Ribbon structure in symmetric pre-monoïdal categories

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\textbf{Abstract.} Let $U(g)$ denote the universal enveloping algebra of a Lie algebra $g$. We show the existence of a ribbon algebra structure in a particular deformation of $U(g)$ which leads to a symmetric pre-monoïdal category of $U(g)$-modules.

1. Introduction

Recently there has been some work\textsuperscript{[1, 2, 3, 4]} on investigating the properties of categories in which the pentagon axiom which is central to the notion of monoïdal categories\textsuperscript{[5]} is forsaken. In particular it was shown in\textsuperscript{[6]} that a certain deformation of the Hopf algebra structure of the universal enveloping algebras of Lie algebras naturally gives rise to a construction for symmetric pre-monoïdal categories as defined in\textsuperscript{[2, 3, 4]}. It is then reasonable to ask to what extent can such categories also be endowed with a ribbon structure, in analogy with the known examples of ribbon categories arising from representations of ribbon quasi-Hopf algebras\textsuperscript{[7]}. Here we note that in a particular specialisation of the deformation given in\textsuperscript{[6]} one can indeed preserve the ribbon algebra structure which provides an initial insight into making the notions of ribbon and pre-monoïdal categories compatible.

2. Symmetric pre-monoïdal categories

We shall begin by defining a pre-monoïdal category, taken from\textsuperscript{[6]}.

\textbf{Definition 1.} A \textit{pre-monoïdal category} is a triple $(\mathcal{C}, \otimes, A)$ where $\mathcal{C}$ is the category of objects, $\otimes$ is a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $A$ is the natural associator isomorphism such that $A : \otimes(id \times \otimes) \to \otimes(\otimes \times id)$.

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In practice we write $a_{U,V,W}$ as the action of the associator such that

$$a_{U,V,W} : U \otimes (V \otimes W) \rightarrow (U \otimes V) \otimes W.$$

We note that there are no conditions imposed on $\mathcal{A}$, but it is important that we should define the natural isomorphism $Q : \otimes (\otimes \times \otimes) \rightarrow \otimes (\otimes \times \otimes)$ via the following diagram:

$$
\begin{array}{cccc}
(U \otimes V) \otimes (W \otimes Z) & \xrightarrow{q_{U,V,W,Z}} & (U \otimes V) \otimes (W \otimes Z) \\
\downarrow & & \downarrow \\
((U \otimes V) \otimes W) \otimes Z & & U \otimes (V \otimes (W \otimes Z)) \\
\downarrow & & \downarrow \\
(U \otimes (V \otimes W)) \otimes Z & \xleftarrow{a_{U,V \otimes W},z} & U \otimes ((V \otimes W) \otimes Z) \\
\end{array}
$$

This box diagram expresses $Q$ in terms of the associator isomorphisms which are defined as:

$$Q((U \otimes V) \otimes (W \otimes Z)) = q_{U,V,W,Z}((U \otimes V) \otimes (W \otimes Z)).$$

(1)

These may be expressed as

$$q_{U,V,W,Z} = a_{U,V,W,1}^{-1}(a_{U,V,W,1}(a_{U,V,W,1}(id \otimes a_{V,W,Z})(id \otimes a_{V,W,Z})(id \otimes a_{V,W,Z}))).$$

(2)

This condition is a generalisation of the pentagon condition used in describing monoidal categories, where $Q = id \otimes id \otimes id \otimes id$. We can examine the significance of $Q$ and its use in distinguishing the coupling of the objects of the category through employing distinct brackets $[,]$. This notation shows that

$$Q([U \otimes V] \otimes [W \otimes Z]) = ([U \otimes V] \otimes [W \otimes Z]).$$

(3)

We describe pre-monoidal categories as being unital if they have an identity object $1 \in \mathcal{C}$ and natural isomorphisms $\rho_U : U \otimes 1 \rightarrow U$ and $\lambda_U : 1 \otimes U \rightarrow U$. The next important class of unital pre-monoidal categories is when the tensor product is commutative up to isomorphism. This leads to the concept of a braided unital pre-monoidal category.

**Definition 2.** A unital pre-monoidal category $\mathcal{C}$ is said to be braided if it is equipped with a natural commutativity isomorphism $\sigma_{U,V} : U \otimes V \rightarrow V \otimes U$ for all objects $U, V \in \mathcal{C}$ such that the following diagrams commute $[4]$:

(i)

$$
\begin{array}{cccc}
(U \otimes (V \otimes W)) & \xrightarrow{a_{U,V,W},1} & (V \otimes W) \otimes U & \xrightarrow{a_{V,W,U}} & V \otimes (W \otimes U) \\
\downarrow & & \downarrow & & \downarrow \\
(U \otimes V) \otimes W & \xrightarrow{a_{U,V \otimes W}} & (V \otimes W) \otimes U & \xrightarrow{a_{V,W,U}} & V \otimes (W \otimes U) \\
\downarrow & & \downarrow & & \downarrow \\
(V \otimes U) \otimes W & \xrightarrow{a_{V,W,U}} & V \otimes (U \otimes W) & & \end{array}
$$
Hereafter we will only be concerned with the case of symmetric categories in which the commutativity isomorphism satisfies the additional condition \( \sigma_{U,V} \circ \sigma_{V,U} = \text{id}_{V \otimes U} \) for all objects \( U, V \in \mathcal{C} \), and \( \circ \) denotes composition of morphisms. In this case the diagrams (i) and (ii) are equivalent.

3. Twining

It is known that the finite-dimensional modules of quasi-triangular quasi-bialgebras give rise to braided monoidal categories, and that these categories are invariant under twisting [8, 9, 10]. Here we need to use a different method to build pre-monoidal categories. We follow the work of [6] where a twining operation was introduced to achieve this.

Define \( A \) as a quasi-triangular quasi-bialgebra by the octuple \( (A, \Delta, \varepsilon, \Phi, \mathcal{R}, S, \alpha, \beta) \) where \( \Delta : A \to A \otimes A \) is the co-product, \( \Phi \) is an invertible element called the co-associator which is defined as \( \Phi = I \otimes I \otimes I \), \( \varepsilon : A \to \mathbb{C} \) is the co-unit, \( \mathcal{R} \in A \otimes A \) is the universal \( \mathcal{R} \)-matrix satisfying

\[
\mathcal{R}\Delta(x) = \Delta^T(x)\mathcal{R} \quad \forall x \in A
\]

where \( \Delta^T \) is the opposite co-product. Further \( S \) is the antipode and \( \alpha, \beta \) are canonical elements satisfying certain properties (see [8, 9, 10] for details). Let \( K \) be an element of the centre of \( A \) and let non-zero \( \gamma \in \mathbb{C} \) be fixed but arbitrary. Following [6] we define

\[
\mathcal{R} = \gamma^{K \otimes K}\mathcal{R} = \mathcal{R} \cdot \gamma^{K \otimes K}
\]

\[
\Phi = \Phi \cdot \gamma^\kappa
\]

such that \( \kappa = K \otimes (I \otimes K + K \otimes I - \Delta(K)) \). In general we write

\[
\Phi = \sum_j X_j \otimes Y_j \otimes Z_j
\]

\[
\Phi^{-1} = \sum_j \bar{X}_j \otimes \bar{Y}_j \otimes \bar{Z}_j
\]

\[
\mathcal{R} = \sum_j a_j \otimes b_j.
\]
Then the following hold:

\[
(id \otimes \Delta) \Delta(a) = \Phi^{-1} (\Delta \otimes id) \Delta(a) \Phi \quad \forall a \in A,
\]

\[
\mathcal{R} \Delta(a) = \Delta^T(a) \mathcal{R},
\]

\[
(\Delta \otimes id) \mathcal{R} = \Phi_{213} \mathcal{R}_{13} \Phi_{123} \mathcal{R}_{23} \Phi_{12}^{-1},
\]

\[
(id \otimes \Delta) \mathcal{R} = \Phi_{213}^{-1} \mathcal{R}_{13} \Phi_{213}^{-1} \mathcal{R}_{12} \Phi_{123} (\gamma^2)^{-1}
\]

and we go on to define

\[
\xi = (\Delta \otimes id \otimes id) \Phi^{-1} (\Phi \otimes I) \cdot (id \otimes \Delta \otimes id) \Phi \cdot (I \otimes \Phi) \cdot (id \otimes id \otimes \Delta) \Phi^{-1}.
\]

These relations show that the category \( \text{mod}_K(A) \) of \( A \)-modules with

\[
a_{U,V,W} = (\pi_U \otimes \pi_V \otimes \pi_W) \Phi,
\]

is a pre-monoidal category as \( \Phi \) fails the pentagon condition. In particular the representation

\[
g_{U,V,W,K} = (\pi_U \otimes \pi_V \otimes \pi_W) \xi
\]

does not act as the identity.

We should note that \( \mathcal{R} \) cannot be used in general as an ingredient in the construction of a braided, pre-monoidal category of \( A \)-modules, as the \( \gamma^2 \) defined above in (4) leads to a violation of the hexagon condition (i) of Definition 2. However, it was observed in [6] that in the case of the universal enveloping algebra \( U(g) \) of a Lie algebra \( g \), it is possible to choose \( \gamma = -1 \) and \( K \in A \) taking integer eigenvalues on all irreducible finite-dimensional \( U(g) \)-modules such that \( \text{mod}_K(A) \) does possess the structure of a symmetric pre-monoidal category. For a given central element \( K \in U(g) \) such that \( \varepsilon(K) = 0 \) we call \( (U(g), \Delta, \varepsilon, \Phi, \mathcal{R}) \) with \( \Phi \) and \( \mathcal{R} \) given by (4) a twining of \( U(g) \).

4. Ribbon algebra structure

In analogy with the definition of ribbon quasi-Hopf algebras [7], we can also investigate the ribbon structure for twined algebras. First recall that each quasi-triangular quasi-Hopf algebra \( A \) possesses a distinguished invertible element \( u \) satisfying

\[
S^0(a) = uav^{-1} \quad \forall a \in A,
\]

\[
S(a)u = \sum_j S(b_j) a a_j.
\]

We then have from [7]:

**Definition 3.** Let \( A \) be a quasi-triangular quasi-Hopf algebra. We say that \( A \) is a ribbon quasi-Hopf algebra if there exists a central element \( v \in A \) such that

i. \( v^2 = u S(u) \)

ii. \( S(v) = v \)
iii. \( \varepsilon(v) = 1 \)

iv. \( \Delta(uv^{-1}) = F^{-1}(S \otimes S)F_{21}(uv^{-1} \otimes uv^{-1}) \)

where \( F \) is the Drinfeld twist [8] defined by the condition

\[
\Delta(a) = F^{-1}((S \otimes S) \Delta^T(S^{-1}(a)))F \quad \forall a \in A.
\]

The universal enveloping algebra \( U(g) \) of a Lie algebra \( g \) acquires the structure of a quasi-bialgebra with mappings:

\[
\begin{align*}
\varepsilon(I) &= 1, \quad \varepsilon(x) = 0, \quad \forall x \in g \\
S(I) &= I, \quad S(x) = -x, \quad \forall x \in g \\
\Delta(I) &= I \otimes I, \quad \Delta(x) = I \otimes x + x \otimes I, \quad \forall x \in g
\end{align*}
\]

(12)

that are extended to all \( U(g) \) such that \( \varepsilon \) and \( \Delta \) are algebra homomorphisms and \( S \) is an anti-automorphism. It is easily checked that \( \Delta \) is co-associative; i.e.

\[
(id \otimes \Delta)\Delta(x) = (\Delta \otimes id)\Delta(x) \quad \forall x \in U(g)
\]

and co-commutative

\[
\Delta(x) = \Delta^T(x) \quad \forall x \in U(g)
\]

and that \( S^3 = \text{id} \). This means that we can equip \( U(g) \) with the structure of a quasi-triangular quasi-Hopf algebra by taking \( \Phi = I \otimes I \otimes I \) for the co-associator of \( U(g) \), \( R = I \otimes I \) as the universal \( R \)-matrix and \( \alpha = \beta = I \). Note that in this instance the Drinfeld twist of \( U(g) \) is trivial, and \( U(g) \) trivially satisfies the conditions of a ribbon Hopf algebra with the choice \( u = v = I \).

Under twining by a central element \( K \) satisfying \( S(K) = -K \) we then have from (5)

\[
\begin{align*}
\hat{R} &= (-1)^{K \otimes K} \\
\hat{\Phi} &= (-1)^{\alpha}
\end{align*}
\]

and moreover it can be shown that \( \alpha = \beta = I \) (cf. [6]). It is easily verified that the choice

\[
v = u = (-1)^{-K^2}
\]

satisfies equations (10) and (11) as well as the conditions of Definition 3. Thus in this instance we can conclude that the twined algebra \( U(g) \) can still be considered as a ribbon algebra.

5. Conclusion

We have shown that a particular case of the twining deformation described in [6] is compatible with the notion of a ribbon structure which can be endowed on quasi-triangular quasi-Hopf algebras. In future work we will explore the compatibility of ribbon categories and symmetric pre-monoidal categories on a general level and in particular investigate the consequences of this for defining traces and inner products on generic symmetric pre-monoidal categories (cf. [11]).
References


