COLOR LAYER

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FOURIER SERIES

- **Fourier Coefficients;**
- **Normalized Extension;**
- **Period $2L$;**
- **Even & Odd Functions, Cosine & Sine Series;**
- **Half-Range Expansions:**

  # Even periodic extension;

  # Odd periodic extension.

FOURIER SERIES.

The idea is to represent periodic functions

$$f(x+2\pi) = f(x), \qquad \forall\ x,$$

by a **trigonometric series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The a_n, b_n are real numbers called **Fourier coefficients,** and the trigonometric series that we consider are **Fourier series.**

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx. \end{aligned}$$

Example 1. Find FS of:

$$f(x) = x, \ -\pi < x < \pi, \ f(x+2\pi) = f(x).$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{-x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi}\sin nx\,dx = 0, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\
&= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\,dx
\end{aligned}$$

$$\begin{aligned} b_n &= -\frac{2}{n}\cos n\pi = (-1)^{n-1}\,\frac{2}{n} \\ &= \begin{cases} 2/n, & n \text{ odd}, \\ -2/n, & n \text{ even}. \end{cases} \end{aligned}$$

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right].$$

We used the formula

$$\cos n\pi \;=\; \begin{cases} -1 & n \;\; \text{odd} \\ \;\;1 & n \;\; \text{even} \end{cases} = (-1)^{n-1}.$$

Example 2. Find FS of

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ 1, & 0 < x < \pi/2, \\ 0, & \pi/2 \le x \le \pi. \end{cases}$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} 1\,dx = \frac{1}{4},\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi/2}\cos nx\,dx\\
&= \left[\frac{1}{n\pi}\sin nx\right]_0^{\pi/2} = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right),\\
a_1 &= \frac{1}{\pi},\ a_2 = 0,\ a_3 = -\frac{1}{3\pi},\ a_4 = 0, \ldots\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi/2}\sin nx\,dx\\
&= \left[-\frac{\cos nx}{n\pi}\right]_0^{\pi/2} = \frac{1}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right).\\
b_1 &= \frac{1}{\pi},\ b_2 = \frac{1}{\pi},\ b_3 = \frac{1}{3\pi},\ b_4 = 0, \ldots
\end{aligned}$$

If $n = 2k$ is even,

$$a_n = a_{2k} = \frac{1}{2k\pi}\sin k\pi = 0.$$

If $n = 2k+1$ is odd,

$$a_{2k+1} = \frac{1}{(2k+1)\pi}\sin(k+1/2)\pi = \frac{(-1)^k}{(2k+1)\pi}.$$

If $n = 2k$ is even,

$$b_{2k} = \frac{1}{2k\pi}(1 - \cos k\pi) = \frac{1 - (-1)^k}{2k\pi}.$$

If $n = 2k + 1$ is odd,

$$b_{2k+1} = \frac{1}{(2k+1)\pi}(1 - \cos(k + 1/2)\pi) = \frac{1}{(2k+1)\pi}$$

Orthogonality Formulas:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$$

Fourier Coefficients:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cos mx \, dx &= a_0 \int_{-\pi}^{\pi} \cos mx \, dx \\ &+ \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx \right] \\ &+ \sum_{n=1}^{\infty} \left[b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx, \right]\end{aligned}$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \pi a_m.$$

Similarly,

$$\int_{-\pi}^{\pi} f(x) \, dx = 2\pi a_0,$$

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \pi b_m.$$

Provided f satisfies certain conditions

- *periodic, period 2π;*

- *piecewise continuous;*

- *a differentiability condition;*

then the FS is **convergent. The sum is $f(x)$, except at discontinuities $x = \xi$, where the sum is**

$$\frac{1}{2}(f(\xi - 0) + f(\xi + 0))$$

Example 3. Fourier Series of

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < 0, \\ x & \text{if} \quad \phantom{-\pi<{}} 0 < x < \pi, \end{cases} \quad f(x+2\pi) = f(x).$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^\pi x\,dx = \frac{\pi}{4}. \\ a_n &= \frac{1}{\pi}\int_0^\pi x\cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^\pi \\ &= \frac{1}{\pi n^2}(\cos n\pi - 1) \\ &= \frac{1}{\pi n^2}((-1)^n - 1) = \begin{cases} -\frac{2}{\pi n^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx \\
&= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= \frac{1}{n}(-\cos n\pi) = \frac{(-1)^{n+1}}{n} \\
&= \begin{cases} 1/n, & n \text{ odd}, \\ -1/n, & n \text{ even}. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \\
&\quad + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} -\frac{2}{\pi}\frac{\cos(2n-1)x}{(2n-1)^2} \\
&\quad + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}\sin nx.
\end{aligned}
$$

Period p=2L.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \\
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx
\end{aligned}
$$

This is obtained from the FS of 2π periodic functions by a change of scale. Put $\xi = \pi x/L$. Then $x = L\xi/\pi$, and the limits of integration $x = \pm\pi$ correspond to $\xi = \pm L$.

Example 4. Find FS of

$$
f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ x-1 & \text{if } \ \ 0 < x < 1, \end{cases}
$$

$f(x+2) = f(x)$, so period $p = 2$ and $L = 1$.

Note that $f(x)$ is **ODD** on $-1 < x < 1$ and so also is $f(x)\cos nx$. Hence,

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx = 0, \\ a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = 0 \end{aligned}$$

But $f(x)\sin n\pi x$ is **EVEN**, so

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin n\pi x\,dx = 2\int_{0}^{1} f(x)\sin n\pi x\,dx \\ &= 2\int_{0}^{1}(x-1)\sin n\pi x\,dx \\ &= 2\left[-\frac{(x-1)\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 = -\frac{2}{n\pi} \end{aligned}$$

$$\begin{aligned} f(x) &= -\frac{2}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\right] \\ &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}\ . \end{aligned}$$

Example 5. Find FS of

$$f(x) = \begin{cases} x+2, & \text{if} \quad -2 < x < 0, \\ 1 & \text{if} \quad 0 < x < 2, \end{cases}$$

$f(x+4) = f(x)$, period $p = 4$, $L = 2$.

$$\begin{aligned} a_0 &= \frac{1}{4}\int_{-2}^{2} f(x)\,dx = 1, \\ a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}\cos\frac{n\pi x}{2}\,dx \\ &= \frac{1}{2}\int_{-2}^{0} x\cos\frac{n\pi x}{2}\,dx = \frac{4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_{-2}^{0} \\ &= \frac{2}{n^2\pi^2}(1-\cos n\pi) = \begin{cases} \frac{4}{n^2\pi^2}, & n \quad \text{odd}, \\ 0, & n \quad \text{even}. \end{cases} \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx \\
&= \frac{1}{2}\int_{-2}^{0}(x+2)\sin\frac{n\pi x}{2}\,dx \\
&\quad +\frac{1}{2}\int_{0}^{2}\sin\frac{n\pi x}{2}\,dx \\
&= \begin{cases} 0, & n \quad \text{odd} \\ -\frac{2}{n\pi}, & n \quad \text{even.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= 1+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2} \\
&\quad -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}\ .
\end{aligned}
$$

Even and Odd Functions.

$$\begin{aligned} \text{If} \quad g(-x) &= g(x)\ \forall x \quad \textbf{is even}, \\ \text{then} \quad \int_{-L}^{L} g(x)\,dx &= 2\int_{0}^{L} g(x)\,dx. \\ \text{If} \quad h(-x) &= -h(x)\ \forall x \quad \textbf{is odd,} \\ \text{then} \quad \int_{-L}^{L} h(x)\,dx &= 0. \end{aligned}$$

Note that a product of two even functions is **even**, of two odd functions is **even** also, but that a product of an even function with an odd function is **odd.**

- FS of an even function of period $2L$ is a **"cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x.$$

- FS of an odd function of period $2L$ is a **"sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x.$$

Example 6. Find FS of

$$f(x) = |x|, \; -\pi < x < \pi, \; f(x+2\pi) = f(x).$$

f even $\Rightarrow b_n = 0$ for all n.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}, \\ a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{x\sin nx}{n}\right]_0^\pi - \frac{2}{n\pi}\int_0^\pi \sin nx\,dx \\ &= \frac{2}{n^2\pi}[\cos nx]_0^\pi = \frac{2}{n^2\pi}(\cos n\pi - 1) \\ &= \frac{2}{n^2\pi}((-1)^n - 1) \end{aligned}$$

$$a_n = \begin{cases} \frac{-4}{(2k+1)^2\pi}, & \text{if } \ n = 2k+1 \\ 0, & \text{if } \ n = 2k, \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] .$$

If $x = 0$, note that

$$\pi^2 = 8(1 + 1/9 + 1/25 + 1/49 + \ldots).$$

Example 7. Redo Example 1:

$$f(x) = x, \ -\pi < x < \pi, \ f(x + 2\pi) = f(x).$$

f odd $\Rightarrow a_0 = a_n = 0, \ n = 1, 2, \ldots.$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi} \cos nx \, dx \\ &= -\frac{2}{n}\cos n\pi = (-1)^{n-1}\frac{2}{n} . \end{aligned}$$

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] .$$

Example 8. Find FS of

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } 1 < x < 3, \end{cases}$$

$f(x+4) = f(x)$, so $p = 2L = 4$. Note $f(x)$ is even on $-2 < x < 2$, so get Fourier Cosine series,

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}, \\ a_n &= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_1^2 1\cdot\cos\frac{n\pi x}{2}\,dx \\ &= \frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_1^2 \\ &= -\frac{2}{n\pi}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-2(-1)^k}{(2k-1)\pi} & \text{if } n = 2k-1 \text{ odd.} \end{cases} \end{aligned}$$

$$\frac{1}{2} - \frac{2}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \ldots\right)$$

Sum of functions $f_1 + f_2$ has a FS which is the sum of the FS of f_1 and the FS of f_2. The FS of cf is c times the FS of f.

Example 9. Find FS of

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0, \\ 2\sin x & \text{if } 0 \le x \le \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all x.

$$f(x) = (\sin x + |\sin x|).$$

Sum of two functions: $f_1 = \sin x$ has a FS $\sin x$, while $f_2 = |\sin x|$ is an even function and has a cosine series.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}, \\
a_n &= \frac{2}{\pi}\int_0^\pi sinx\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^\pi (\sin(n+1)x - \sin(n-1)x)\,dx \\
&= \begin{cases} \frac{-1}{\pi}[\cos 2x]_0^\pi \qquad (=0), & n=1, \\ \frac{-1}{\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^\pi, & n>1, \end{cases}
\end{aligned}
$$

But $\cos(n+1)\pi = (-1)^{n+1}$, so

$$
\begin{aligned}
a_n &= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n-1}\right] \\
&= \frac{-1}{\pi}\left[\frac{-2(-1)^{n+1}}{n^2-1} + \frac{2}{n^2-1}\right] \\
&= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right] \\
&= \begin{cases} 0, & n \ \text{odd}, \\ \frac{-4}{\pi(n^2-1)}, & n \ \text{even}. \end{cases}
\end{aligned}
$$

Answer:

$$f(x) = \sin x + \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1} \ .$$

Half-Range Expansions.
It is often necessary to use a periodic extension of a function $f(x)$, $0 \le x \le L$ in a FS. This is done in two ways:

- Extend as an **even** function on $-L \le x \le L$ and then in a **cosine series**;

- Extend as an **odd** function on $-L \le x \le L$ and then in a **sine series.**

Example 10.

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 < x < \frac{\pi}{2}, \\ 1 & \text{if} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$

Even Periodic Extension: $b_n = 0$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\pi/2}^{\pi} dx = \frac{1}{2}, \\ a_n &= \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos nx \, dx = \left[\frac{2}{n\pi}\sin nx\right]_{\pi/2}^{\pi} \\ &= -\frac{2}{n\pi}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & \text{if} \quad n = 2k \\ -\frac{2(-1)^{k-1}}{(2k-1)\pi}, & \text{if} \quad n = 2k-1. \end{cases} \end{aligned}$$

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right] .$$

Odd Periodic Extension: $a_n = 0$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin nx\,dx = \left[-\frac{2}{n\pi}\cos nx\right]_{\pi/2}^{\pi} \\ &= -\frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) \\ &= \begin{cases} \frac{2}{n\pi}, & n \text{ odd}, \\ \frac{2}{2k\pi}[(-1)^k - 1], & n = 2k. \end{cases} \end{aligned}$$

$$f(x) = \frac{2}{\pi}\left[\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{0\cdot\sin 4x}{4} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \frac{\sin 7x}{7} + \frac{0\cdot\sin 8x}{8} + \dots\right].$$

Example 11. Find Fourier Sine series:

$$f(x) = L - x,\ 0 < x < L.$$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L (L-x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\frac{L}{n\pi}\left\{\left[-L\cos\frac{n\pi x}{L}\right]_0^L - \left[-x\cos\frac{n\pi x}{L}\right]_0^L \right. \\ &\qquad \left. -\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L\right\} \\ &= \frac{2}{n\pi}\{-L(\cos n\pi - 1) + L\cos n\pi\} \\ &= \frac{2L}{n\pi} \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{2L}{\pi}\left(\frac{1}{1}\sin \pi x L + \frac{1}{2}\sin\frac{2\pi x}{L}\right. \\ &\qquad \left. + \frac{1}{3}\sin\frac{3\pi x}{L} + \dots\right) \end{aligned}$$

Example 12. Find Fourier Cosine series:

$$f(x) = \sin\frac{\pi x}{L}\ , \quad 0 < x < L.$$

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{1}{L}\left[-\frac{L}{\pi}\cos\frac{\pi x}{L}\right]_0^L \\
&= \frac{1}{\pi}(1-\cos\pi) = \frac{2}{\pi}, \\
a_n &= \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{2}\,dx \\
&= \frac{2}{L}\int_1^L \frac{1}{2}\left(\sin\frac{(n+1)\pi x}{L} - \sin\frac{(1-n)\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(\frac{L}{(n+1)\pi}\left[-\cos\frac{(n+1)\pi x}{L}\right]_0^L\right. \\
&\qquad \left. + \frac{L}{(n-1)\pi}\left[-\cos\frac{(n-1)\pi x}{L}\right]_0^L\right)
\end{aligned}$$

$$= \frac{1}{(n+1)\pi}\left(1-\cos(n+1)\pi\right) + \frac{1}{(n-1)\pi}\left(1-\cos(n-1)\pi\right)$$

$$= \frac{1}{(n^2-1)\pi}\Big((n-1)+(n+1) - (n-1+n+1)\cos(n+1)\pi\Big)$$

$$= \frac{2n}{(n^2-1)\pi}\left(1-\cos(n+1)\pi\right).$$

When n is odd, $a_n = 0$; when n is even,

$$a_n = \frac{4n}{(n^2-1)\pi}, n \quad \text{even},$$

FS: $$\frac{2}{\pi} - \frac{4}{\pi}\Big(\frac{2}{1\cdot 3}\cos\frac{2\pi x}{L} + \frac{4}{3\cdot 5}\cos 4\pi x L + \frac{6}{5\cdot 7}\cos\frac{6\pi x}{L} + \ldots\Big)$$

Using Fourier Series Formulas.

Example 13. Find FS of

$$f(x) = \begin{cases} k, & -\pi/2 < x < \pi/2, \\ 0, & \pi/2 < x < 3\pi/2. \end{cases}$$

Hence show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

The FS is a cosine series because this is an even function.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} k\,dx = \frac{k}{2} \\ a_n &= \frac{2}{\pi}\int_0^{\pi/2} k\cos nx\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2} . \end{aligned}$$

So $a_2 = a_4 = \ldots = 0$, and the series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots\right).$$

Now, put $x = 0$, and obtain

$$\begin{aligned} k &= \frac{k}{2} + \frac{2k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \\ \frac{1}{2} &= \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \end{aligned}$$

from which the formula follows.

Example 14. Find FS of

$$f(x) = \frac{x^2}{4}, \quad -\pi < x < \pi.$$

Hence show the formulas

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots,$$
$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \ .$$

This is a cosine series as in the last example

$$f(x) = \frac{\pi^2}{12} - \cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \ldots$$

First, put $x = 0$ and

$$0 = \frac{\pi^2}{12} - 1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \ldots \quad .$$

Then, if $x = \pi$,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} - (-1) + \frac{1}{4} - \left(-\frac{1}{9}\right) + \frac{1}{16} + \ldots$$