Physical Meaning of \textit{div}

\textbf{Fluid Dynamics}: measure of rate of decrease of density at a point:

\[ \nabla \cdot \mathbf{u} = \frac{\partial \rho}{\partial t} \]

The density. Put \( \nabla \cdot \mathbf{u} = \rho \mathbf{u} \)

\[ \nabla \cdot \mathbf{u} = -\frac{\partial \rho}{\partial t} \]

"Continuity equation". Incompressible fluid \( \Rightarrow \nabla \cdot \mathbf{u} = 0 \).

\textbf{Electromagnetic Fields}: electric force vector \( \mathbf{E} \) satisfies

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \]

\( \rho = \text{charge density} \)

No charge \( \Rightarrow \nabla \cdot \mathbf{E} = 0 \).

\textbf{Geometric Definition}: consider at time \( t = 0 \) a box of sides \( \varepsilon_i, \varepsilon_j, \varepsilon_k \)

Field \( \mathbf{F} \) carries box along \( \mathbf{F} \) at time \( t \)

We have a parallelipiped:

\[ \text{Vol} \mathbf{F} (x, y, z) = \frac{1}{V(0)} \frac{d}{dt} V(t) \bigg|_{t=0} \]
Recap of Defns & Difficulties

1. Double Integral

\[ \int \int_D f(x, y) \, dA \]

= \[ \int_a^b \int_{B(x)}^{C(x)} f(x, y) \, dy \, dx \]

= \[ \int_c^d \int_{E(y)}^{G(y)} f(x, y) \, dx \, dy \]

What it is: # area of a region if \( f = 1 \)

# Volume over Dunder

\[ z = f(x, y) \]

Difficulties: # Seeing what the region is from \( \int_a^b \int_{B(x)}^{C(x)} \) so as to find the limits \( E(y) \) & \( G(y) \) to reverse order of integration
2. **Triple Integrals**

\[ \iiint_R f(x, y, z) \, dV \]

**What it is:**
- Volume of the region \( R \) if \( f = 1 \)
- Centre of mass, moment of inertia, other physically important quantities.

**Difficulties:**
- Working out bounding surfaces
  
  e.g. \( x^2 + y^2 = ax + by \)

- Visualising the region \( R \)
  
  e.g. \( y^2 + y^2 = by \) & \( x^2 + y^2 + z^2 = b^2 \)

- Finding limits in the iterated integral for first integration
  (between surfaces)

- Projecting 3-D volume to a 2-D area for second integration
  (between curves)
3. Change of Variable

\[ \iint_D f(x, y) \, dA_{xy} = \iint_{D'} F(u, v) \left| J \right| \, dA_{uv} \]

What it is: Makes impossible integrals easy to integrate.

Difficulties: What variables to use? e.g. \( x = \cos \theta, y = \sin \theta; u = xy, v = xy \).
* Finding the Jacobian \( J \)
* Visualising \( D' \) in \( uv \) plane & finding limits of integration.

4. Change of Variable

\[ \iiint_R f(x, y, z) \, dV_{xyz} = \iiint_{R'} F(u, v, w) \left| J \right| \, dV_{uvw} \]

What it is: See above.

Difficulties: Use spherical or cylindrical?
* Geometry of \( R \) & form of \( f(x, y, z) \) determines this.
  * Finding \( J \) in \( r \) & \( \theta \) or \( p \)
* Surfaces in polars?
  * Cones, Planes, Paraboloids, Cylinders, Spheres.
5. Vector Calculus

\[ \nabla \phi = \nabla \phi = \phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k} \]

\[ \text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \]

**What they are:** \( \nabla \phi \) is normal to surface \( \phi(x,y,z) = 0 \)

**difficulties:** Born easy to calculate.

6. Line Integrals

\[ \int_C P \, dx + Q \, dy, \quad \int_C F_1 \, dx + F_2 \, dy + F_3 \, dz, \]

\[ \int_C \mathbf{F} \cdot d\mathbf{s} \]

**What they are:** Work done moving along curve \( C \) through the vector field \( \mathbf{F} \).

**Difficulties:** # Parametrizing \( C \)
# Working out the integral.

6. Green's Theorem in Plane

\[ \int_C P \, dx + Q \, dy = \oint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

**What it is:**

**Difficulties:**
\[ \int_{C} (3x^2 - 6yz) \, dx + (2y + 3xz) \, dy + (1-4xy^2) \, dz \]

\[ = \int_{x=0}^{2/2} (3x^2 - 6x^3) \, dt + (2t^3 + 3x^2 t^3) \, dx \, dt \]

\[ + (1-4x \cdot t^4 (t^3)^2) \, 3x^2 \, dx \, dt \]

\[ = 2 \cdot 1 \]

**Physical Interpretations**

# If \( F \) is a force field and a particle \( P \) of unit mass takes the path \( C \) from \( A \) to \( B \),

\[ W = \int_{C} F \cdot dx \]

is the **work done**.

# If \( F \) is a velocity field (e.g. of a fluid flow),

\[ \text{div} F \]

is the flux through a "box" of sides \( dx, dy, dz \).
7. Path Independence

\[ \int_{A}^{B} (P \, dx + Q \, dy) \]

is independent of path taken (path smooth, region has 'no holes')

\[ (P, Q) = \nabla \phi(x, y) \]

**What it is:** Work done against a **conservative field** is independent of path between two points \(A\) and \(B\).

**Differential:** Gradient fields \(\nabla \phi\) are conservative fields.

**Difficulties:** Too many applications in Engineering.
Proof:
\[ C = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \]
\[ \nabla \phi = \phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k} \]
\[ A: t = t_0 \]
\[ B: t = t_f \]
\[ \nabla \phi \cdot dx = (\phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k}) \cdot (\hat{i} \, dt + \hat{j} \, dt + \hat{k} \, dt) \]
\[ = (\phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k}) \, dt \]

By chain rule:
\[ \frac{d}{dt} \phi(x(t), y(t), z(t)) \, dt \]
\[ \Rightarrow \int_C \nabla \phi \cdot dx = \int_{t_0}^{t_f} \left( \frac{d}{dt} \phi \right) \, dt \]
\[ = \phi(x(t_f)) - \phi(x(t_0)) \]
\[ = \phi(B) - \phi(A) \]
A vector field \( \mathbf{F}(x, y, z) \) is a conservative field if it is a gradient, \( \mathbf{F} = \nabla \phi \).

\( \phi \) is called a potential function.

Example \( \phi(x, y, z) = g R^r / r \)

\[ \nabla \phi = gR^r \nabla \left( \frac{1}{r} \right) \]

\[ = -\frac{gR^r}{r^2} \nabla r = -\frac{gR^r}{r^2} \left( \nabla \sqrt{x^2+y^2+z^2} \right) \]

\[ = -\frac{gR^r}{r^3} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \]

Recall that we showed before that \( \text{div} \left( -\frac{gR^r}{r^3} \mathbf{r} \right) = 0 \).

That is \( \nabla \cdot \nabla \phi = 0 \)

\[ \nabla^2 \phi = \frac{2}{\partial x^2} + \frac{2}{\partial y^2} + \frac{2}{\partial z^2} \]

Laplace's equation