of the form

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where $d_j(\lambda) = m_j - \sum_{i \in C} m_i (\lambda + \mu) \phi_{ij}(\lambda), \quad j \in C$,

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then *all* Q-processes for which m is μ-invariant can be constructed in this way by varying e in the range

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Exactly one of these is honest

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