COLOR LAYER

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PHASE PLANE ANALYSIS

- **Linear Systems:**

  # Classification;

  # Stability.

- **Nonlinear Systems:**

  # Critical points;

  # Linearization;

  # Global picture.

Phase Plane – Linear Systems

Solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ can be drawn as **solution paths** $y_1(t)$, $y_2(t)$ in the **phase plane**. An **equilibrium point** or **critical point** is where $\mathbf{y}' = \mathbf{0}$ and for linear systems is always $\mathbf{0}$. With nonlinear systems, other points apart from the origin can be critical points.

Types are **node, saddle, centre, focus**. Which one depends on the eigenvalues λ_1, λ_2 of $\mathbf{A} = (a_{ij})$, and thus on the characteristic eqn

$$\lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A} = 0.$$

Put $p = a_{11} + a_{22}$, $q = \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ and $\Delta = p^2 - 4q$, so that

$$p = \lambda_1 + \lambda_2, \quad q = \lambda_1\lambda_2$$

and Δ is the discriminant of the quadratic characteristic equation. That is, roots are real if $\Delta \geq 0$ and complex if $\Delta < 0$.

Critical point P of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is a

- **node** if $q > 0$ and $\Delta \geq 0$;

- **saddle point** if $q < 0$;

- **centre** if $p = 0$ and $q > 0$;

- **focus** if $p \neq 0$ and $\Delta < 0$.

To see this: if $q = \lambda_1 \lambda_2 > 0$, the eigenvalues are real when ($\Delta \geq 0$) of the same sign, or complex conjugates. The first gives a node, while if the second and $p = 0$, then the roots are pure imaginary and a centre results. If $q < 0$, eigenvalues are real, of opposite sign and there is a saddle. If $\Delta < 0$ the eigenvalues are complex, not pure imaginary when $p \neq 0$, so a focus.

Stability

A critical point P is **stable** if all solution paths close enough to P remain close for all future time.

P is **stable and attractive** if P is stable and every path sufficiently close to P approaches P as $t \to \infty$.

Say P **unstable** if it is not stable.

The critical point P is:

- **stable and attractive** if $p < 0$ and $q > 0$;

- **stable** if $p \leq 0$ and $q > 0$;

- **unstable** if $p > 0$ or $q < 0$.

$p < 0$, $q > 0 \Rightarrow \lambda_1$, λ_2 both negative or complex conjugate with negative real part, hence stable and attractive node or focus. $p \leq 0$, $q > 0 \Rightarrow \lambda_1$, λ_2 both zero or pure imaginary, hence stable (centre). If $q < 0$, then λ_1, λ_2 are real and of opposite sign (saddle); if $p > 0$, then either λ_1, λ_2 are both real with at least one positive (unstable saddle or node), or they are conjugates with positive real part (unstable focus).

Stability chart:

Example 1

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix} \mathbf{y},$$

where $p = a_{11} + a_{22} = -3,\ q = \det \mathbf{A} = 2,$
$\Delta = p^2 - 4q = 1$. So the critical point is a node, and this is stable and attractive,$(\lambda = -2, -1)$.

Example 2

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -2 & -3 \\ 1 & -1 \end{bmatrix} \mathbf{y},$$

where $p = a_{11} + a_{22} = -3,\ q = \det \mathbf{A} = 5,$
$\Delta = p^2 - 4q = -11$ .So the critical point is a focus, and this is stable and attractive.

Example 3

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & -3 \\ -5 & 3 \end{bmatrix} \mathbf{y},$$

where $p = a_{11} + a_{22} = 4,\ q = \det \mathbf{A} = -9,$
So the critical point is a saddle , and this is unstable, $(\lambda = -2, 6)$.

Example 4 Mass on a spring.

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0, \text{ or, } \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{y}$$

and $p = -c/m,\ q = k/m,\ \Delta = (c/m)^2 - 4k/m.$

No damping: $c = 0,\ p = 0,\ q > 0$, **centre**;

Underdamping: $c^2 < 4mk,\ p < 0,\ q > 0,\ \Delta < 0$, **stable attracting focus**;

Critical damping: $c^2 = 4mk,\ p < 0,\ q > 0,\ \Delta = 0$, **stable attracting node;**

Overdamping: $c^2 > 4mk,\ p < 0,\ q > 0,\ \Delta > 0$, **stable attracting node.**

Phase Plane – Nonlinear Systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix} \tag{1}$$

autonomous system (not involving t explicitly). A point $P:\ (a, b)$ of the phase plane, at which $f_1(a, b) = f_2(a, b) = 0$, is a **critical point** or equilibrium. If (1) has several critical points (a, b), for analysis of each we translate the point to the origin $\mathbf{0}$ by $\widetilde{y_1} = y_1 - a$, $\widetilde{y_2} = y_2 - b$. So, for discussion, we might as well suppose that $\mathbf{0}$ is the critical point.

Near $\mathbf{0}$ we **linearize** the nonlinear system and consider the stability characteristics of the resulting linear system.

Assuming that $f_1(y_1, y_2),\ f_2(y_1, y_2)$ have Taylor expansions around the critical point $P = \mathbf{0}$, they have **no constant terms** because $f_1(\mathbf{0}) = f_2(\mathbf{0}) = 0$ Then (19) becomes

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(y) = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2) \\ a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2) \end{bmatrix}.$$

If $\det \mathbf{A} \neq 0$ **the type and stability of** P **coincides with that of** $\mathbf{0}$ **of the linear system**

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 \\ a_{21}y_1 + a_{22}y_2 \end{bmatrix},$$

provided that $h_1,\ h_2$ are small near $\mathbf{0}$. Except if the eigenvalues of $\mathbf{A}$ are equal or pure imaginary, when the nonlinear system may have a focus.

Example: damped pendulum. mass m, rod of length L. DE $mL\theta'' + mg \sin\theta = 0$. Or,

$$\theta'' + k \sin\theta = 0, \quad k = \frac{g}{L}.$$

Introduce a damping term $c\theta'$:

$$\theta'' + c\theta' + k \sin\theta = 0, \;\; k > 0, \;\; c \geq 0.$$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k \sin y_1 - c y_2 \end{aligned}$$

Critical points $(0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, $\ldots$.
First, $(0,0)$: linearizing $\sin y_1 \approx y_1$,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \mathbf{y}.$$

For undamped ($c = 0$), a centre, for small damping a focus. Since $\sin y_1$ is periodic with period 2π, the critical points $(2n\pi, 0)$, $n = \pm 1, \pm 2, \ldots$ have this same behaviour.
Second, $(\pi, 0)$: set $\theta - \pi = y_1$, $y_1' = \theta' = y_2$ and linearize

$$\sin\theta = \sin(y_1 + \pi) = -\sin y_1 \approx -y_1,$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \mathbf{y}.$$

Now, $p = -c,\ q = -k,\ \Delta = c^2 + 4k$ and

No damping. $c = 0,\ p = 0,\ q < 0,\ \Delta > 0$, which is a **saddle**;

Damping $c > 0,\ p < 0,\ q < 0,\ \Delta > 0$, **saddle**.

Transform to 1st order DE in phase plane

Above, to draw the phase plane trajectories, we eliminated t from $y_1(t)$, $y_2(t)$. Another way is to take $y = y_1$ as the independent variable directly in the DE $F(y, y', y'') = 0$, set $y' = y_2$ and use

$$y'' = y_2' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1}y_2.$$

The result is a first order DE in the phase plane:

$$F\left(y_1, y_2, \frac{dy_2}{dy_1}y_2\right) = 0.$$

Example. Undamped pendulum $\theta'' + k\sin\theta = 0$. Putting $\theta = y_1$, $\theta' = y_2$, $\theta'' = y_2\frac{dy_2}{dy_1}$ gives

$$\frac{dy_2}{dy_1}y_2 = -k\sin y_1 \Rightarrow \frac{1}{2}y_2^2 = k\cos y_1 + C.$$

Additional Example: §4.5 #2. $y'' + y - y^3 = 0$; Find type and stability of the critical points and sketch the solution curves in the phase plane.

Critical Points: First make a system.
$y_1 = y,\ y_2 = y',\ y_2' = y'' = -y + y^3$.

$$\begin{aligned} y_1' &= y_2 = f_1 \\ y_2' &= -y_1 + y_1^3 = f_2. \end{aligned}$$

For critical points, $f_1 = 0,\ f_2 = 0$. So $y_2 = 0$ and $-y_1 + y_1^3 = 0,\ y_1(-1 + y_1^2) = 0$ and $y_1 = 0,\ +1,\ -1$.

$$(0\ 0), \quad (1\ 0), \quad (-1, 0).$$

Linearization: At a critical point $\mathbf{y}^*$ the system is approximated by the
LINEAR SYSTEM $\quad \mathbf{y}' = \mathbf{J}\mathbf{y}$,

$$\mathbf{J} = \begin{bmatrix} \partial f_1/\partial y_1 & \partial f_1/\partial y_2 \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 \end{bmatrix}_{\mathbf{y}^*}$$

That is, $\mathbf{J} = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \end{bmatrix}_{\mathbf{y}^*}$

Study linearization at each $\mathbf{y}^*$:

$$
\begin{aligned}
(0\ 0): \quad \mathbf{J} &= \begin{bmatrix} 0 & 1 \\ -1+3y_1^2 & 0 \end{bmatrix}_{0\ 0} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\end{aligned}
$$

So $p = 0,\ q = 1$, **CENTRE.**

$$
(\pm 1\ 0): \quad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}
$$

Here, $p = 0,\ q = -2$, **SADDLE.** Eigenvalues are $\lambda = \pm\sqrt{2}$, eigenvector eqn $-\lambda x_1 + x_2 = 0$ and get $(1\ \lambda)^T$.

Global Picture: Find the solution curves in the form of $y_2 =$ a function of y_1.

$$\begin{aligned}
\frac{dy_2}{dy_1} &= \frac{f_2(y_1,y_2)}{f_1(y_1,y_2)} \\
&= \frac{-y_1+y_1^3}{y_2} \\
\int y_2\,dy_2 &= \int(-y_1+y_1^3)\,dy_1 + C \\
\frac{1}{2}y_2^2 &= -\frac{1}{2}y_1^2 + \frac{1}{4}y_1^4 + C \\
2y_2^2 &= -2y_1^2 + y_1^4 + 4C \\
&= (1-y_1^2)^2 + 4C - 1, \text{ or} \\
2y_1^2 + 2y_2^2 &= y_1^4 + 4C.
\end{aligned}$$

If y_1 is small, this is approximately like a circle. If y_1 is large, $2y_1^2 - (1-y_1^2)^2 = K$ is approximately like a hyperbola.

van der Pol equation

$$y'' - \mu(1 - y^2)y' + y = 0, \quad \mu = constant > 0$$

Damping coefficient $\mu(1 - y^2) < 0$ for small oscillations, $y^2 < 1$, producing negative damping. It is positive for larger oscillations, $y^2 > 1$, producing positive damping. There must be a periodic solution separating the regions of different damping, called a **limit cycle**.

Put $y = y_1,\ y' = y_2,\ y'' = y_2\, dy_2/dy_1$:

$$\frac{dy_2}{dy_1} y_2 - \mu(1 - y_1^2)y_2 + y_1 = 0.$$

Isoclines are the curves $dy_2/dy_1 = k = const$,

$$\begin{aligned} \frac{dy_2}{dy_1} &= \mu(1 - y_1^2) - \frac{y_1}{y_2} = k \Rightarrow \\ y_2 &= \frac{y_1}{\mu(1 - y_1^2) - k}. \end{aligned}$$

Critical Points: Another Example

Find and classify the critical points of

$$\begin{aligned}\frac{dx}{dt} &= f(x,y) = -6y + 2xy - 8\\ \frac{dy}{dt} &= g(x,y) = \quad y^2 - x^2.\end{aligned}$$

Critical points where $y^2 = x^2$ and $-6y + 2xy - 8 = 0$. From the first, $y = \pm x$. If $y = x$, $2x^2 - 6x - 8 = 0$, and $x = -1,\ 4$. Substituting $y = -x$ gives $2x^2 - 6x + 8 = 0$, which has no real solution. Hence, the only critical points are $(-1,-1)$, $(4,4)$.

The linearised matrix at a critical point $P = (x_0, y_0)$ is

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]_{(x_0,y_0)} = \left[\begin{array}{cc} 2y_0 & 2x_0 - 6 \\ -2x_0 & 2y_0 \end{array}\right].$$

The linearised matrix at $(-1,-1)$ is thus

$$\begin{bmatrix} -2 & -8 \\ 2 & -2 \end{bmatrix},\ p = a + d = -4,\ q = ad - bc = 20,$$

and $\Delta = p^2 - 4q = -64$. This is a **focus**, and it is **stable** because $p < 0$. The linearised matrix at $(4,4)$ is

$$\begin{bmatrix} 8 & 2 \\ -8 & 8 \end{bmatrix},\ p = 16,\ q = 80,\ \Delta = -64$$

and this is also a **focus**, but **unstable** because $p > 0$.

CHAOS

Some systems which occur in physical processes are difficult to analyse because of

- Apparently random behaviour;

- Exceptional sensitivity to initial conditions ("butterfly effect").

These systems behave in a very erratic manner as shown in the drawings below. Also, if a very small error is made in the initial conditions, it is magnified by the system as time goes on and drifts away from the "true" solution at an exponential rate.

Lorenz equations

$$\begin{array}{l} x' = \sigma(y - x) \\ y' = \rho x - y - xz \ , \\ z' = -\beta z + xy \end{array} \qquad \begin{array}{l} (x, y, z) \in \Re^3 \\ \sigma, \rho, \beta > 0. \end{array}$$

This system was used to describe the instability of weather patterns, but was later found to be chaotic for wide ranges of values of the parameters σ, ρ, β.