1. (a) Find the half-range Fourier expansion of the odd periodic extension of

$$f(x) = \begin{cases} \frac{2}{\pi} x & \text{if} \quad 0 < x \le \frac{\pi}{2} \\ \frac{2}{\pi}(\pi - x) & \text{if} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$= \frac{4}{\pi^2}\left\{\int_0^{\pi/2} x \sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right\}$$

$$= \frac{4}{\pi^2}\,\mathrm{Im}\left\{\int_0^{\pi/2} x e^{inx}\,dx + \int_{\pi/2}^{\pi}(\pi - x)e^{inx}\,dx\right\}$$

$$= \frac{4}{\pi^2}\,\mathrm{Im}\left\{\int_0^{\pi/2} x e^{inx}\,dx + \int_{\pi/2}^{0}\xi\, e^{in(\pi-\xi)}(-d\xi)\right\}$$

$$= \frac{4}{\pi^2}\,\mathrm{Im}\left\{\int_0^{\pi/2} x e^{inx}\,dx + e^{in\pi}\int_0^{\pi/2}\xi e^{-in\xi}\,d\xi\right\}$$

Consider the integrals inside $\{\cdots\}$ & integrate by pts

$$\left[\frac{x e^{inx}}{in}\right]_0^{\pi/2} - \left[\frac{e^{inx}}{(in)^2}\right]_0^{\pi/2} + e^{in\pi}\left[\frac{\xi e^{-in\xi}}{-in}\right]_{\pi/2}^{\pi} - e^{in\pi}\left[\frac{e^{-in\xi}}{(-in)^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{-\pi i e^{in\pi/2}}{2n} + \frac{1}{n^2}\left(e^{in\pi/2} - 1\right) + \frac{\pi e^{in\pi} i}{n}\left(e^{-in\pi} - \frac{1}{2}e^{-in\pi/2}\right)$$

$$+ \frac{e^{in\pi}}{n^2}\left(e^{-in\pi} - e^{-in\pi/2}\right) = \frac{\pi}{n}\left(i - \frac{ie^{in\pi/2}}{2} - \frac{e^{in\pi} i e^{-in\pi/2}}{2}\right)$$

$$e^{in\pi/2} = \begin{cases} +i, \\ -1, \\ -i, \\ 1, \end{cases} \quad e^{in\pi} \begin{cases} = -1, & n = 4k-3 \\ = 1, & n = 4k-2 \\ = -1, & n = 4k-1 \\ = 1 & n = 4k \end{cases} \qquad k = 1, 2, 3, \ldots$$

$$\{\text{integrals}\} = \begin{cases} +2i/n^2, & n = 4k-3 \\ -4/n^2, & n = 4k-2 \\ -2i/n^2, & n = 4k-3 \\ 0, & n = 4k \end{cases}$$

Question 1(a) working space on next page

So $b_n = \frac{4}{\pi^2} I_m \{ \text{integrals} \}$

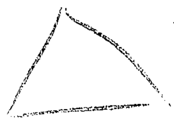
$$= \begin{cases} 0, & n \text{ even} \\ +8/\pi(4k-3)^2, & n = 4k-3 \\ -8/\pi(4k-1)^2, & n = 4k-1 \end{cases} \qquad k = 1, 2, 3, \dots$$

$$= \begin{cases} 0, & n = 2m \\ \dfrac{(-1)^{m+1}}{\pi(2m-1)^2} & n = 2m-1 \end{cases} \qquad m = 1, 2, \dots$$

$$f(x) = \frac{8}{\pi^2} \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots$$

$$= \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(2m-1)x}{(m-1)^2}$$

Note $a_n = \begin{cases} -16/(4k-2)^2\pi^2, & k = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$

& $a_0 = \frac{1}{2} 2 \times$ area under 

$$= \frac{1}{\pi} \cdot \pi = 1$$

& F. cosine series is

$$1 - \frac{16}{\pi^2} \left(\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} + \dots \right.$$

$$= 1 - \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(4k-2)x}{(4k-2)^2}.$$

Question 1(b) on next page

(b) Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

corresponding to the initial deflection

$$u(x,0) = \begin{cases} \frac{2}{\pi}x & \text{if} \quad 0 < x \leq \frac{\pi}{2} \\ \frac{2}{\pi}(\pi - x) & \text{if} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$

and initial velocity $u_t(x,0) = 0$.

Let $u(x,t) = F(x)\,G(t)$

$u_{tt} = F\ddot{G}$, $u(x,0) = f(x)$

$u_{xx} = F''G$ $\quad u_t(x,0) \Rightarrow \dot{G}(0) = 0$

ends fixed $\quad u(0,t) = u(0,\pi) = 0$

$F(0) = F(\pi) = 0$

SEPARATE VARIABLES

$F\ddot{G} = 4F''G$

fn of t only $\quad \ddot{G}/4G = F''/F = \text{const} = -p^2$ for nontrivial solutions

fn of x only

$F'' + p^2 F = 0 \Rightarrow F(x) = A\cos px + B\sin px$

ADMISSIBLE CONSTS

$F(0) = 0 \Rightarrow A = 0$, $F(\pi) = 0 \Rightarrow \sin p\pi = 0$,

so $p_n = n$, $n = 1, 2, \ldots$, $F_n(x) = B_n \sin nx$

EIGENVALS & EIGENFNS

$\ddot{G}/4G = -p^2 = -n^2$

$\ddot{G} + (2n)^2 G = 0$, $G(t) = a\cos 2nt + b\sin 2nt$

$\dot{G}(t) = 0 \Rightarrow b = 0$, $G(t) = \cos 2nt$

$u_n(x,t) = F_n(x)\,G_n(t) = B_n \sin nx \cos 2nt$.

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin nx \cos 2nt$

Question 1(b) working space on next page

INITIAL CONDITION

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

$$= f(x)$$

$$= \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(2m-1)x}{(2m-1)^2}$$

So $B_n = \begin{cases} 0 & , n \text{ even} \\ \frac{8}{\pi^2} \frac{(-1)^{m+1}}{(2m-1)^2} \end{cases}$

$$u(x,t) = \frac{8}{\pi^2} \sum \frac{(-1)^{m+1}}{(2m-1)^2} \sin nx \cos 2nt$$

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2. Find the solution of the heat equation in a bar of length π with both ends insulated, that is, of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

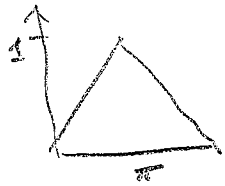
subject to boundary conditions $u_x(0,t) = 0, \quad u_x(\pi,t) = 0$ for all $\geq 0, t$ and initial temperature given by

$$u(x,0) = \begin{cases} \frac{2}{\pi}x & \text{if} \quad 0 < x \leq \frac{\pi}{2} \\ \frac{2}{\pi}(\pi - x) & \text{if} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$

(Hint: look at the even periodic extension of the function in Question 1(a)).

From calculations in Qn 1,

$$a_n = \begin{cases} -16/(4k-2)^2\pi^2, & n = 4k-2 \\ 0, & \text{otherwise} \end{cases}$$

& $a_0 = \frac{1}{2} \times 2 \times$ area under 

$= \frac{1}{\pi}\pi = 1$

So $u(x,0) = f(x) = 1 - \frac{16}{\pi^2}\left(\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} + \dots\right)$

$= 1 - \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(4k-2)x}{(4k-2)^2}$

$= 1 - \frac{4}{\pi^2} \sum_1^{\infty} \frac{\cos(4k-2)x}{(2k-1)^2}$

SEPARATE

$u(x,t) = F(x)G(t)$ $\quad F'(0) = F'(\pi) = 0$

$u_t = F\dot{G}, \quad u_{xx} = F''$

Question 2 working on next page

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$F\dot{G} = F''G$

$F''/F = \dot{G}/G = \text{const} = -k^2$ for nontrivial solns.

$F'' + k^2 F = 0$, $F(x) = A\cos kx + B\sin kx$

$F'(0) = 0 \Rightarrow B = 0$, $F'(\pi) = 0 \Rightarrow \sin k\pi = 0$
$k = 1, 2, \ldots$

$F_n(x) = A_n \cos nx$

EIGENVALS & EIGENFNS

$\dot{G} = -n^2 G$, $G(t) = e^{-n^2 t}$

$u_n(x,t) = A_n \cos nx \, e^{-n^2 t}$

$u(x,t) = \sum_0^\infty A_n \cos nx \, e^{-n^2 t}$

INITIAL CONDN

$u(x,0) = A_0 + \sum_1^\infty A_n \cos nx$
$= f(x)$

So $A_n = \begin{cases} -16/(4k-2)^2\pi^2, & n = 4k-2 \\ 0, & \text{otherwise} \end{cases}$

& $A_0 = a_0 = 1$

$u(x,t) = 1 - \frac{16}{\pi^2} \sum_{k=1}^\infty \frac{\cos(4k-2)x}{(4k-2)^2}$

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3. Find the displacement u(r,t) in a vibrating circular elastic membrane of radius 1 that satisfies the boundary condition

$$u(1,t) = 0, \qquad t \geq 0,$$

and the initial conditions

$$u(r,0) = 0, \qquad \frac{\partial}{\partial t}u(r,0) = 1 - r^4, \qquad 0 \leq r \leq 1.$$

Hint: Use a Fourier-Bessel Series and recall that the Wave Equation, in polar coordinates independent of θ, is

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), \qquad 0 < r < 1,\ t \geq 0.$$

For simplicity take $c = 1$ in the Wave Equation.

SEPARATE VARIABLES

$$u(r,t) = F(r)\,G(t)$$

$$u_{tt} = F\ddot{G}$$

$$u_{rr} = F''G$$

$$F\ddot{G} = c^2(F'' + \frac{1}{r}F')G$$

$$\frac{\ddot{G}}{G} = (F'' + \frac{1}{r}F')/F = -p^2$$

for $\ddot{G}/G$ = const to have nonzero solns

$$\ddot{G} + p^2 G = 0, \qquad G(t) = A\cos pt + B\sin pt$$

$$u(r,0) = 0 \Rightarrow G(0) = 0 \Rightarrow A = 0, \qquad G(t) = B\sin pt$$

$$r^2F'' + rF' + p^2r^2F = 0 \qquad \text{Set } s = pr$$

$$F' = dF/dr = dF/ds\,ds/dr = p\,dF/ds, \quad F'' = p\frac{d}{ds}\left(\frac{dF}{ds}\right)\frac{ds}{dr} = p^2\frac{d^2F}{ds^2}$$

$$\frac{s^2}{p^2}p^2\frac{d^2F}{ds^2} + \frac{s}{p}p\frac{dF}{ds} + s^2F = 0$$

$$s^2 d^2F/ds^2 + s\,dF/ds + s^2F = 0$$

Bessel's equation with $\nu = 0$

$$F(s) = aJ_0(s) + bY_0(s)$$

But $\lim_{s\to 0} Y_0(s) = \infty$, so for finite vibrations have $b = 0$

Question 3 working space on next page

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$F(s) = a J_0(s)$

But $u(1,t) = 0 \Rightarrow F(r) = 0$ when $r = 1$

That is, $J_0(p) = 0$. Hence,

$p = \alpha_n$, the zeros of J_0.

Hence $G_n(t) = B_n \sin p_n t = B_n \sin \alpha_n t$

$F_n(\cancel{s}) = J_0(\alpha_n r)$, $u_n(x,t) = J_0(\alpha_n r) \cancel{B_n} \sin \alpha_n t$

$u(\cancel{x}t) = \sum_{n=1}^{\infty} B_n J_0(\alpha_n r) \sin \alpha_n t$

$\frac{\partial u}{\partial t} = \sum_1^{\infty} B_n J_0(\alpha_n r)\, \alpha_n \cos \alpha_n t$

$\frac{\partial u}{\partial t}(r,0) = \sum_1^{\infty} B_n \alpha_n J_0(\alpha_n r) = 1 - r^4$

So the $\alpha_n B_n$ are the ~~Bes~~ cfts in the Fourier-Bessel series for $1 - r^4$