MODELLING BISTABILITY IN TELECOMMUNICATIONS NETWORKS

by

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A SYMMETRIC FULLY-CONNECTED NETWORK

N - nodes

$K = \binom{N}{2} = \frac{1}{2}N(N-1)$ - links

C - circuits on each link

Poisson traffic offered at rate ν

Exponentially distributed holding times (mean 1)

THE GHK MODEL

Let $n_j^{(K)}(t)$ be the number of links with j circuits in use at time t (for a network with K links).

Let $n^{(K)} = (n_0^{(K)}, n_1^{(K)}, \ldots, n_C^{(K)})$. Then, $(n^{(K)}(t),\ t \geq 0)$ is a continuous-time Markov chain which takes values in

$$S^{(K)} = \left\{ n \in \{0, 1, \ldots\}^{C+1} \ : \ \sum_{i=0}^{C} n_i = K \right\}.$$

The transition rates, $Q^{(K)} = (q^{(K)}(n, n'), n, n' \in S^{(K)})$, of the process are given by

$$q^{(K)}(n, n + e_{j+1} - e_j) = \nu n_j, \qquad 0 \le j \le C - 1,$$

$$q^{(K)}(n, n + e_{j-1} - e_j) = j n_j, \qquad 1 \le j \le C,$$

$$q^{(K)}(n, n + e_{i+1} - e_i + e_{j+1} - e_j) = \nu n_C \frac{n_i n_j}{\binom{K}{2}},$$

$$i > j, 0 \le i, j \le C - 1,$$

$$q^{(K)}(n, n + 2(e_{j+1} - e_j)) = \nu n_C \frac{\binom{n_j}{2}}{\binom{K}{2}},$$

$$0 \le j \le C - 1,$$

where e_i is the unit vector with 1 as its i^{th} entry.

THE BEHAVIOUR AS THE NETWORK BECOMES LARGE (Theorem 4.1)

Let $X^{(K)} = (X_0^{(K)}, X_1^{(K)}, \ldots, X_C^{(K)})$, where $X_j^{(K)}(t) = n_j^{(K)}(t)/K$ is the *proportion* of links with j circuits in use at time t.

If, as $K \to \infty$, $X^{(K)}(0) \Rightarrow x_0$, then $X^{(K)}(\cdot) \Rightarrow X(\cdot, x_0)$, where $(X(t, x_0),\ t \geq 0)$ is a deterministic process with initial point $X(0, x_0) = x_0$ and which satisfies

$$\frac{d}{dt}X_0(t) = X_1(t) - (\nu + \lambda(t))X_0(t),$$

$$\frac{d}{dt}X_j(t) = (\nu+\lambda(t))X_{j-1}(t) - (\nu+\lambda(t)+j)X_j(t)$$

$$+(j+1)X_{j+1}(t), \quad 1 \leq j \leq C-1,$$

$$\frac{d}{dt}X_C(t) = (\nu + \lambda(t))X_{C-1}(t) - CX_C(t),$$

where $\lambda(t) = 2\nu X_C(t)(1 - X_C(t))$.

THE EQUILIBRIUM POINTS

If $x_0 = (p_0, p_1, \ldots p_C)$ is an equilibrium point it must be of the form given by

$$p_j = \frac{\xi^j}{j!}\left(\sum_{i=0}^{C}\frac{\xi^i}{i!}\right)^{-1}, \qquad 0 \le j \le C,$$

where ξ solves

$$\xi = \nu + 2\nu E(\xi, C)\left(1 - E(\xi, C)\right).$$

The quantity $E(\xi, C)$, given by

$$E(\xi, C) = \frac{\xi^C}{C!}\left(\sum_{i=0}^{C}\frac{\xi^i}{i!}\right)^{-1},$$

is Erlang's formula for the loss probability of a single link with C circuits and with Poisson traffic offered at rate ξ. It is usually more convenient to calculate the equilibrium points by setting $b = E(\xi, C)$ and solving the equation

$$b = E\left(\nu + 2\nu b(1-b), C\right).$$

MODELLING RANDOM FLUCTUATIONS

The central limit law (Theorem 4.2) shows that the random fluctuations about any given equilibrium point, x_0, are Gaussian. Moreover, it shows that these fluctuations can be approximated by an Ornstein-Uhlenbeck (OU) process.

Let x_0 be an equilibrium point. Then, if

$$\lim_{K \to \infty} \sqrt{K} \left(X^{(K)}(0) - x_0 \right) = z,$$

the family of processes $\{Z^{(K)}(\cdot)\}$, defined by

$$Z^{(K)}(s) = \sqrt{K} \left(X^{(K)}(s) - x_0 \right), \quad 0 \le s \le t,$$

converges weakly to an Ornstein-Uhlenbeck process with initial value $Z(0) = z$ and with a local drift matrix, B, and a local covariance matrix, G, which can be determined from the parameters of the model.

In particular, $Z(s)$ is normally distributed with mean

$$\mu_s = e^{Bs} z$$

and covariance matrix

$$\Sigma_s = \int_0^s e^{Bu} G e^{B^T u} du = \Sigma - e^{Bs} \Sigma e^{B^T s},$$

where Σ, the stationary covariance matrix, satisfies

$$B\Sigma + \Sigma B^T + G = 0.$$

We can conclude that, for K large, $X^{(K)}(s)$ has an approximate normal distribution for each s, and an approximation for the mean and the covariance matrix of $X^{(K)}(s)$ is given by

$$\mathsf{E} X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0)$$

and

$$\mathrm{Cov}\, X^{(K)}(s) \simeq K^{-1} \left(\Sigma - e^{Bs} \Sigma e^{B^T s} \right).$$