MT 253/256 Solutions 8

2. $u_{tt} = u_{xx}$, $u(x,0) = 0.01\sin 3x$, $u_t(x,0) = 0$,
$(L = \pi)$ $u(0,t) = u(L,t) = 0$, all t.

(I). $u(x,t) = F(x)\,G(t)$ separation of variables

$$u_{tt} = F\ddot{G} = F''G = u_{xx}$$

$$\ddot{G}/G = F''/F$$

LHS depends only on t (not x), RHS depends only on x (not t). For equality to hold, each must be constant

$$\ddot{G}/G = F''/F = k, \quad k \neq 0 \text{ here (otherwise, zero soln)}$$

$F'' - kF = 0$, $k = -p^2$; $\ddot{G} - kG = 0$

(II) $\Rightarrow F = A\cos px + B\sin px$ $\qquad k = -p^2 = -n^2$

$F(0) = F(L) = 0$ (bdy condn) $\qquad \ddot{G} + n^2 G = 0$

$\Rightarrow A = 0$ & $B\sin pL = 0$ $\qquad G_n(t) = B_n\cos nt + B_n^*\sin nt$

$\Rightarrow pL = n\pi$, $n = 1, 2, \cdots$

Since $L = \pi$, $p = n$.

$F_n(x) = \sin nx$

$n = 1, 2, \ldots$

So $u_n(x,t) = F_n(x)G_n(t) = (B_n\cos nt + B_n^*\sin nt)\sin nx$.

$n = 1, 2, \cdots$.

(III) $\forall x$, $u(x,0) = 0.01\sin 3x = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{1}^{\infty} B_n\sin nx$.

Since the B_n's are the coefficients of the Fourier sine series of $0.01\sin 3x$ (which is its own sine series),

$$B_n = \begin{cases} 0, & n \neq 3 \\ 0.01, & n = 3. \end{cases}$$

Also, since $u_t(x,0) = 0$ all t, $B_n^* = 0$, $n = 1, 2, \ldots$.

Hence, $u(x,t) = 0.01\cos 3t\sin 3x$.