Given any \( M > 0 \), there exists \( N > 0 \) such that \( a_n > M \) for all \( n > N \). \( (a_n < -M \) for all \( n > N) \)

Require \( \frac{n^3 + 1}{n} > M \) for \( n > N \).

Well, if \( \frac{n^3 + 1}{n} = \frac{n}{M} \), take \( N = M \).

Proof by induction: \( x_1 \geq \frac{x_n}{x_{n-1}} \), so

\[
\frac{z_{n+1}}{z_n} = \frac{x_1 \cdots x_n}{y_1 \cdots y_n} < \frac{x_n}{y_n}.
\]

Suppose that \( \frac{x_1 \cdots x_k}{y_1 \cdots y_k} \leq \frac{x_{k+1}}{y_{k+1}} \leq \frac{x_k}{y_k} \).

Then

\[
\frac{z_{k+1}}{z_k} = \frac{x_1 \cdots x_k + x_{k+1}}{y_1 \cdots y_k + y_{k+1}} \leq \frac{x_k + x_{k+1}}{y_k + y_{k+1}}.
\]

\[
\Rightarrow \frac{x_k + x_{k+1}}{y_k + y_{k+1}} = \frac{z_{k+1}}{z_k}.
\]

\& result follows.

For \( k, n \) arbitrarily large,

\[
\left| x_{nk} - x_n \right| \leq \left| x_{nk} - x_{nk-1} \right| + \left| x_{nk-1} - x_{nk-2} \right| + \cdots + \left| x_{n+1} - x_n \right| \leq \frac{1}{3} \cdot 3^{-k} + \frac{1}{3} \cdot 3^{-k+1} + \cdots + \frac{1}{3} \cdot 3^{-n} = 3^{-n} \left( \frac{1}{3} \cdot 3^{-k} + \cdots + \frac{1}{3} \cdot 3^{-n} \right) = \frac{1}{2} \cdot 3^{-n} \left( 1 - \frac{1}{3} \right) \left( 1 - \frac{1}{3} \right)
\]

So \( \{x_n\} \) is a Cauchy sequence \& converges.

No. First note that one cannot show that \( \{x_n\} \) is bounded above. Second, consider the sequence \( x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}, \ n \geq 0 \).

Then \( \{x_n\} \) is now, i.e., \( x_{n+1} - x_n = (1/n+1) < 1/n \), yet \( \{x_n\} \) is divergent, since \( \Sigma 1/n \) is a divergent series.