Question 1

(a) Find the general solution of the differential equation
\[ y'' + 4y' + 4y = \cos 2x + x^2. \]

(b) Find the general solution of the differential equation
\[ y'' - 4y' + 3y = 8e^{-3x} + e^{3x}. \]

Question 2

(a) Find the Laplace transforms of the functions
\begin{align*}
&\text{(i)} \quad f(x) = x^4e^x, \\
&\text{(ii)} \quad f(t) = t \cos 4t.
\end{align*}

(b) Find the functions whose Laplace transforms are
\begin{align*}
&\text{(i)} \quad \frac{s}{s^2 + 2s + 2}, \\
&\text{(ii)} \quad \frac{1}{s(s^2 + 4)}.
\end{align*}

(c) Solve the initial value problem
\[ y'' - y' - 6y = 5 \cos 2t, \quad y(0) = 0, \quad y'(0) = 1. \]

Question 3

Let \( f(x) = \begin{cases} 
0 & \text{if } -\pi \leq x \leq 0, \\
\sin x & \text{if } 0 \leq x \leq \pi,
\end{cases} \)
and \( f(x) = f(x + 2\pi) \) for all \( x \). Show that
\[ f(x) = \frac{1}{2}(|\sin x| + \sin x). \]
Hence or otherwise find the Fourier series of \( f \). From the Fourier series deduce that
\[
\sum_{m=1}^{\infty} \frac{1}{4m^2-1} = \frac{1}{2},
\]
\[
\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1} = \left( \frac{1}{2} - \frac{\pi}{4} \right).
\]

**Question 4**

(a) Given that \( x \) is a solution of the differential equation
\[
x^2 y'' + (x^2 - 2x)y' - (x - 2)y = 0,
\]
find a second linearly independent solution. Find the general solution of
\[
x^2 y'' + (x^2 - 2x)y' - (x - 2)y = x^3.
\]

(b) Find the Laplace transform of the function
\[
f(t) = t^2, \quad 0 < t < 2\pi, \quad f(t) = f(t + 2\pi), \quad t > 0.
\]

**Question 5**

(a) Let \( y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \). Find the general solution of the system
\[
y' = \begin{bmatrix} -1 & 1 \\ -6 & -6 \end{bmatrix} y
\]
Identify the nature of the equilibrium point at the origin and give a rough sketch of some trajectories.

(b) Convert the differential equation
\[
y'' + y' - 4y + y^3 = 0
\]
to a system. Find the critical points of the system and determine their type by linearization. (*Do not sketch the trajectories.*)