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LAPLACE TRANSFORM

- **Transform & Inverse Transform;**
- **Transform of Derivatives;**
- **Initial Value Problems;**
- **Transform of Integrals;**
- **Partial Fraction Expansions;**
- **First Shift Theorem;**
- **Unit Step Functions & Second Shift Theorem.**

LAPLACE TRANSFORM. Overview:

Laplace transforms solve DEs and initial value, boundary value problems. They do this by reducing differential equations to **linear algebraic problems**, not involving derivatives.

• DE transformed to an algebraic equation (**subsidiary equation**);

• Subsidiary equation involves the transform of the unknown function **linearly** and is easily solved;

• This solution is transformed back to give the solution of the original problem.

• Direct method: no need for general solution to solve initial value problems; no need to work out homog soln first for nonhomogeneous DEs.

The transformed function

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

is called the **Laplace transform of** f.

The original function $f(t)$ is the **inverse transform**

$$f(t) = \mathcal{L}^{-1}(F).$$

Example 1. $f(t) = 1,\ t \geq 0.$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty 1\dot{e}^{-st}\, dt \;=\; -\left[\frac{-1}{s} e^{-st}\right]_0^\infty$$
$$= \frac{1}{s},\ s > 0.$$

Example 2. $f(t) = e^{-at},\ t \geq 0.$

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt \;=\; \frac{-1}{s+a} e^{-(s-a)t}\Big|_0^\infty$$
$$= \frac{1}{s-a},\ s + a > 0.$$

Linearity. The Laplace transform is a linear transformation: for any functions $f(t)$ and $g(t)$ whose transforms exist, and any constants a, b,

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g).$$

Easy to see:

$$\begin{aligned}\mathcal{L}(af + bg) &= \int_0^\infty e^{-st}(af(t) + bg(t))\,dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= a\mathcal{L}(f) + b\mathcal{L}(g).\end{aligned}$$

Example 3. $f(t) = \sinh at = (e^{at} - e^{-at})/2$.

$$\begin{aligned}\mathcal{L}(\sinh at) &= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) \\ &= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \\ &= \frac{a}{s^2 - a^2},\ s > a \geq 0.\end{aligned}$$

Example 3. $f(t) = \cos t,\ \sin t$.

Recall that

$$e^{i\omega t} = \cos t + i \sin t.$$

Observe that the real part is $\cos t$ and the imaginary part is $\sin t$. Now,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2}.$$

Equating real and imaginary parts and using the linearity of the Laplace transform,

$$\begin{aligned} \mathcal{L}(\cos t) &= \frac{s}{s^2 + \omega^2}\ , \\ \mathcal{L}(\sin t) &= \frac{1}{s^2 + \omega^2}\ . \end{aligned}$$

Example 4. $F(s) = 1/\{(s-a)(s-b)\},\ a \neq b.$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)\right) \\ &= \frac{1}{a-b}\left(e^{at} - e^{bt}\right). \end{aligned}$$

($\mathcal{L}^{-1}$ is also linear!)

Note: PARTIAL FRACTIONS

$$\begin{aligned} \frac{1}{(s-a)(s-b)} &= \frac{A}{s-a} + \frac{B}{s-b} \\ 1 &= A(s-b) + B(s-a), \\ s = a &\Rightarrow 1 = A(a-b) \\ s = b &\Rightarrow 1 = -B(a-b). \end{aligned}$$

Some Laplace Transforms

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$t^n, \; n = 2, \; 3, \ldots$	$\frac{n!}{s^{n+1}}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$t^a, \; a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$

Gamma Function

$$\Gamma(a+1) = \int_0^\infty e^{-x} x^a \, dx, \; a > 0.$$

If $a = n > 0$ is an integer, $\Gamma(n+1) = n!$

Do transforms always exist?

- $f(t)$ **piecewise continuous** on finite intervals $[0, T]$;

- $|f(t)| \leq Me^{\gamma t}, \ \forall t \geq 0.$

Then $F(s) = \mathcal{L}(f)$ exists for all $s > \gamma$.

Is $\mathcal{L}(f)$ unique? (so $\mathcal{L}^{-1}$ makes sense)

- $\mathcal{L}(f)$ exists $\Rightarrow$ uniquely determined;

- If f, g have the same Laplace transform then, practically speaking, they are the same function. **So $\mathcal{L}^{-1}$ is unique.**

Transform of Derivatives

Laplace transform replaces operations of calculus, like differentiation by algebraic operations on transforms. For example:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \quad f \text{ continuous},$$

provided $|f(t)| \leq Me^{\gamma t}$, and the formula is then true for $s > \gamma$. This is easily seen from

$$\begin{aligned}
\mathcal{L}(f') &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \\
&= -f(0) + s\mathcal{L}(f).
\end{aligned}$$

Apply the formula to $f''(t)$:

$$\begin{aligned}
\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0) \\
&= s(s\mathcal{L}(f) - f(0)) - f'(0), \\
\mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0).
\end{aligned}$$

If f and all the derivatives are continuous,

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0).$$

Example 1. $f(t) = t\cos\omega t$. $\mathcal{L}(f) =?$
Consider $g(t) = te^{i\omega t}$. Now,

$$\begin{aligned} g'(t) &= e^{i\omega t} + i\omega t e^{i\omega t},\ g(0) = 0; \\ &= e^{i\omega t} + i\omega g(t) \\ \mathcal{L}(g') &= \frac{1}{s - i\omega} + i\omega\mathcal{L}(g) \\ &= s\mathcal{L}(g) - 0, \end{aligned}$$

$$\begin{aligned} (s - i\omega)\mathcal{L}(g) &= \frac{1}{s - i\omega} \\ \mathcal{L}(g) &= \frac{1}{(s - i\omega)^2} = \frac{(s + i\omega)^2}{(s^2 + \omega^2)^2} \\ &= \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} + \frac{2i\omega s}{(s^2 + \omega^2)^2}\ . \end{aligned}$$

Equating real & imaginary parts

$$\begin{aligned} \mathcal{L}(t\cos\omega t) &= (s^2 - \omega^2)/(s^2 + \omega^2)^2\,, \\ \mathcal{L}(t\sin\omega t) &= 2\omega s/(s^2 + \omega^2)^2\,. \end{aligned}$$

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Initial Value Problems.

$$y'' + ay' + by = r(t),\ y(0) = K_0,\ y'(0) = K_1.$$

Here, $r(t)$ is called the **input**, and $y(t)$ the **output.**

Step 1. Take the Laplace transform of the diff eqn, and write $Y = \mathcal{L}(y),\ R = \mathcal{L}(r)$ to get

$$(s^2Y - sy(0) - y'(0)) + a(sY - y(0)) + bY = R(s),$$

which is called the **subsidiary equation**. Rearranging,

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Divide by s^2+as+b and use **transfer function**

$$Q(s) = \frac{1}{s^2 + as + b}$$

gives the solution of subsidiary eqn:

$$Y(s) = ((s+a)y(0) + y'(0))Q(s) + R(s)Q(s).$$

In the special case $y(0) = y'(0) = 0$, $Y(s) = R(s)Q(s)$,

$$Q(s) = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}.$$

Note that $Q(s)$ depends only on a, b, and not on the input r, nor the initial conditions.

Step 3. Reduce the expression for Y by partial fractions to terms in $1/(s-\alpha)$ and $1/(s^2+\omega^2)$ and find inverses from table. This gives the solution $y(t) = \mathcal{L}^{-1}(Y)$.

Example 2. $y'' + y = t,\ y(0) = 1,\ y'(0) = 1.$
Step 1: Subsidiary equation

$$\begin{aligned} s^2Y - sy(0) - y'(0) + Y &= \frac{1}{s^2} \\ (s^2+1)Y &= s + 1 + \frac{1}{s^2}. \end{aligned}$$

Step 2: Transfer function $Q(s) = 1/(s^2+1)$, so

$$\begin{aligned} Y &= \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} \\ &= \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1} \\ &= \frac{s}{s^2+1} + \frac{1}{s^2} \ . \end{aligned}$$

Step 3: Inverse transform gives

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= \cos t + t. \end{aligned}$$

Example 3.

$$y'' - 4y = 5\sin t,\ y(0) = 1,\ y'(0) = 0.$$

Subsidiary equation is:

$$s^2Y - s - 0 - 4Y = \frac{5}{s^2+1}.$$

Solving for Y and using partial fractions,

$$\begin{aligned} Y &= \frac{s}{s^2-4} + \frac{5}{(s^2-4)(s^2+1)} \\ &= \frac{s}{s^2-4} + \frac{1}{s^2-4} - \frac{1}{s^2+1} \\ y(t) &= \cosh 2t + \frac{1}{2}\sinh 2t - \sin t. \end{aligned}$$

Example 4

$$y'' - 3y' + 2y = \cos t,\ \ y(0) = 1,\ \ y'(0) = 3.$$

Subsidiary equation is:

$$s^2Y - s - 3 - 3(sY - 1) + 2Y = \frac{s}{s^2+1}\ .$$

Solving for Y and using partial fractions,

$$\begin{aligned} Y &= \frac{s^3+2s}{(s-1)(s-2)(s^2+1)} \\ &= -\frac{3}{2}\frac{1}{s-1} + \frac{12}{5}\frac{1}{s-2} \\ &\quad + \frac{1}{10}\frac{s}{s^2+1} - \frac{3}{10}\frac{1}{s^2+1}\ , \end{aligned}$$

$$y(t) = -\frac{3}{2}e^t + \frac{12}{5}e^{2t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t.$$

Transforms of Integrals.

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f(t)), \quad s > 0,\ s > \gamma,$$

provided f is piecewise continuous and $|f(t)| \leq Me^{\gamma t}$. Conversely, a useful formula is:

$$\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau.$$

Example 5. $\mathcal{L}(f) = \frac{1}{s(s^2-a^2)}$. $f(t) =?$

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s^2-a^2}\right) &= \frac{1}{a}\sinh at. \\
\mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s^2-a^2}\right) &= \frac{1}{a}\int_0^t \sinh a\tau\,d\tau \\
&= \frac{1}{a^2}(1-\cosh at).
\end{aligned}$$

Example 6. $\mathcal{L}(f) = \frac{1}{s^2(s^2-a^2)}$. $f(t) =?$
Integrating the above answer again,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{s^2-a^2}\right) &= \frac{1}{a^2}\int_0^t (1-\cosh a\tau)\,d\tau \\
&= \frac{1}{a^2}\left(t - \frac{\sinh at}{a}\right).
\end{aligned}$$

Partial Fraction Expansions.

Example 7. $F(s) = \dfrac{s-2}{s^2-s-6}, \quad f(t) = ?$

$$\frac{s-2}{s^2-s-6} = \frac{s-2}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$s-2 = A(s-3) + B(s+2), \quad \forall s$$

$$s = 3 \Rightarrow B = \frac{1}{5}, \quad s = -2 \Rightarrow A = \frac{4}{5}$$

$$\begin{aligned} F(s) &= \frac{4}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3} \\ f(t) &= \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t}. \end{aligned}$$

Example 8. $F(s) = \dfrac{s^2-4s+7}{(s-1)^4}$,

$$F(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4} \ .$$

$$s^2-4s+7 = A(s-1)^3+B(s-1)^2+C(s-1)+D.$$

$s=1 \Rightarrow D=4;\quad s^3:\ A=0;$
$s^2:\ B=1;\quad s:\ -2B+C=-4,\ C=-2.$

$$F(s) = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}$$

$$f(t) = te^t - t^2e^t + \frac{2}{3}t^3e^t.$$ *need 1st Shift Thm*

Example 9.

$$\begin{aligned} F(s) &= \frac{1}{s(s^2+1)} \\ &= \frac{A}{s} + \frac{Bs+c}{s^2+1} \end{aligned}$$

$$1 = A(s^2+1) + Bs^2 + Cs$$

$s=0 \Rightarrow A=1;\quad s=i \Rightarrow B=-1,\ C=0.$

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{s}{s^2+1}\ , \\ f(t) &= 1-\cos t. \end{aligned}$$

Example 10. Initial Value Problem.

$$y''' + y' = t + 1, \quad y(0) = y'(0) = y''(0) = 3.$$

$$s^3Y - s^2.3 - s.3 - 3 + sY - 3 = \frac{1}{s^2} + \frac{1}{s},$$

$$\begin{aligned}(s^3 + s)Y &= \frac{1}{s^2} + \frac{1}{s} + 3s^2 + 3s + 6 \\ &= \frac{3s^4 + 3s^3 + 6s^2 + s + 1}{s^2}\end{aligned}$$

$$\begin{aligned}Y &= \frac{3s^4 + 3s^3 + 6s^2 + s + 1}{s^3(s^2 + 1)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 1}\end{aligned}$$

$$(As^2 + Bs + C)(s^2 + 1) + Ds^4 + Es^3 \\ = 3s^4 + 3s^3 + 6s^2 + s + 1$$

$s = 0 \Rightarrow C = 1;\ s = i \Rightarrow D - iE = -2 - 2i,$
$D = -2,\ E = 2;$
$s^4:\ A + D = 3\ A = 5; \quad s:\ B = 1.$
Hence, $y(t) = 5 + t + t^2/2 - 2\cos t + 2\sin t$

Shift Theorems; Unit Step Function.
First Shift Theorem: s-shifting

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

If $F(s) = \mathcal{L}(f)$, the transform of $e^{at}f(t)$ is found by **shifting on the s-axis**.

Example 11. $y'' - 2y' + 10y = 0$,
$y(0) = 3,\ y'(0) = 6$.

$$s^2Y - 3s - 6 - 2(sY - 3) + 10Y = 0,$$

$$\begin{aligned} Y(s) &= \frac{3s}{(s-1)^2 + 3^2} \\ &= 3\frac{s-1}{(s-1)^2+3^2} + \frac{3}{(s-1)^2+3^2}, \\ y(t) &= \mathcal{L}^{-1}(Y) = e^t(3\cos 3t + \sin 3t). \end{aligned}$$

Example 12.

$$y'' - 4y' + 4y = e^{2t} + t,\ \ y(0) = 1,\ \ y'(0) = 0.$$

$$(s^2Y - s) - 4(sY - 1) + 4Y = \frac{1}{s-2} + \frac{1}{s^2},$$

$$\begin{aligned} (s-2)^2 Y &= s - 4 + \frac{1}{s-2} + \frac{1}{s^2} \\ Y &= \frac{s-4}{(s-2)^2} + \frac{1}{(s-2)^3} + \frac{1}{s^2(s-2)^2}. \end{aligned}$$

Apply First Shift Theorem: first term

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{s-4}{(s-2)^2}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s-2} - 2\frac{1}{(s-2)^2}\right) \\ &= e^{2t} - 2te^{2t}. \end{aligned}$$

The inverse of second term is $t^2e^{2t}/2$. Last term by partial fractions is

$$\frac{1}{s^2(s-2)^2} = \frac{A}{(s-2)^2} + \frac{B}{s-2} + \frac{C}{s^2} + \frac{D}{s},$$

$$1 = As^2 + Bs^2(s-2) + C(s-2)^2 + Ds(s-2)^2$$

$s = 0 \;\Rightarrow\; C = 1/4,\; s = 1 \Rightarrow A = 1/4.$
Equating s^3 terms $D + B = 0$. Equating s terms, $-4C + 4D = 0 \;\Rightarrow\; D = C = 1/4$, $B = -D = -1/4$. So last term gives

$$\mathcal{L}^{-1}\left(\frac{1/4}{(s-2)^2} - \frac{-1/4}{s-2} + \frac{1/4}{s^2} + \frac{1/4}{s}\right)$$
$$= te^{2t}/4 + e^{2t}/4 + t/4 + 1/4 \;.$$

Adding all this together,

$$y(t) = \mathcal{L}^{-1}(Y) = \frac{1}{2}t^2e^{2t} - \frac{7}{4}te^{2t} + \frac{5}{4}e^{2t} + \frac{1}{4}t + \frac{1}{4}.$$

Second Shift Theorem: t-shifting $a \geq 0$.

$$\mathcal{L}^{-1}(e^{-as}F(s)) = \widetilde{f}(t) = \begin{cases} 0 & \text{if } \; t < a, \\ f(t-a) & \text{if } \; t > a, \end{cases} .$$

The function $\widetilde{f}$ has been **shifted on the t-axis** and find $\mathcal{L}^{-1}(e^{-as}F(s))$ by this shift.

Unit Step Function $u(t-a)$.

$$u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t > a, \end{cases} \qquad a \geq 0.$$

$$\widetilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ f(t-a) & \text{if } t > a, \end{cases}$$

which is the function f for $t > 0$, shifted a to the right.

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s).$$

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)u(t-a).$$

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}.$$

Example 13. $\mathcal{L}^{-1}(e^{-2s}/s^2)$
Since $\mathcal{L}^{-1}(1/s^2) = t$, Second Shift Theorem gives

$$\begin{aligned} \mathcal{L}^{-1}(e^{-2s}/s^2) &= (t-2)u(t-2) \\ &= \begin{cases} 0 & \text{if } t < 2, \\ (t-2) & \text{if } t > 2. \end{cases} \end{aligned}$$

Example 14. Inverse transform of

$$\begin{aligned} F(s) &= \frac{1}{s^2}(1-e^{-s})^2 \\ &= \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} \end{aligned}$$

$$\begin{aligned} f(t) &= t - 2(t-1)u(t-1) + (t-2)u(t-2) \\ &= \begin{cases} t & \text{if } 0 < t < 1, \\ 2-t & \text{if } 1 < t < 2, \\ 0 & \text{if } t > 2. \end{cases} \end{aligned}$$

Example 15. Find transform of

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi, \\ 0 & \text{if } \pi < t < 2\pi, \\ \sin t & \text{if } t > 2\pi. \end{cases}$$

In step functions, f is just $2u(t)$ for $(0, \pi)$, then we subtract $2u(t-\pi)$ to get 0 in $(\pi, 2\pi)$, and add in $u(t-2\pi)\sin t$ after $t = 2\pi$,

$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin t.$$

Since $\sin t = \sin(t - 2\pi)$,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}.$$