PDEs

Laplace's Eqn

\[ \nabla^2 \phi = 0 \]
\[ R^3: \ n^2 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
\[ R^2: \ \Delta^2 = \frac{\partial^2}{\partial x^2} \]
\[ R^2: \ \Delta^2 = \frac{\partial^2}{\partial x^2} \]

Vibrating Membrane

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \Delta^2 u \]

Heat Equation Diff.

\[ \frac{\partial u}{\partial t} = c^2 \Delta u \]
Example \[ L = \pi, \quad c = 1 \]
\[ u(x,0) = f(x) = x \]

Bar length \[ L \]

\[ u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \exp \left( -\left( \frac{c n \pi}{L} \right)^2 t \right) \]

\[ L = \pi \Rightarrow \]

\[ u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n x \exp \left( -\left( \frac{c n \pi}{L} \right)^2 t \right) \]

Cosine series of \( f(x) = x, \quad 0 < x < \pi \)

\[ x = A_0 + \sum_{n=1}^{\infty} A_n \cos n x \]

\[ A_0 = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{\pi}{2} \]

\[ A_n = \frac{2}{\pi} \int_0^\pi x \cos n x \, dx \]

\[ = \frac{2}{\pi} \left[ \frac{xs \sin nx}{n} \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \frac{\sin nx}{n} \, dx \]

\[ = \frac{2}{n^2} \left[ \sin nx \right]_0^\pi = \frac{2}{n^2} (\sin \pi - \sin 0) \]

\[ = \frac{2}{n^2} \]
\[
A_m = \begin{cases} 
0, & m \text{ even} \\
-\frac{\alpha}{\pi m^2}, & m \text{ odd} \\
\end{cases}
\]

\[
u(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} e^{-c(2k-1)^2 t}
\]

(we'll accept an answer of form)

\[
u(t) = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos x}{1} e^{-c x t} + \frac{\cos 3x}{3} e^{-3c x t} + \right.
\]

steady state as \( t \to \infty \)

\[
u(t) \to \frac{\pi}{2}
\]

That is, uniform temperature in the bar

Boundary conditions in 1-D situation

\[
u(0, t) = \ldots
\]

\[
u(L, t) = \ldots
\]

\[
u(0, t) = \ldots
\]

\[
u_x(L, t) = \ldots
\]
\[ u_{x_t} = c^2 u_{xx} \] (*)

Make a change of variables:
\[ u = u(x, t) \]
\[ v = x - ct \]
\[ z = x + ct \]

Chain rule:
\[ u_{x_t} = u_{x} \frac{\partial x}{\partial x} + u_{t} \frac{\partial x}{\partial t} = u_{x} + u_{t} \]
\[ u_{xx} = \frac{\partial^2}{\partial x^2} (u_{x} + u_{t}) = \frac{\partial^2 u_{x}}{\partial x^2} + \frac{\partial^2 u_{t}}{\partial x^2} \]
\[ u_{x_t} = c^2 (u_{x} - u_{t}) + 2 u_{x} z \]

\[ u_{x} = u_{x_t} \frac{\partial x}{\partial x_t} + u_{z} \frac{\partial z}{\partial x_t} = c (u_{x} - u_{t}) \]

\[ u_{x_t} = c^2 \frac{\partial}{\partial z} (u_{x} - u_{t}) - c^2 \frac{\partial}{\partial z} (u_{x} - u_{t}) \]
\[ = c^2 (u_{x} - u_{t}) \]

Equate these in (*):
\[ c^2 (u_{x} - u_{t}) = 0 \]
\[ \Rightarrow u_{x} = h(x) \]
\[ u = \int h(x) \, dx + \psi(z) \]
\[ u = \phi(v) + \psi(z) \]
\[ = \phi(x+ct) + \psi(x-ct) \]

**Note** special technique which works for the wave equation, but not in general.

Initial conditions:

1. \( u(x,0) = f(x) \), \( u_t(x,0) = g(x) \)
2. \( u_t(x,t) = c \phi'(x+ct) - c \psi'(x-ct) \)
3. \( u_t(x,0) = c \phi'(x) - c \psi'(x) = g(x) \)
4. \( u(x,0) = \phi(x) + \psi(x) = f(x) \)

\[ c \phi(x) - c \psi(x) = c \phi(x_0) - c \psi(x_0) + \int_{x_0}^{x} g(x) \, dx \]
\[ = k(x_0) + \int_{x_0}^{x} g(x) \, dx \]

**Invoking** (3)

(3) \( 2 \phi(x) = f(x) + \frac{k(x_0)}{c} + \frac{1}{c} \int_{x_0}^{x} g(x) \, dx \)
(4) \( 2 \psi(x) = f(x) - \frac{k(x_0)}{c} - \frac{1}{c} \int_{x_0}^{x} g(x) \, dx \)
\[ u(x, t) = \phi(x + ct) + \psi(x - ct) \]

\[ = \frac{1}{2} \left\{ \phi(x + ct) + \phi(x - ct) \right\} \]

\[ + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(x) \, dx - \frac{1}{2c} \int_{x_0}^{x_{-ct}} \phi(x) \, dx \]

\[ = \frac{1}{2} \left\{ \phi(x + ct) + \phi(x - ct) \right\} \]

\[ + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(x) \, dx \]

In particular, if the initial velocity \( \psi'(x) = 0 \), then

\[ u(x, t) = \frac{1}{2} \left\{ \phi(x + ct) + \phi(x - ct) \right\} \]
Classification of 2nd order PDEs

\[ A u_{xx} + 2B u_{xy} + C u_{yy} = F(x, y, u, u_x, u_y) \]

\( A, B, C \) may be functions of \( x, y \).

\[ AC - B^2 < 0 \quad \text{HYPERBOLIC} \]
\[ AC - B^2 = 0 \quad \text{PARABOLIC} \]
\[ AC - B^2 > 0 \quad \text{ELLiptic} \]

\text{LAPLACE EQN} \quad u_{xx} + u_{yy} = 0 \quad \text{ELLiptic} \]

\text{HEAT EQN} \quad u_t = c^2 u_{xx} \quad \text{PARABOLIC} \]

\text{WAVE EQN} \quad u_{tt} - c^2 u_{xx} = 0 \quad \text{HYPERBOLIC} \]

2 2nd derivs in \( t, x \) \quad A = 1, \quad C = -c^2, \quad B = 0
Example: Tricomi Eqn

\[ y u_{xx} + u_{yy} = 0 \]

Seek a solution by separation of variables.

\[ u(x, y) = F(x) G(y) \]

\[ y F''(x) G(y) + F(x) G''(y) = 0 \]

\[ \frac{G''(y)}{G'(y)} = -\frac{F''(x)}{F(x)} = k \]

\[ F''(x) = -kF(x) \]

\[ G''(y) - ky G'(y) = 0 \]

Airy Eqn

Solutions of this (use series solns method for ODEs) called Airy functions.
Heat Equation

\[ u_t = c^2 \nabla^2 u \quad \frac{c^2}{\rho} = \frac{k}{\rho} \]

\( k \): Thermal conductivity
\( \rho \): Density
\( c \): Specific heat

\( u_t = c^2 \nabla^2 u \) (1-D, \( u_t = c^2 u_{xx} \))

**Case I** Bar of length \( L \)
\( x = 0, x = L \), kept at constant temperature, \( 0 \) \( \text{C} \); temp. distn. in bar

**Boundary Conditions**
\[ u(0, t) = u(L, t) = 0 \]

**Initial Temp. Distn**
\[ u(x, 0) = f(x) \]

1. Separation of Variables
\[ u(x, t) = F(x) G(t) \]
\[ u_t = F \dot{e}, \quad u_{xx} = F^{\prime \prime} e \]

\[ F \dot{e} = c^2 F^{\prime \prime} \]

\[ \frac{F}{F^\prime} = \frac{c^2 \dot{e}}{\dot{e}} = k \] and

\[ F^{\prime \prime} - k F = 0 \]

\[ \dot{e} - kc^2 e = 0 \]

2. Use the boundary conditions to obtain information about \( k \).

Let \( F(x) = Ax + B \)

\( k = 0 \), \( F(x) = A x + B \)

boundary conditions \( A \& B = 0 \), impossible.

Let \( k = m^2 \), \( F(x) = ce^{mx} + de^{-mx} \)

boundary conditions \( c \& d = 0 \), no nontrivial solutions, impossible.

Let \( k < 0 \), \( k = -p^2 \), \( F^{\prime \prime} + p^2 F = 0 \)

\[ F(x) = A \cos px + B \sin px \]

\[ F(0) = 0 \Rightarrow A = 0 \]

\[ F(L) = 0 \Rightarrow B \sin pL = 0 \]

cannot have \( B = 0 \) \( \Rightarrow \sin pL = 0 \)

\[ pL = m\pi, \quad m = 1, 2, \ldots \]

\[ p = \frac{m\pi}{L}, \quad m = 1, 2, \ldots \]
That is, the $B_n$ are the Fourier coefficients in the Fourier sine expansion of $f(x)$ in $[-L, L]$.

$$B_n = \frac{2}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.$$ 

Example 4, p. 608

If $P = 10.6 \, g/cm^2$, $L = 5$

$K = 1.0 \, cal/cm^2 \cdot sec$

$s = 0.056$, $C = \sqrt{s/\rho}$

$f(x) = ks \sin \left( 0.2 \pi x \right)$

$c = 1.752$

Know that

$$u(x, t) = \sum_{n=1,3,5,\ldots} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$\lambda_n = cnt^2 / 5$

$$\sum_{n=1,3,5,\ldots} B_n \sin \frac{n\pi x}{5} = ks \sin \left( \frac{\pi x}{5} \right)$$

$B_1 = A$, $B_n = 0$, $n \geq 2$

$$u(x, t) = ks \sin \left( \frac{\pi x}{5} \right) e^{- \frac{1.752 \pi^2 x^2}{25}}$$

(better $\exp(-1.752 \pi^2 x^2 / 25)$)
**CASE II** \( \text{Imulated results} \)

\[ u_x(0, t) = u_x(L, t) = 0. \]

As before

\[ F''(x) - \lambda c F = 0, \quad G''(x) - \lambda c G = 0. \]

\( \lambda = 0 \) is possible,
\( \lambda = m^2 \),
\[ F(x) = A e^{mx} + B e^{-mx} \]
\[ F'(0) = F'(L) = 0 \]
\[ m A - m B = 0 \]
\[ m A e^{mL} - m B e^{-mL} = 0 \]

\( \Rightarrow A = B = 0 \) impossible

\( k = -\mu^2 \)
\[ F(x) = A \cos \mu x + B \sin \mu x \]
\[ F'(x) = -\mu A \sin \mu x + \mu B \cos \mu x \]
\[ F'(0) = 0 \Rightarrow B = 0 \]
\[ F'(L) = 0 \Rightarrow \sin \mu L = 0 \]
\[ \mu L = n \pi, \quad n = 0, 1, \ldots \]
\[ \mu = n \pi / L, \quad n = 0, 1, \ldots \]

\[ F_m(x) = A_m \cos \frac{n \pi x}{L} \]
\[ m = 0, 1, 2, \ldots \]
\[ \dot{G} + (\rho - c)^2 G = 0 \]

\[ G(x) = \exp(-\lambda_n^2 x) \]

\[ \lambda_n = \beta_n c = \frac{c \pi n}{L}, \quad n = 0, 1, \ldots \]

\[ u_m(x, t) = A_m \cos \frac{m \pi x}{L} \exp(-\lambda_n^2 t) \]

\[ u(x, t) = \sum_{m=0}^\infty A_m \cos \frac{m \pi x}{L} \exp(-\lambda_n^2 t) \]

\[ u(x, 0) = f(x) \quad \text{gives} \]

\[ A_0 + \sum_{m=1}^\infty A_m \cos \frac{m \pi x}{L} = f(x) \]

\[ \text{Fourier cosine series} \]

\[ A_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m \pi x}{L} \, dx \]

\[ A_0 = \frac{1}{L} \int_0^L f(x) \, dx. \]
1. Find the temperature $u(x, t)$ in a bar of silver (length 10 cm, constant cross section of area 1 cm$^2$, density 10.6 gm/cm$^3$, thermal conductivity 1.04 cal/cm sec °C, specific heat 0.056 cal/gm °C) that is perfectly insulated laterally and whose ends are kept at temperature 0 °C, whose initial temperature distribution is $f(x) = 5 - |x - 5|$ °C.

2. Find the temperature in a bar insulated at both ends with

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad u(x, 0) = f(x),$$

where

$$f(x) = \begin{cases} 
1 & \text{if } 0 < x < \frac{\pi}{2}, \\
0 & \text{if } \frac{\pi}{2} < x < \pi.
\end{cases}$$

3. Find the temperature $u(x, t)$ in a bar of length $L$ that is kept at zero temperature at $x = 0$, assuming that the end $x = L$ is perfectly insulated, the initial temperature is a constant $U_0$ and $u_x(L, t) = 0$ (because of perfect insulation there).