$$u(x,y) = F(x)G(y)$$

$$u(t,x,y) = F(t,x)\,G(y)$$

Up until now, & etc., we've had simple 1^{ST} & 2^{ND} order ODEs for F, G with constant coefficients. This has largely been the case because we've looked at strings, bars & rectangles.

As soon as we change the geometry, you can get 2nd order ODEs with nonconstant coefficients

Bessel's Equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

ν const

This has a series solution but needn't be a power series solution, namely

$$y(x) = x^r \sum_0^\infty a_m x^m$$

where r satisfies INDICIAL equation

$$r(r-1) + r - \nu^2 = 0$$

$$r^2 - \nu^2 = 0$$

$$r = \pm \nu$$

WILL DO LATER λ, ν const

eg (1) $x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$

can be transformed into Bessel's DE by changing the dependent variable to

$$z = \lambda x$$

$$y' = dy/dx = \frac{dy}{dz}\frac{dz}{dx} = \lambda \frac{dy}{dz}$$

$$y'' = \frac{d(y')}{dx} = \frac{d(y')}{dz}\frac{dz}{dx} = \lambda \frac{d(y')}{dz}$$

$$= \lambda^2 \frac{d^2 y}{dz^2}$$

$$\lambda^{-2} z^2 \lambda^2 d^2y/dz^2 + \lambda^{-1} z \lambda \, dy/dz$$

$$+ (z^2 - \nu^2) y = 0$$

BESSEL

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0$$

$$x^2 y'' + x y' + (4x^4 - 1/4) y = 0$$

$$z = x^2$$

$$\frac{dx}{dz} = 1/\frac{dz}{dx} = \frac{1}{2x}$$

$$y' = \frac{dy}{dz} \cdot 2x$$

$$y'' = \frac{d(y')}{dz} \cdot 2x = \frac{d}{dz}\left(\frac{dy}{dz} \cdot 2x\right) \cdot 2x$$

$$= 2x \left\{ \frac{d^2 y}{dz^2} \cdot 2x + \frac{dy}{dz}\, 2 \cdot \frac{1}{2x} \right\}$$

$$= 4x^2 \ddot{y} + 2\dot{y}$$

$$4x^4 \ddot{y} + 2x^2 \dot{y} + x \cdot 2x \dot{y}$$

$$+ (4z^2 - 1/4) y = 0$$

$$4z^2 \ddot{y} + 4z \dot{y} + 4(z^2 - 1/16) y = 0$$

$$z^2 \ddot{y} + z \dot{y} + (z^2 - 1/16) y = 0$$

$$\nu = 1/4.$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad (*)$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$= \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y'(x) = \sum_{0}^{\infty} a_m (m+r) x^{m+r-1}$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r}$$

$$+ \sum^{\infty} (m+r) a_m x^{m+r} \qquad = 0$$

$$- \nu^2 \sum^{\infty} a_m x^{m+r} + \sum a_m x^{m+r+2}$$

Compare cfts of x^{m+r}

$m = 0 \qquad r(r-1) a_0 + r a_0 - \nu^2 a_0 = 0$

$a_0 \neq 0 \qquad r^2 - \nu^2 = 0$ INDICIAL EQN

$m = 1 \quad (r+1) r a_1 + (r+1) a_1 - \nu^2 a_1 = 0$

$m \geq 2 \left\{ (m+r)(m+r-1) + (m+r) - \nu^2 \right\} a_m$

$+ a_{m-2} = 0$

INDICIAL EQN $r = \pm \nu$

$$a_1(r^2+2r+1-\nu^2) = 0$$

$$a_1(2r+1) = 0 \Rightarrow a_1 = 0$$

$$((m+r)^2-\nu^2)a_m + a_{m-2} = 0$$

$$a_m = \frac{-a_{m-2}}{(m^2+2rm)} \qquad m(m+2r)$$

$$a_3 = 0, a_5 = 0, \ldots, a_{2k-1} = 0$$

Only the even cfts are nonzero & their sign will oscillate

$$a_2 = \frac{-a_0}{2 \cdot 2(1+r)}$$

$$a_4 = \frac{-a_2}{2^2(2^2+2r)} = \frac{a_0}{2^5(1+r)(2+r)}$$

&c.

$$a_{2k} = \frac{-a_{2k-2}}{2^2k(k+r)}$$

Two special cases

$\nu = 0$, $\nu = 1$

$\nu = 0$: $\quad a_2 = \dfrac{-a_0}{2^2}$ $\qquad J_\nu(x)$

$r = 0$

$$a_4 = \frac{-a_2}{2^4} = \frac{a_0}{2^4 \, 1.2.}$$

$$a_6 = \frac{-a_4}{2^2 \, 3^2} = \frac{-a_0}{(1.2.3)^2 \, 2^6}$$

$$a_{2k} = \frac{(-1)^k a_0}{(k!)^2 \, 2^{2k}}$$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k a_0 x^{2k}}{(k!)^2 \, 2^{2k}}$$

$\underline{\underline{ex}}$ do $J_1(x)$

\# $J_{-n}(x) = (-1)^n J_n(x)$

if n is a positive integer.

PROBLEM! Only have one soln

Awful : $\ln x \, J_n(x) + \sum_m b_m x^m$

$= Y_n(x)$

Bessel fn

of 2ND type