Identifying Markov chains with a given invariant measure

by

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PRELIMINARIES

State-space: $S = \{0, 1, \dots\}$

Transition functions: A set of real-valued functions $P(\cdot) = (p_{ij}(\cdot),\ i, j \in S)$ defined on $(0, \infty)$ is called a *process* (or *transition function*) if

- $p_{ij}(t) \geq 0$,
- $\sum_j p_{ij}(t) \leq 1$, and
- $p_{ij}(s+t) = \sum_k p_{ik}(s) p_{kj}(t)$.

It is called *standard*

- $\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij}$

and *honest* if

- $\sum_j p_{ij}(t) = 1$,

for some (and then for all) $t > 0$.

THE Q-MATRIX

For a standard process P the right-hand derivative $p'_{ij}(0+) = q_{ij}$ exists and defines a *q-matrix* $Q = (q_{ij}, i, j \in S)$ whose entries satisfy

- $0 \leq q_{ij} < \infty, \quad j \neq i$, and
- $\sum_{j \neq i} q_{ij} \leq -q_{ii} \leq \infty$.

We set $q_i = -q_{ii}$, $i \in S$.

Suppose that Q is given. Assume that Q is *stable*, that is $q_i < \infty$ for all i in S. A standard process P will then be called a *Q-process* if its q-matrix is Q.

THE KOLMOGOROV DIFFERENTIAL EQUATIONS

For simplicity, we shall assume that Q is *conservative*, that is

$$\sum_{j \neq i} q_{ij} = q_i$$

for all $i \in S$. Under this condition, every Q-process P satisfies the *backward equations*,

$$\text{BE}_{ij} \qquad {p_{ij}}'(t) = \sum_k q_{ik} p_{kj}(t), \quad t > 0,$$

but might not satisfy the *forward equations*,

$$\text{FE}_{ij} \qquad {p_{ij}}'(t) = \sum_k p_{ik}(t) q_{kj}, \quad t > 0.$$

STATIONARY DISTRIBUTIONS

A collection of positive numbers $\pi = (\pi_j, j \in S)$ is a *stationary distribution* if $\sum_j \pi_j = 1$ and

$$\sum_i \pi_i p_{ij}(t) = \pi_j. \tag{2}$$

Recipe: Find a collection of strictly positive numbers $m = (m_j, j \in S)$ such that

$$\sum_i m_i q_{ij} = 0.$$

Such an m is called an *invariant measure for Q*. If $\sum_i m_i < \infty$, then we set

$$\pi_j = m_j / \sum_i m_i$$

and *hope* that π satisfies (2).

BIRTH-DEATH PROCESSES

Transition rates:

$q_{i,i+1} = \lambda_i$ ($\uparrow$ - *birth rates*)

$q_{i,i-1} = \mu_i$ ($\downarrow$ - *death rates*) ($\mu_0 = 0$)

Solve: $\sum_{i \geq 0} m_i q_{ij} = 0$, $j \geq 0$, that is,

$$-m_0\lambda_0 + m_1\mu_1 = 0,$$

and, for $j \geq 1$,

$$m_{j-1}\lambda_{j-1} - m_j(\lambda_j + \mu_j) + m_{j+1}\mu_{j+1} = 0.$$

Solution: $m_0 = 1$ and

$$m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.$$

MILLER'S EXAMPLE

Rates: fix $r > 0$ and set

$$\lambda_i = r^{2i}, \quad i \geq 0,$$

$$\mu_i = r^{2i-1}, \quad i \geq 1.$$

Solution: $m_0 = 1$ and

$$m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.$$

So,

$$m_j = \rho^j, \qquad \text{where } \rho = 1/r,$$

and hence if $r > 1$,

$$\pi_j = (1 - \rho)\rho^j, \quad j \geq 0.$$

WHAT IS GOING WRONG?

Rates:

$$\lambda_j = r^{2j}, \quad j \geq 0,$$
$$\mu_j = r^{2j-1}, \quad j \geq 1.$$

The relative proportion of births to deaths is r and so, if $r > 1$, the "process" is clearly *transient*.

In fact, the "process" is *explosive*: Q is not regular. R.G. Miller* showed that Q needs to be regular for the recipe to work.

*Miller, R.G. Jr. (1963) Stationary equations in continuous time Markov chains. *Trans. Amer. Math. Soc.* 109, 35–44.

THE MOTIVATING QUESTION

If Q is regular, then there exists uniquely a Q-process, namely the minimal process: the minimal solution $F(\cdot) = (f_{ij}(\cdot),\ i,j \in S)$ to BE$_{ij}$.

If Q is not regular there are infinitely many Q-processes, infinitely many of which are honest.

Question: Suppose that there exists a collection of strictly positive numbers $\pi = (\pi_j, j \in S)$ such that

$$\sum_i \pi_i = 1 \quad \text{and} \quad \sum_i \pi_i q_{ij} = 0.$$

Does π admit an interpretation as a stationary distribution for any of these processes?

AN INVARIANCE RESULT

Let $m = (m_i, i \in S)$ be a collection of strictly positive numbers. We call m a *subinvariant measure for Q* if

$$\sum_i m_i q_{ij} \le 0,$$

and an *invariant measure for Q* if

$$\sum_i m_i q_{ij} = 0.$$

It is called an *invariant measure for P* if

$$\sum_i m_i p_{ij}(t) = m_j.$$

Theorem: Let P be an arbitrary Q-process. If m is invariant for P, then m is subinvariant for Q, and invariant for Q *if and only if* P satisfies the forward equations FE$_{ij}$ over S.

Corollary: If m is invariant for the minimal process F, then m is invariant for Q.

A CONSTRUCTION PROBLEM

Suppose that Q is a stable and conservative q-matrix, and that m is subinvariant for Q.

Problem 1: Does there exist a Q-process for which m is invariant?

Problem 2: Does there exist an *honest* Q-process for which m is invariant?

Problem 3: When such a Q-process exists, is it unique?

Problem 4: In the case of non-uniqueness, can one identify all Q-processes (or perhaps all honest Q-processes) for which m is invariant?

THE RESOLVENT

Let P be a transition function. If we write

$$\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt, \qquad \lambda > 0,$$

for the Laplace transform of $p_{ij}(\cdot)$, then $\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$ enjoys the following (analogous) properties:

- $\psi_{ij}(\lambda) \geq 0$,
- $\sum_j \lambda \psi_{ij}(\lambda) \leq 1$, and
- $\psi_{ij}(\lambda) - \psi_{ij}(\mu) + (\lambda - \mu) \sum_k \psi_{ik}(\lambda)\psi_{kj}(\mu) = 0$.

Ψ is called the *resolvent* of P. Indeed, if Ψ is a given resolvent, in that it satisfies these properties, then there exists a *standard* (!) process P with Ψ as its resolvent*.

*Reuter, G.E.H. (1967) Note on resolvents of denumerable submarkovian processes. *Z. Wahrscheinlichkeitstheorie* 9, 16–19.

IDENTIFYING Q-PROCESSES USING RESOLVENTS

Now, if one is given a stable and conservative q-matrix Q, together with a resolvent Ψ which satisfies the *backward equations*,

$$\lambda\psi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik}\psi_{kj}(\lambda), \qquad \lambda > 0,$$

then Ψ determines a standard *Q-process*. In particular, as $\lambda \to \infty$,

- $\lambda\psi_{ij}(\lambda) \to \delta_{ij}$, and
- $\lambda(\lambda\psi_{ij}(\lambda) - \delta_{ij}) \to q_{ij}$.

One can also use the resolvent to determine whether or not the Q-process is *honest*, for this happens when and only when

$$\sum_j \lambda\psi_{ij}(\lambda) = 1, \qquad i \in S,\ \lambda > 0.$$

IDENTIFYING INVARIANT MEASURES USING RESOLVENTS

Theorem: Let P be an arbitrary process and let Ψ be its resolvent. Then, m is invariant for P if and only if it is invariant for Ψ, that is,

$$\sum_i m_i p_{ij}(t) = m_j$$

if and only if

$$\sum_i m_i \lambda \psi_{ij}(\lambda) = m_j.$$

THE EXISTENCE OF A Q-PROCESS WITH A GIVEN INVARIANT MEASURE

Theorem: Let Q be a stable and conservative q-matrix, and suppose that m is a subinvariant measure for Q. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$ be the resolvent of the minimal Q-process and define $z(\cdot) = (z_i(\cdot), i \in S)$ and $d(\cdot) = (d_i(\cdot), i \in S)$ by

$$z_i(\lambda) = 1 - \textstyle\sum_j \lambda\phi_{ij}(\lambda),$$

and

$$d_i(\lambda) = m_i - \textstyle\sum_j m_j\lambda\phi_{ji}(\lambda).$$

Then, if $d = 0$, m is invariant for the minimal Q-process. Otherwise, if

$$\textstyle\sum_i d_i(\lambda) \le \sum_i m_i z_i(\lambda) < \infty,$$

for all $\lambda > 0$, there exists a Q-process P for which m is invariant.

Theorem continued: The resolvent of one such process is given by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)}, \tag{4}$$

and this process is honest if and only if

$$\sum_i d_i(\lambda) = \sum_i m_i z_i(\lambda),$$

for all $\lambda > 0$. A sufficient condition for there to exist an honest Q-process for which m is invariant is that m satisfies

$$\sum_j m_j (1 - \lambda \phi_{jj}(\lambda)) < \infty,$$

for all $\lambda > 0$.

Corollary: If m is a subinvariant *probability distribution* for Q, then there exists an honest Q-process with stationary distribution m. The resolvent of one such process is given by (4).

THE SINGLE-EXIT CASE

Suppose that Q is a *single-exit* q-matrix, that is, the space of bounded, non-negative vectors $\xi = (\xi_i, i \in S)$ which satisfy

$$\sum_j q_{ij}\xi_j = \alpha\xi_i, \qquad \alpha > 0,$$

has dimension 1. (The minimal process has only one available "escape route" to infinity). Then, the condition

$$\textstyle\sum_i d_i(\lambda) \leq \sum_i m_i z_i(\lambda) < \infty$$

is *necessary* for the existence of a Q-process for which the specified measure is invariant; the Q-process is then determined *uniquely* by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda\sum_k m_k z_k(\lambda)}.$$

NON-UNIQUENESS

Consider a *pure-birth process* with strictly positive birth rates $(q_i, i \geq 0)$, but imagine that we have *two distinct* sets of birth rates, $(q_i^{(0)}, i \geq 0)$ and $(q_i^{(1)}, i \geq 0)$, which satisfy

$$\sum_{i=0}^{\infty} 1/q_i^{(r)} < \infty, \qquad r = 0, 1.$$

Let $S = \{0, 1\} \times \{0, 1, \ldots\}$ and define $Q = (q_{xy}, x, y \in S)$ by

$$q_{(r,i)(s,j)} = \begin{cases} q_i^{(r)}, & \text{if } j = i + 1 \text{ and } s = r, \\ -q_i^{(r)}, & \text{if } j = i \text{ and } s = r, \\ 0, & \text{otherwise,} \end{cases}$$

for $r = 0, 1$ and $i \geq 0$. The measure $m = (m_x, x \in S)$, given by

$$m_{(r,i)} = 1/q_i^{(r)}, \qquad r = 0, 1, \ i \geq 0,$$

is subinvariant for Q.

NON-UNIQUENESS

The resolvents of *two distinct* Q-processes for which m is invariant are given by

$$\psi_{(r,i)(s,j)}(\lambda) = \delta_{rs}\phi_{ij}^{(r)}(\lambda) + \frac{z_i^{(r)}(\lambda)\phi_{0j}^{(s)}(\lambda)}{2 - \{z_0^{(0)}(\lambda) + z_0^{(1)}(\lambda)\}}$$

and

$$\psi_{(r,i)(s,j)}(\lambda) = \begin{cases} \phi_{ij}^{(r)}(\lambda) + \frac{z_i^{(r)}(\lambda)z_0^{(1-r)}(\lambda)\phi_{0j}^{(r)}(\lambda)}{1-z_0^{(0)}(\lambda)z_0^{(1)}(\lambda)}, & s = r \\ \\ \frac{z_i^{(r)}(\lambda)\phi_{0j}^{(1-r)}(\lambda)}{1-z_0^{(0)}(\lambda)z_0^{(1)}(\lambda)}, & s \neq r. \end{cases}$$

Interpretation: The first process chooses between $(0,0)$ and $(1,0)$ with equal probability as the starting point following an explosion, no matter which was the most recently traversed path, and the second process traverses *alternate paths* following successive explosions.

THE REVERSIBLE CASE

Suppose that Q is *symmetrically reversible* with respect to m, that is, $m_i q_{ij} = m_j q_{ji}$, $i, j \in S$. Then, $d_i(\lambda) = m_i z_i(\lambda)$, and so we arrive at the following corollary due to Hou and Chen*.

Corollary: If Q is reversible with respect to m, then there exists uniquely a Q-function P for which m is invariant *if and only if* $\sum_j m_j z_j(\lambda) < \infty$, for all $\lambda > 0$. It is honest and its resolvent is given by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) m_j z_j(\lambda)}{\lambda \sum_{k \in S} m_k z_k(\lambda)}.$$

Moreover, P is reversible with respect to m in that $m_i p_{ij}(t) = m_j p_{ji}(t)$ (or, equivalently, $m_i \psi_{ij}(\lambda) = m_j \psi_{ji}(\lambda)$).

*Hou Chen-Ting and Chen Mufa (1980) Markov processes and field theory. *Kexue Tongbao* 25, 807–811.

BIRTH-DEATH PROCESSES

Suppose that the birth rates $(\lambda_i, i \geq 0)$ and death rates $(\mu_i, i \geq 1)$ are strictly positive. The q-matrix Q is then regular if and only if

$$\sum_{i=0}^{\infty} \frac{1}{\lambda_i m_i} \sum_{j=0}^{i} m_j = \infty. \tag{6}$$

Proposition: Let $m = (m_i,\ i \in S)$ be the essentially unique invariant measure for Q.

(i) m is invariant for the minimal Q-process if and only if (6) holds.

(ii) When (6) fails, there exists uniquely a Q-process P for which m is invariant if and only if m is finite, in which case P is the unique, honest Q-process which satisfies FE_{ij}; P is positive recurrent and its stationary distribution is obtained by normalizing m.

μ-INVARIANCE

Suppose that $S = \{0\} \cup C$, where 0 is an absorbing state and C is irreducible (for F). Let $\mu \geq 0$. A collection $m = (m_i, i \in C)$ of strictly positive numbers is called a *μ-subinvariant measure for Q* if

$$\sum_{i \in C} m_i q_{ij} \leq -\mu m_j, \quad j \in C,$$

and a *μ-invariant measure for Q* if

$$\sum_{i \in C} m_i q_{ij} = -\mu m_j, \quad j \in C.$$

It is called a *μ-invariant measure for P*, where P is any transition function, if

$$\sum_{i \in C} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in C.$$

Proposition: A *probability distribution* $\pi = (\pi_i, i \in C)$ is a μ-invariant measure for some $\mu > 0$ if and only if it is a *quasistationary distribution*: for $j \in C$,

$$p_j(t) = \sum_{i \in C} m_i p_{ij}(t) \Rightarrow p_j(t) / \sum_{k \in C} p_k(t) = m_j.$$

μ-INVARIANCE FOR F

Theorem: If m is μ-invariant for P, then m is μ-subinvariant for Q, and μ-invariant for Q *if and only if* P satisfies the forward equations *over* C. For example, if m is μ-invariant for the minimal process, then it is μ-invariant for Q.

Theorem: If m is μ-invariant for Q, then it is μ-invariant for F *if and only if* the equations

$$\sum_{i \in C} y_i q_{ij} = -\nu y_j, \quad 0 \leq y_i \leq m_i, \; i \in C,$$

have no non-trivial solution for some (and then all) $\nu < \mu$.

Theorem: If m is a *finite* μ-invariant measure for Q, then

$$\mu \sum_{i \in C} m_i a_i^F \leq \sum_{i \in C} m_i q_{i0}, \tag{8}$$

where $a_i^F = \lim_{t \to \infty} f_{i0}(t)$, and m is μ-invariant for F *if and only if* equality holds in (8).

Q-PROCESSES WITH A GIVEN μ-INVARIANT MEASURE

Theorem: Suppose that Q is single-exit and that m is a *finite* μ-subinvariant measure for Q. Then, there exists a Q-process for which m is μ-invariant if and only if

$$\sum_{i\in C} m_i q_{i0} \le \mu \sum_{i\in C} m_i.$$

The resolvent Ψ of any Q-process for which m is μ-invariant must be of the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{(\lambda+\mu)\sum_{k\in C} m_k z_k(\lambda)}, i,j \in S,$$

where

$$d_j(\lambda) = m_j - \sum_{i\in C} m_i(\lambda+\mu)\phi_{ij}(\lambda), \quad j \in C,$$

$$d_0(\lambda) = e/\lambda - \sum_{i\in C} m_i(\lambda+\mu)\phi_{i0}(\lambda),$$

and e satisfies $\sum_{i\in C} m_i q_{i0} \le e \le \mu \sum_{i\in C} m_i$.

Q-PROCESSES WITH A GIVEN μ-INVARIANT MEASURE

Theorem continued: Conversely, if

$$\sum_{i \in C} m_i q_{i0} \leq \mu \sum_{i \in C} m_i,$$

then *all* Q-processes for which m is μ-invariant can be constructed in this way by varying e in the range

$$\sum_{i \in C} m_i q_{i0} \leq e \leq \mu \sum_{i \in C} m_i.$$

Exactly one of these is honest; this is obtained by setting $e = \mu \sum_{i \in C} m_i$. And, exactly one satisfies the forward equations FE_{i0} over $i \in C$; this is obtained by setting $e = \sum_{i \in C} m_i q_{i0}$.