Limits Theorems for Population Networks with Patch Dependent Extinction Probabilities

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AUSTRALIAN RESEARCH COUNCIL
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Collaborator

Ross McVinish
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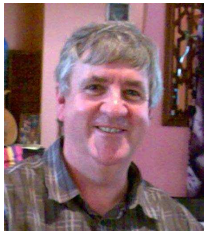
Networks at UQ

Phil Pollett

Mathematical modelling, stochastic process theory and applications: ecology, epidemiology, parasitology, telecommunications and chemical kinetics.



A current project: *Limits theorems for population networks with patch dependent extinction probabilities*.

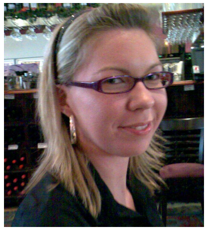
Ross McVinish

Lévy processes and stochastic processes displaying long memory, Bayesian nonparametrics, computation for Bayesian statistics and time series analysis.

A current project: *Estimation in partially observed large metapopulations*.

PhD Projects

Fionnuala Buckley (April 2007 –)



Discrete-time Stochastic Metapopulation Models

Robert Cope (July 2009 –)

Animal Movement Between Populations Deduced from Family Trees

Dejan Jovanović (March 2009 –)

Fault Detection in Complex and Distributed Systems

PhD Projects

Aminath Shausan (July 2010 –)

Stochastic Models for Epidemics in Population Networks

Andrew Smith (July 2009 –)

Models for Spatially Structured Metapopulations

Metapopulations

- A metapopulation is a population that is confined to a network of geographically separated habitat patches (for example a group of islands).
- Individual patches may suffer local extinction.
- Recolonization can occur through dispersal of individuals from other patches.

A Stochastic Patch Occupancy Model (SPOM)

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Suppose that there are n patches.

Let $X_t^{(n)} = (X_{1,t}^{(n)}, \ldots, X_{n,t}^{(n)})$, where $X_{i,t}^{(n)}$ is a binary variable indicating whether or not patch i is occupied.

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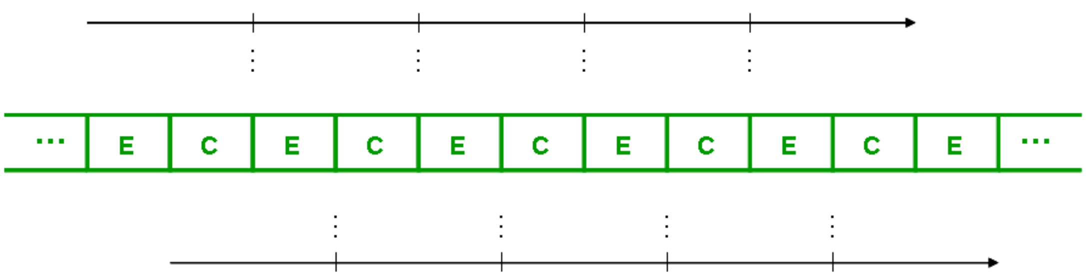
Colonization and extinction happen in distinct, successive phases.



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Thus, we have a *Chain Bernoulli* structure:

$$X_{i,t+1}^{(n)} \stackrel{d}{=} \textit{Bin}\left(X_{i,t}^{(n)} + \textit{Bin}\left(1 - X_{i,t}^{(n)}, f\left(\frac{1}{n}\sum_{j=1}^{n} X_{j,t}^{(n)}\right)\right), S_i\right)$$

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Notation: $\textit{Bin}(m, p)$ is a binomial random variable with m trials and success probability p.

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Compare this with the *homogenous case*, where $S_i = s$ (non-random) is the same for each i, and we merely count the *number* $N_t^{(n)}$ of occupied patches at time t.

We have the following *Chain Binomial* structure:

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A deterministic limit

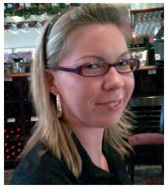
Theorem* If $N_0^{(n)}/n \xrightarrow{p} x_0$ (a constant), then

$$N_t^{(n)}/n \xrightarrow{p} x_t \quad \text{for all } t \geq 1,$$

where (x_t) is determined by

$$x_{t+1} = s(x_t + (1 - x_t)f(x_t)).$$

*Buckley, F.M. and Pollett, P.K. (2010) Limit theorems for discrete-time metapopulation models. Probability Surveys 7, 53-83.



Stability

$$x_{t+1} = s(x_t + (1 - x_t)f(x_t))$$

- *Stationarity*: $f(0) > 0$. There is a unique fixed point $x^* \in [0,1]$. It satisfies $x^* \in (0,1)$ and is stable.
- *Evanescence*: $f(0) = 0$ and $1 + f'(0) \leq 1/s$. Now 0 is the unique fixed point in $[0,1]$. It is stable.
- *Quasi stationarity*: $f(0) = 0$ and $1 + f'(0) > 1/s$. There are two fixed points in $[0,1]$: 0 (unstable) and $x^* \in (0,1)$ (stable).

A deterministic limit

Returning to the general case, where patch survival probabilities are random and patch dependent, and we keep track of which patches are occupied ...

$$X_{i,t+1}^{(n)} \stackrel{d}{=} \textit{Bin}\Big(X_{i,t}^{(n)} + \textit{Bin}\Big(1 - X_{i,t}^{(n)}, f\Big(\tfrac{1}{n}\textstyle\sum_{j=1}^{n} X_{j,t}^{(n)}\Big)\Big), S_i\Big)$$

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First, $\ldots$

Notation: If σ is a probability measure on $[0, 1)$ and let $\bar{s}_k$ denote its k-th moment, that is,

$$\bar{s}_k = \int_0^1 \lambda^k \sigma(d\lambda).$$

A deterministic limit

Theorem Suppose that there is a probability measure σ and deterministic sequence $\{d(0,k)\}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} S_i^k \xrightarrow{p} \bar{s}_k \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} S_i^k X_{i,0}^n \xrightarrow{p} d(0,k)$$

for all $k = 0, 1, \ldots, T$. Then, there is a (deterministic) triangular array $\{d(t,k)\}$ such that, for all $t = 0, 1, \ldots, T$ and $k = 0, 1, \ldots, T - t$,

$$\frac{1}{n}\sum_{i=1}^{n} S_i^k X_{i,t}^n \xrightarrow{p} d(t,k),$$

where

$$d(t+1,k) = d(t,k+1) + f\left(d(t,0)\right)\left(\bar{s}_{k+1} - d(t,k+1)\right).$$

Remarks

- Typically, we are only interested in $d(t,0)$, being the asymptotic proportion of occupied patches.
- However, we may still interpret the ratio $d(t,k)/d(t,0)$ $(k \geq 1)$ as the k-th moment of the conditional distribution of the patch survival probability given that the patch is occupied. (From these moments, the conditional distribution could then be reconstructed.)

Remarks

- When $\bar{s}_k = \bar{s}_1^k$ for all k, that is the patch survival probabilities are the same, then it is possible to simplify

$$d(t+1,k) = d(t,k+1) + f\left(d(t,0)\right)\left(\bar{s}_{k+1} - d(t,k+1)\right).$$

We can show by induction that $d(t,k) = \bar{s}_1^k x_t$, where

$$x_{t+1} = \bar{s}_1\left(x_t + (1-x_t)\, f(x_t)\right).$$

(Compare this with the earlier result.)

Stability

Theorem The fixed points are given by

$$d(k) = \int_0^1 \frac{f(\psi)\lambda^{k+1}}{1-\lambda+f(\psi)\lambda}\sigma(d\lambda),$$

where ψ solves

$$R(\psi) = \int_0^1 \frac{f(\psi)\lambda}{1-\lambda+f(\psi)\lambda}\sigma(d\lambda) = \psi. \qquad (1)$$

If $f(0) > 0$, there exists a unique $\psi > 0$. If $f(0) = 0$ and

$$f'(0) \int_0^1 \frac{\lambda}{1-\lambda}\sigma(d\lambda) \leq 1,$$

then $\psi = 0$ is the unique solution to (1). Otherwise, (1) has two solutions, one of which is $\psi = 0$.

Stability

Theorem If $f(0) = 0$ and

$$f'(0) \int_0^1 \frac{\lambda}{1-\lambda} \sigma(d\lambda) \leq 1,$$

then $d(k) \equiv 0$ is a stable fixed point. Otherwise, the non-zero solution to

$$R(\psi) = \int_0^1 \frac{f(\psi)\lambda}{1-\lambda+f(\psi)\lambda} \sigma(d\lambda) = \psi$$

provides the stable fixed point through

$$d(k) = \int_0^1 \frac{f(\psi)\lambda^{k+1}}{1-\lambda+f(\psi)\lambda} \sigma(d\lambda).$$