2. USING LAPLACE TRANSFORMS

This is a useful technique when one variable has the whole positive axis as its domain. For a 2-D PDE

Take the Laplace transform with respect to one of the 2 variables. This gives an ODE with respect to the other variable.
Solve that ODE to obtain the transformed unknown.
Take the inverse transform.

EXAMPLE $\quad \frac{\partial w}{\partial x} + x\frac{\partial w}{\partial t} = 0,$

$x, t \geq 0, \quad w(x,0) = 0, \quad w(0,t) = t.$

(I) Take the Laplace transform with respect to t:

$$\mathcal{L}(w_x) + x\left(sW_L^{(x,s)} - w(x,0)\right) = 0.$$

$$\int_0^\infty e^{-st}\frac{\partial w}{\partial x}(x,t)\,dt + xsW = 0$$

Under the conditions that you want w to satisfy, integration and $\frac{\partial}{\partial x}$ can be interchanged

$$\int_0^\infty e^{-st} \frac{\partial}{\partial x} w(x,t)\,dt = \frac{\partial}{\partial x} \int_0^\infty e^{-st} w(x,t)\,dt$$

$$= \frac{\partial}{\partial x} W(x,s).$$

Hence, we get

$$\boxed{\frac{\partial}{\partial x} W(x,s) + sx\,W(x,s) = 0.}$$

ODE in the variable x, BUT the "arbitrary ~~constant~~" is constant wrt x, so will involves.

$$w(0,t) = t \Rightarrow W(0,s) = 1/s^2$$

is ~~the~~ initial condition for the ODE.

$$\frac{\partial}{\partial x} W = -sxW$$

$$W = c(s)\,e^{-sx^2/2}$$

$x=0$

$$\Rightarrow W(0,s) = \frac{1}{s^2} = c(s)$$

$$W(x,s) = \frac{1}{s^2} e^{-sx^2/2}$$

~~$y' = axy$~~

$$y' = axy$$

$$\frac{dy}{y} = ax\,dx$$

$$y = c\,e^{ax^2/2}$$

Recall $\mathcal{L}^{-1}(F(s)e^{-as}) = u(t-a)f(t-a)$

by 2ND SHIFT THM

3. $F(s) = 1/s^2 \implies f(t) = t$

$a = x^2/2$

$$w(x,t) = u(t - x^2/2)\,(t - x^2/2)$$

$$= \begin{cases} 0, & t < x^2/2 \\ t - x^2/2, & t > x^2/2 \end{cases}$$

check that it really is a solution.

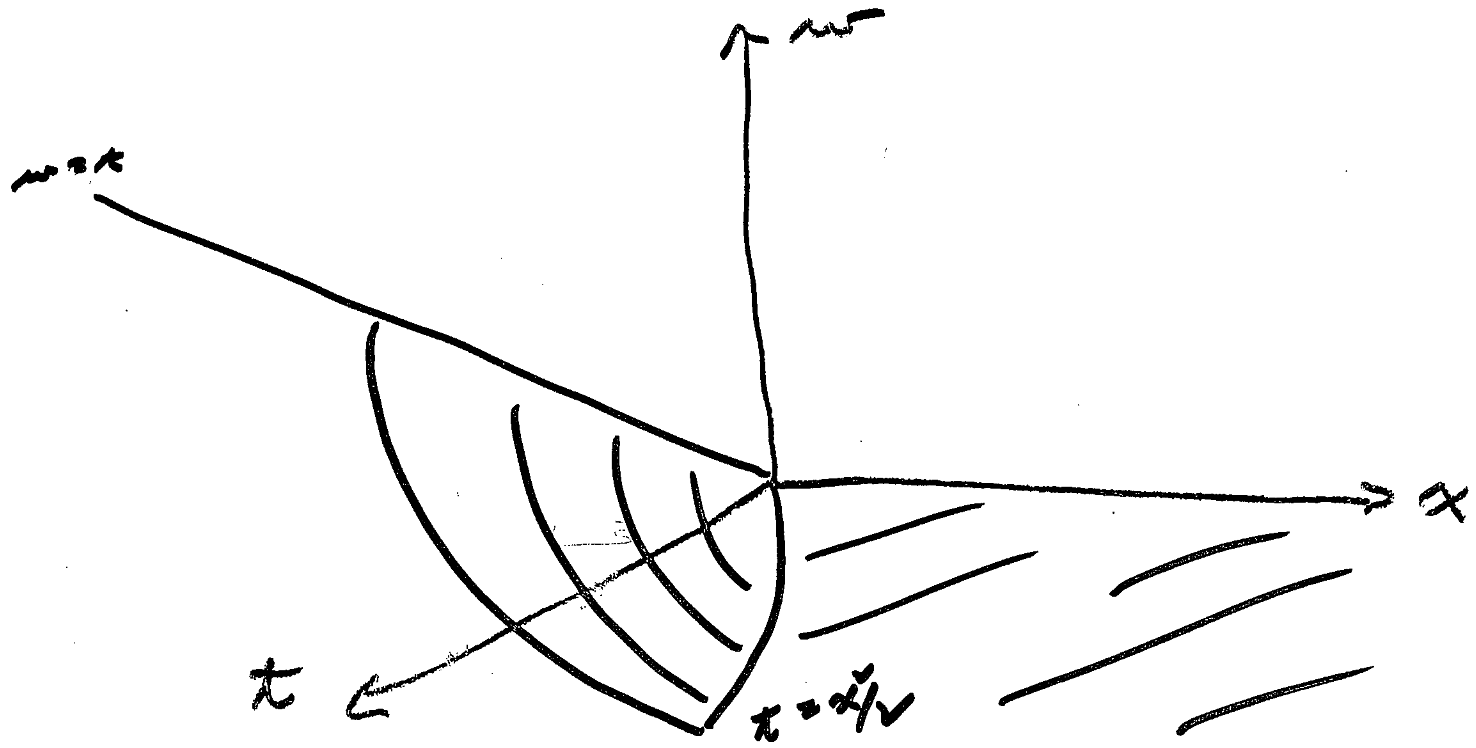
(1) $t < x^2/2$, $w \equiv 0$, $w_t \equiv 0$, $w_x \equiv$

Trivially $\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial t} = 0$

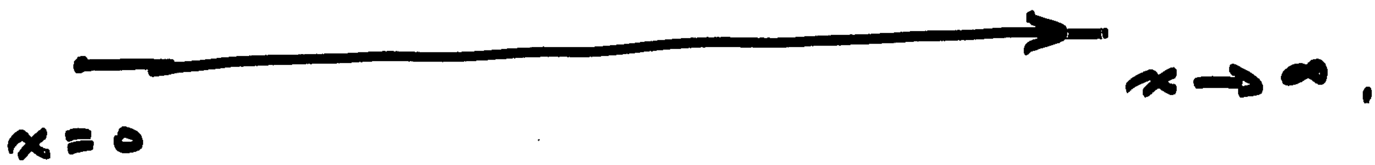
(2) $t > x^2/2$, $w_t = 1$, $w_x = -x$

$$\partial w/\partial x + x\,\partial w/\partial t = -x + x \cdot 1 = 0$$

This is a solution



EXAMPLE 2 Find the temperature $w(x,t)$ on a semi-infinite laterally insulated bar.
(Model or approximation of a very long bar or wire)



$w(x,0)=0$, $\quad w(x,t) \to 0$ as $x \to \infty$

$w(0,t) = f(t)$, $\quad 0 \le t < \infty$

$$\partial w / \partial t = c^2 \, \partial^2 w / \partial x^2$$

Take Laplace transform wrt t

$$sW(x,s) - w(x,0) = c^2 \frac{\partial^2 W(x,s)}{\partial x^2}$$

$$\frac{\partial^2 W}{\partial x^2} - \frac{s}{c^2} W = 0$$

Again, arby consts involves

$$W(x,s) = A(s)\, e^{\sqrt{s}\, x/c} + B(s)\, e^{-\sqrt{s}\, x/c}$$

$W(x,s) \to 0$ as $x \to \infty$ $\left(\begin{array}{c}\text{since} \\ w(x,t) \to 0\end{array}\right)$

$$\Rightarrow A(s) = 0$$

$$W(x,s) = B(s)\, e^{-\sqrt{s}\, x/c}$$

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\dot{G}_n + (p_n c)^2 G_n = 0$$

$$G_n = e^{-\lambda_n^2 t}$$

$(G_n = Ke^{\cdot})$

$u_n = F_n G_n$

$u_n = KB_n)$

$\ldots\ldots$

$$\lambda_n = \frac{c\, n\pi}{L}$$

$n = 1, 2, \ldots\ldots$

$$u_n(x,t) = F_n(x)\, G_n(t) = B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}$$

3. Superposition of all the solutions u_n corresponding to the eigenvalues λ_n:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}$$

Initial condition $u(x,0) = f(x)$

$0 \le x \le L$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

MT253 Assignment 9, MT256 Assignment 4
To be handed in Friday, 13 October, 2000

1. Find the temperature $u(x,t)$ in a bar of silver (length 10cm, constant cross section of area 1 cm^2, density 10.6 gm/cm^3, thermal conductivity 1.04 cal/cm sec °C, specific heat 0.056 cal/gm °C) that is perfectly insulated laterally and whose ends are kept at temperature 0 °C, whose initial temperature distribution is $f(x) = 5 - |x-5|$ °C.

2. Find the temperature in a bar insulated at both ends with

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad u(x,0) = f(x),$$

where

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < \frac{\pi}{2}, \\ 0 \text{ if } \frac{\pi}{2} < x < \pi. \end{cases}$$

3. Find the temperature $u(x,t)$ in a bar of length L that is kept at zero temperature at $x = 0$, assuming that the end $x = L$ is perfectly insulated, the initial temperature is a constant U_0 and $u_x(L,t) = 0$ (because of perfect insulation there).

5. $w(0,t) = f(t) \Longrightarrow W(0,s) = F(s)$

$W(0,s) = B(s)$

$$\Longrightarrow W(x,s) = F(s)\, e^{-\sqrt{s}\, x/c}$$

Can't use second shift thm here because of the $\sqrt{s}$ & have to use the convolution theorem.

From Kreyszig's Table:

$$\mathcal{L}^{-1}\left(e^{-k\sqrt{s}}\right) = \frac{k}{2\sqrt{\pi t^3}}\, e^{-k^2/4t}$$

$$\mathcal{L}^{-1}\left(e^{-\sqrt{s}\, x/c}\right) = \frac{x}{2c\sqrt{\pi}\, t^{3/2}}\, e^{-x^2/4c^2 t}$$

$$= g(t)$$

Convolution gives

$$\mathcal{L}^{-1}\left(F(s)\, G(s)\right) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$w(x,t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \frac{x}{2c\sqrt{\pi}} \int_0^t f(t-\tau)\, \tau^{-3/2} e^{-x^2/4c^2\tau}\, d\tau$$

$*$

1. Using the convolution theorem & noting that (see Kreyszig tables)

$$\mathcal{L}^{-1}\left(e^{-k\sqrt{s}}\right) = \frac{k}{2\sqrt{\pi}\, t^{3/2}}\, e^{-k^2/4t}$$

$$\Rightarrow \mathcal{L}^{-1}\left(e^{-\sqrt{s}\, x/c}\right) = \frac{x}{2c\sqrt{\pi}\, t^{3/2}}\, e^{-x^2/4c^2 t}$$

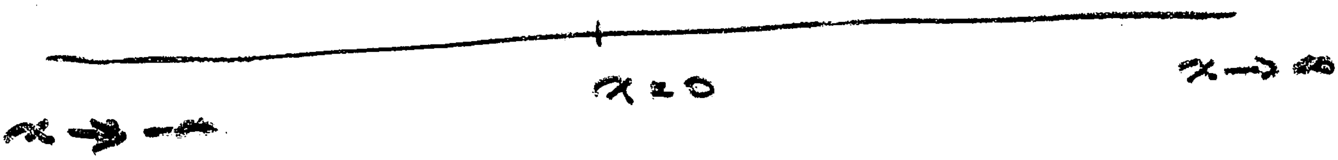
$$= g(t)$$

Convolution gives

$$u(x,t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

giving the formula (*)

What if we have a long bar with the heat source (or a diffusion) at the centre



What sort of boundary conditions make sense <u>physically</u>?

2.

$$u(x,0) = f(x)$$

$$u(x,t) \longrightarrow 0, \quad |x| \longrightarrow \infty$$

$$u_x(x,t) \longrightarrow 0, \quad |x| \longrightarrow \infty.$$

Possible to show (by Fourier transforms)

$$u(x,t) = \frac{1}{2c\sqrt{\pi}} \int_{-\infty}^{\infty} f(p) \underbrace{\frac{e^{-\frac{(x-p)^2}{4c^2t}}}{\sqrt{t}}}_{\substack{\text{heat kernel} \\ \text{diffusion kernel} \\ \text{Green's function}}} dp$$

<u>Exercise</u> Check by direct differentiation that $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

But how would you "guess" such a result anyway? We'll derive it using the Fourier Integral & Fourier Transform methods.

Whereas Fourier Series methods involve PERIODIC FUNCTIONS, Fourier Integral / Transform techniques give us a way of analysing NONPERIODIC FUNCTIONS.

3. **Fourier Integrals**

Consider any function $f_L(x)$ of period $2L$, with a F.S. repn.

$$f_L(x) = a_0 + \sum_n{}' (a_n \cos p_n x + b_n \sin p_n x)$$

$$p_n = n\pi/L$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_L(v)\cos p_n v\,dv$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_L(v)\sin p_n v\,dv$$

Subst. for a_n, b_n in the F.S.

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum{}' \Big(\cos p_n x \int_{-L}^{L} f_L(v)\cos p_n v\,dv + \sin p_n x \int_{-L}^{L} f_L(v)\sin p_n v\,dv\Big)$$

Let $L \to \infty$, $\Delta p = p_{n+1} - p_n$
$= (n+1)\pi/L - n\pi/L = \pi/L$

So $1/L = \Delta p/\pi$ & so, formally,

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv$$

$$+ \frac{1}{\pi}\sum_n{}' \cos p_n x\,\Delta p \int_{-L}^{L} f_L(v)\cos p_n v\,dv$$

$$+ \frac{1}{\pi}\sum_n{}' \sin p_n x\,\Delta p \int_{-L}^{L} f_L(v)\sin p_n v\,dv$$

As $L \to \infty$, $f_L(x) \to f(x)$

& assume that $|f(x)|$ integrable on $(-\infty, \infty)$; note $\Delta p = \pi/L \to 0$

$$\Rightarrow \sum_{n=1}^{\infty}{}' \cos p_n x\,\Delta p \longrightarrow \int_0^\infty \cos px\,dp$$

$$\sum_{n=1}^{\infty}{}' \sin p_n x\,\Delta p \longrightarrow \int_0^\infty \sin px\,dp$$

(Essentially, this is equivalent to partitioning $[0,\infty)$ by

$p_1 < p_2 < \cdots\cdots < p_n < \cdots$ &

using Riemann sum, followed by taking $L \to \infty$ & get improper integrals which converge)

$\int_{-\infty}^{\infty} f(x)\,dx$ is finite, so $\frac{1}{2L}\int_{-\infty}^{\infty} f(x)\,dx \longrightarrow 0$

as $L \to \infty$

5. In the limit

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos px \Big(\overbrace{\int_{-\infty}^\infty f(v) \cos pv \, dv}^{\pi A(p)}\Big)dp$$

$$+\frac{1}{\pi}\int_0^\infty \sin px \Big(\underbrace{\int_{-\infty}^\infty f(v) \sin pv \, dv}_{\pi B(p)}\Big)dp$$

$$f(x) = \int_0^\infty \big(A(p)\cos px + B(p)\sin px\big)\,dp$$

(REAL) Fourier Integral representation of $f(x)$, $-\infty < x < \infty$.

We'll use this for solving PDE on $\mathbb{R}$ which have bdy conditions $\to 0$ as $|x| \to \infty$.

Note that, looking at the double integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v) \left\{ \begin{array}{l} \cos pv \cos px \\ + \sin pv \sin px \end{array} \right\} dv\,dp$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v) \cos p(v-x) \, dv\,dp$$

But $\int_{-\infty}^\infty f(v) \cos p(v-x)\, dv$ is an EVEN function of p. Call it $F(p)$.

Recall, for even functions,

$$\int_0^\infty F(p)\,dp = \frac{1}{2}\int_{-\infty}^{\infty} F(p)\,dp$$

So, we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cos p(x-v)\,dv\right)dp.$$

On the other hand, since

$\int_{-\infty}^{\infty} f(v)\sin p(x-v)\,dv$ is an odd function of p, call it $G(p)$

& $\int_{-\infty}^{\infty} G(p)\,dp = 0$. Use

$$e^{ip(x-v)} = \cos p(x-v) + i\sin p(x-v)$$

to obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\,e^{ip(x-v)}\,dv\,dp$$

(COMPLEX) Fourier Integral representation.

1. FOURIER TRANSFORM

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ipx} dx$$

Examples 1. Find F.T. of

$$f(x) = \begin{cases} e^{x}, & x < 0 \\ 0, & x > 0 \end{cases}$$

Soln $\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{ipx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1+ip)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1+ip)x}}{1+ip} \right]_{-\infty}^{0}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+ip} \quad \left(= \frac{1}{\sqrt{2\pi}} \frac{1-ip}{1+p^2} \right)$$

note $|e^{(1+ip)L}| = |e^{L}| \underbrace{|e^{ipL}|}_{=1}$

$= e^{L}$

$\longrightarrow 0$ as $L \longrightarrow -\infty$