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Ribbon structure in symmetric pre-monoidal categories

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Abstract. Let $U(g)$ denote the universal enveloping algebra of a Lie algebra g. We show the existence of a ribbon algebra structure in a particular deformation of $U(g)$ which leads to a symmetric pre-monoidal category of $U(g)$-modules.

1. Introduction

Recently there has been some work [1, 2, 3, 4] on investigating the properties of categories in which the pentagon axiom which is central to the notion of monoidal categories [5] is forsaken. In particular it was shown in [6] that a certain deformation of the Hopf algebra structure of the universal enveloping algebras of Lie algebras naturally gives rise to a construction for symmetric pre-monoidal categories as defined in [2, 3, 4]. It is then reasonable to ask to what extent can such categories also be endowed with a ribbon structure, in analogy with the known examples of ribbon categories arising from representations of ribbon quasi-Hopf algebras [7]. Here we note that in a particular specialisation of the deformation given in [6] one can indeed preserve the ribbon algebra structure which provides an initial insight into making the notions of ribbon and pre-monoidal categories compatible.

2. Symmetric pre-monoidal categories

We shall begin by defining a pre-monoidal category, taken from [6].

Definition 1. *A pre-monoidal category is a triple* $(\mathcal{C}, \otimes, \mathcal{A})$ *where* $\mathcal{C}$ *is the category of objects,* $\otimes$ *is a bifunctor* $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ *and* $\mathcal{A}$ *is the natural associator isomorphism such that* $\mathcal{A} : \otimes(\mathrm{id} \times \otimes) \to \otimes(\otimes \times \mathrm{id})$.

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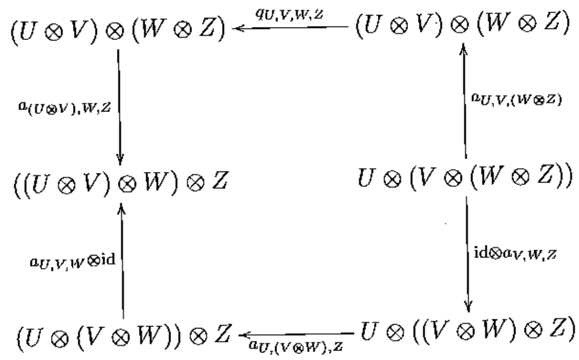
In practice we we write $a_{U,V,W}$ as the action of the associator such that

$$a_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W.$$

We note that there are no conditions imposed on $\mathcal{A}$, but it is important that we should define the natural isomorphism $Q : \otimes(\otimes \times \otimes) \to \otimes(\otimes \times \otimes)$ via the following diagram:



This box diagram expresses Q in terms of the associator isomorphisms which are defined as:

$$Q((U \otimes V) \otimes (W \otimes Z)) = q_{U,V,W,Z}((U \otimes V) \otimes (W \otimes Z)). \tag{1}$$

These may be expressed as

$$q_{U,V,W,Z} = a^{-1}_{(U\otimes V),W,Z}(a_{U,V,W} \otimes \mathrm{id})a_{U,(V\otimes W),Z}(\mathrm{id} \otimes a_{V,W,Z})a^{-1}_{U,V,(W\otimes Z)}. \tag{2}$$

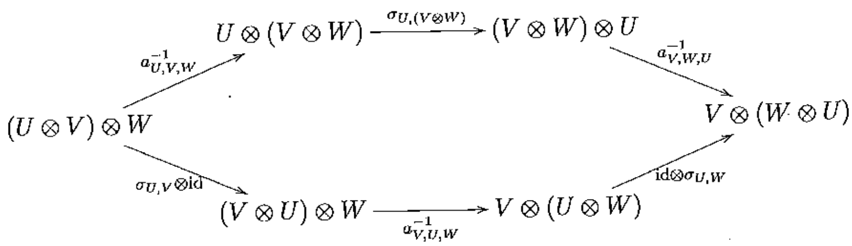
This condition is a generalisation of the pentagon condition used in describing monoidal categories, where $Q = \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}$. We can examine the significance of Q and its use in distinguishing the coupling of the objects of the category through employing distinct brackets $[], \{\}$. This notation shows that

$$Q([U \otimes V] \otimes \{W \otimes Z\}) = (\{U \otimes V\} \otimes [W \otimes Z]). \tag{3}$$

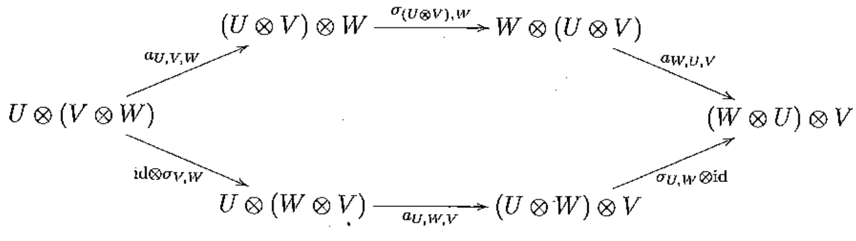
We describe pre-monoidal categories as being unital if they have an identity object $\mathbf{1} \in \mathcal{C}$ and natural isomorphisms $\rho_U : U \otimes \mathbf{1} \to U$ and $\lambda_U : \mathbf{1} \otimes U \to U$. The next important class of unital pre-monoidal categories is when the tensor product is commutative up to isomorphism. This leads to the concept of a braided unital pre-monoidal category.

Definition 2. *A unital pre-monoidal category $\mathcal{C}$ is said to be braided if it is equipped with a natural commutativity isomorphism $\sigma_{U,V} : U \otimes V \to V \otimes U$ for all objects $U, V \in \mathcal{C}$ such that the following diagrams commute [4]:*

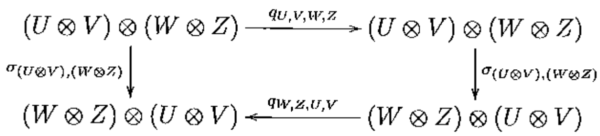
(i)



(ii)



(iii)



Hereafter we will only be concerned with the case of *symmetric* categories in which the commutativity isomorphism satisfies the additional condition $\sigma_{U,V} \circ \sigma_{V,U} = \mathrm{id}_{V\otimes U}$ for all objects $U,\ V \in \mathcal{C}$, and $\circ$ denotes composition of morphisms. In this case the diagrams (i) and (ii) are equivalent.

3. Twining

It is known that the finite-dimensional modules of quasi-triangular quasi-bialgebras give rise to braided monoidal categories, and that these categories are invariant under twisting [8, 9, 10]. Here we need to use a different method to build pre-monoidal categories. We follow the work of [6] where a twining operation was introduced to achieve this.

Define A as a quasi-triangular quasi-bialgebra by the octuple $(A, \Delta, \varepsilon, \Phi, \mathcal{R}, S, \alpha, \beta)$ where $\Delta : A \to A \otimes A$ is the co-product, Φ is an invertible element called the co-associator which is defined as $\Phi = I \otimes I \otimes I$, $\varepsilon : A \to \mathbb{C}$ is the co-unit, $\mathcal{R} \in A \otimes A$ is the universal R-matrix satisfying

$$\mathcal{R}\Delta(x) = \Delta^T(x)\mathcal{R} \quad \forall x \in A$$

where Δ^T is the opposite co-product. Further S is the antipode and α, β are canonical elements satisfying certain properties (see [8, 9, 10] for details). Let K be an element of the centre of A and let non-zero $\gamma \in \mathbb{C}$ be fixed but arbitrary. Following [6] we define

$$\begin{aligned} \tilde{\mathcal{R}} &= \gamma^{K\otimes K}\mathcal{R} = \mathcal{R}\cdot\gamma^{K\otimes K} \\ \tilde{\Phi} &= \Phi\cdot\gamma^{\kappa} \end{aligned} \tag{4}$$

such that $\kappa = K \otimes (I \otimes K + K \otimes I - \Delta(K))$. In general we write

$$\begin{aligned} \tilde{\Phi} &= \sum_j X_j \otimes Y_j \otimes Z_j \\ \tilde{\Phi}^{-1} &= \sum_j \bar{X}_j \otimes \bar{Y}_j \otimes \bar{Z}_j \\ \tilde{\mathcal{R}} &= \sum_j a_j \otimes b_j. \end{aligned} \tag{5}$$

Then the following hold:

$$
\begin{aligned}
&(\mathrm{id}\otimes\Delta)\Delta(a) = \tilde{\Phi}^{-1}(\Delta\otimes\mathrm{id})\Delta(a)\tilde{\Phi} \quad \forall\, a\in A,\\
&\tilde{\mathcal{R}}\Delta(a) = \Delta^{T}(a)\tilde{\mathcal{R}},\\
&(\Delta\otimes\mathrm{id})\tilde{\mathcal{R}} = \tilde{\Phi}_{312}\tilde{\mathcal{R}}_{13}\tilde{\Phi}_{132}\tilde{\mathcal{R}}_{23}\tilde{\Phi}_{123}^{-1},\\
&(\mathrm{id}\otimes\Delta)\tilde{\mathcal{R}} = \tilde{\Phi}_{213}^{-1}\tilde{\mathcal{R}}_{13}\tilde{\Phi}_{213}^{-1}\tilde{\mathcal{R}}_{12}\tilde{\Phi}_{123}(\gamma^{2\kappa})_{123}^{-1}
\end{aligned}
\tag{6}
$$

and we go on to define

$$\xi = (\Delta\otimes\mathrm{id}\otimes\mathrm{id})\tilde{\Phi}^{-1}(\tilde{\Phi}\otimes I)\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\tilde{\Phi}\cdot(I\otimes\tilde{\Phi})\cdot(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\tilde{\Phi}^{-1}. \tag{7}$$

These relations show that the category $\mathbf{mod}_K(A)$ of A-modules with

$$a_{U,V,W} = (\pi_U\otimes\pi_V\otimes\pi_W)\tilde{\Phi}, \tag{8}$$

is a pre-monoidal category as $\tilde{\Phi}$ fails the pentagon condition. In particular the representation

$$q_{U,V,W,Z} = (\pi_U\otimes\pi_V\otimes\pi_Z)\xi \tag{9}$$

does not act as the identity.

We should note that $\tilde{\mathcal{R}}$ cannot be used in general as an ingredient in the construction of a braided, pre-monoidal category of A-modules, as the $\gamma^{2\kappa}$ defined above in (4) leads to a violation of the hexagon condition (i) of Definition 2. However, it was observed in [6] that in the case of the universal enveloping algebra $U(g)$ of a Lie algebra g, it is possible to choose $\gamma = -1$ and $K\in A$ taking integer eigenvalues on all irreducible finite-dimensional $U(g)$-modules such that $\mathbf{mod}_K(A)$ does possess the structure of a symmetric pre-monoidal category. For a given central element $K\in U(g)$ such that $\varepsilon(K)=0$ we call $(U(g),\Delta,\varepsilon,\tilde{\Phi},\tilde{\mathcal{R}})$ with $\tilde{\Phi}$ and $\tilde{\mathcal{R}}$ given by (4) a *twining* of $U(g)$.

4. Ribbon algebra structure

In analogy with the definition of ribbon quasi-Hopf algebras [7], we can also investigate the ribbon structure for twined algebras. First recall that each quasi-triangular quasi-Hopf algebra A possesses a distinguished invertible element u satisfying

$$S^2(a) = uau^{-1} \quad \forall\, a\in A, \tag{10}$$

$$S(\alpha)u = \sum_j S(b_j)\alpha a_j. \tag{11}$$

We then have from [7]:

Definition 3. *Let A be a quasi-triangular quasi-Hopf algebra. We say that A is a ribbon quasi-Hopf algebra if there exists a central element $v\in A$ such that*

i. $v^2 = uS(u)$

ii. $S(v) = v$

iii. $\varepsilon(v) = 1$

iv. $\Delta(uv^{-1}) = \mathcal{F}^{-1}(S \otimes S)\mathcal{F}_{21}(uv^{-1} \otimes uv^{-1})$

where $\mathcal{F}$ is the Drinfeld twist [8] defined by the condition

$$\Delta(a) = \mathcal{F}^{-1}((S \otimes S)\Delta^T(S^{-1}(a)))\mathcal{F} \quad \forall a \in A.$$

The universal enveloping algebra $U(g)$ of a Lie algebra g acquires the structure of a quasi-bialgebra with mappings:

$$\begin{aligned} &\varepsilon(I) = 1,\ \varepsilon(x) = 0,\ \forall x \in g \\ &S(I) = I,\ S(x) = -x,\ \forall x \in g \\ &\Delta(I) = I \otimes I,\ \Delta(x) = I \otimes x + x \otimes I,\ \forall x \in g \end{aligned} \tag{12}$$

that are extended to all $U(g)$ such that ϵ and Δ are algebra homomorphisms and S is an anti-automorphism. It is easily checked that Δ is co-associative; i.e.

$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x) \quad \forall x \in U(g)$$

and co-commutative

$$\Delta(x) = \Delta^T(x) \quad \forall x \in U(g)$$

and that $S^2 = \mathrm{id}$. This means that we can equip $U(g)$ with the structure of a quasi-triangular quasi-Hopf algebra by taking $\Phi = I \otimes I \otimes I$ for the co-associator of $U(g)$, $\mathcal{R} = I \otimes I$ as the universal R-matrix and $\alpha = \beta = I$. Note that in this instance the Drinfeld twist of $U(g)$ is trivial, and $U(g)$ trivially satisfies the conditions of a ribbon Hopf algebra with the choice $u = v = I$.

Under twining by a central element K satisfying $S(K) = -K$ we then have from (5)

$$\begin{aligned} \tilde{\mathcal{R}} &= (-1)^{K \otimes K} \\ \tilde{\Phi} &= (-1)^{\kappa} \end{aligned}$$

and moreover it can be shown that $\alpha = \beta = I$ (cf. [6]). It is easily verified that the choice

$$v = u = (-1)^{-K^2}$$

satisfies equations (10) and (11) as well as the conditions of Definition 3. Thus in this instance we can conclude that the twined algebra $U(g)$ can still be considered as a ribbon algebra.

5. Conclusion

We have shown that a particular case of the twining deformation described in [6] is compatible with the notion of a ribbon structure which can be endowed on quasi-triangular quasi-Hopf algebras. In future work we will explore the compatibility of ribbon categories and symmetric pre-monoidal categories on a general level and in particular investigate the consequences of this for defining traces and inner products on generic symmetric pre-monoidal categories (cf. [11]).

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