Efficient Sequential Importance Sampling for Counting Vertex Covers via Stochastic Relaxation Technique

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Overview

1. Counting vertex covers in a graph
2. Sequential Importance Sampling
3. Stochastic Graph Relaxation
4. Numerical Study
5. Future plans
A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set.

\[
\text{VC} = \{\{v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_2, v_3\}\}.
\]
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Hardness of the vertex cover problem

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- Counting all vertex covers is \#P.

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A basic SIS Algorithm

Given a graph $G = (V, E)$, $|V| = n$, execute the following steps.

1. Define a graph vertex ordering $(v_1, \ldots, v_n)$, set $C = \emptyset$, and set $X = 1$.
2. For each $i = 1, \ldots, n$, flip a $\tilde{p}_i - \tilde{q}_i$ coin.
   - If heads: take $v_i$ to the cover ($C = C \cup \{v_i\}$), and set $X = \tilde{p}_i - 1$.
   - else, set $X = \tilde{q}_i - 1$.
3. If $C$ does not form a valid vertex cover, set $X = 0$. Return $X$ as an estimator of the number of vertex covers in the $G = (V, E)$ graph.

Theorem (Estimator unbiasedness)
For any $\tilde{p}_i$ and $\tilde{q}_i = 1 - \tilde{p}_i$ ($i = 1, \ldots, n$), it holds that:

$$E(X) = \text{VC}(G) = \text{# of vertex covers in } G.$$
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The Importance Sampling

Choose:

\[ \tilde{p}_i^* = \frac{|\{\text{valid covers which contain } C \cup \{v_i\}\}|}{|\{\text{valid covers which contain } C\}|} \quad \forall i = 1, \ldots, n. \]
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For example, suppose we generate \( v_1 \rightarrow v_2 \rightarrow v_3 \). In this case,

\[ X = \tilde{p}_1^{-1} \cdot \tilde{p}_2^{-1} \cdot \tilde{q}_3^{-1} = \frac{5}{3} \cdot \frac{3}{2} \cdot \frac{2}{1} = 5. \]
Unfortunately, calculating

\[ \tilde{p}_i^* = \frac{|\{\text{valid covers which contain } C \cup \{v_i\}\}|}{|\{\text{valid covers which contain } C\}|} \quad \forall i = 1, \ldots, n, \]

is hard. To find \( \tilde{p}_i^* \), we have to calculate the number of vertex covers in \( G \); but, this is the problem we tried to solve in the first place...
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Can we choose \( \tilde{p}_i = 1/2 \)? – Yes, **BUT** the estimator will generally result in high variance...
Our problem summary

- Unfortunately, calculating

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- Can we choose \( \tilde{p}_i = 1/2 \)? – Yes, **BUT** the estimator will generally result in high variance...

- This is a very important problem, and we will try to solve it using stochastic graph relaxation technique.
Let $G = (V, E)$ be a graph and let $|V| = n$. Introduce some vertex ordering $(v_1, v_2, \cdots, v_n)$ and denote by $d_i = \{j \mid (v_i, v_j) \in E, j > i\}$ a neighbors set of $v_i$ such that each neighbor $v_j$ satisfies $j > i$. 

**Definition (Induced probability vector)**

A probability vector induced by $G$ is given by $P = (p_1, \cdots, p_n) = (|d_1| n - 1, |d_2| n - 2, \cdots, |d_{n-1}| 1, 0)$. 

$$(v_1, v_2, v_3, v_4) \Rightarrow P = (2, 3, 2, 2, 1, 1, 0).$$
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**Definition (Induced probability vector)**

A probability vector induced by $G$ is given by

$$P = (p_1, \cdots, p_n) = \left(\frac{|d_1|}{n-1}, \frac{|d_2|}{n-2}, \cdots, \frac{|d_{n-1}|}{1}, 0\right).$$

For the example given:

$$(v_1, v_2, v_3, v_4) \Rightarrow P = \left(\frac{2}{3}, \frac{2}{2}, \frac{1}{1}, 0\right).$$
Consider a probability space $\Omega_G$ of all random graphs $G' = (V', E')$ where the set of vertexes remains the same as in $G$ i.e. $V' = V$ but each edge $(v_i, v_j)$ $i < j$ is present with probability $p_i = \frac{|d_i|}{n-i}$. In particular, the following graphs can be generated from $P = \left(\frac{2}{3}, \frac{2}{2}, \frac{1}{1}, 0\right)$. 

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![Graphs generated from P = (2/3, 2/2, 1/1, 0)]
Let $G'$ be a random variable such that $G' \in \Omega_G$, and note that $\mathbb{P}(G' \in \Omega_G)$ is well defined by probability vector $P$ we can write the expected number of vertex covers under $\Omega_G$ as

$$
\mathbb{E}_{\Omega_G}(|V\!C(G')|) = \sum_{G'' \in \Omega_G} \mathbb{P}(G'') |V\!C(G')|.
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Proposed approximation to the number of covers in $G$

We propose to use the $\mathbb{E}_{\Omega_G}(|V\!C(G')|)$ value, as an approximation to the real number of covers in the original graph $G$. 

Theorem

There exists a deterministic polynomial time Dynamic Programming Algorithm that calculates $\mathbb{E}_{\Omega_G}(|V\!C(G')|)$ analytically.
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$$\mathbb{E}_{\Omega_G}(|\text{VC}(G')|) = \sum_{G'' \in \Omega_G} \mathbb{P}(G'')|\text{VC}(G')|.$$ 

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Stochastic Graph Relaxation (3)

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There exists a deterministic polynomial time Dynamic Programming Algorithm that calculates $\mathbb{E}_{\Omega_G} (|\text{VC}(G')|)$ analytically.
Lemma

Given that an instance $G = (V, E)$ induce a probability vector $P = (p_1, \ldots, p_n)$ where each $p_i$ satisfies $p_i \in \{0, 1\}$, the Dynamic Programming Algorithm provides the exact answer to the number of vertex covers, i.e. $|VC(G)| = \mathbb{E}_{\Omega_G}[|VC(G')|]$. 
Lemma

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Proof.

Notice that there is only one graph in $\Omega_G$ when $G$ induce a vector $P$ where $p_i \in \{0, 1\}$, this follows immediately from the construction process of random graph under this particular $\Omega_G$. In other words, we have that $\forall G'(V, E') \in \Omega_G, G'(V, E') = G(V, E)$ so $|VC(G')| = \mathbb{E}_{\Omega_G} (|VC(G')|)$.
Figure: Algorithm A — Rassmussen Fully Polynomial Randomized Approximation Scheme. Algorithm B — SIS.
**Model 1**

A graph with $|V| = 100$ and $|E| = 2,432$. The graph was generated in the following way. We defined the number of vertexes to be 100 and each edge $(v_i, v_j)$ was generated with probability $Ber(p)$ while $p$ is also a random variable such that $p \sim Uni(0,1)$.

- *cachet* delivers an exact solution of 244,941 in 0.75 seconds.
- We ran *SampleSearch* for 10 times and it provides an average of 192,251.25 using 60 seconds time limit.
- The SIS Algorithm ($N = 100$) delivered $2.440 \times 10^5$ in 1.698 seconds. The RE is $1.614 \times 10^{-2}$

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

**Figure**: Histogram of 1,000 Importance Weights for Model 1.
A graph with $|V| = 300$ and $|E| = 21,094$. The graph was generated in the following way. We defined the number of vertexes to be 300 and each edge $(v_i, v_j)$ was generated with probability $Ber(p_i)$ while $p_i$ is also a random variable such that $p_i \sim Uni(0, 1)$. The results are self explanatory.

- *cachet* delivers an exact solution of $1.306 \times 10^{14}$ in about 17 minutes.
- We ran *SampleSearch* for 10 times and it provides an average of $6.001 \times 10^{13}$ using 1,200 seconds time limit.
- The SIS Algorithm ($N = 100$) delivered $1.387 \times 10^{14}$ in 56.64 seconds. The RE is $4.171 \times 10^{-2}$

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

**Figure:** Histogram of 1,000 Importance Weights for Model 2.
A graph with $|V| = 1,000$ and $|E| = 64,251$. The graph was generated in the following way. We defined the number of vertexes to be 1,000 and each edge $(v_i, v_j)$ was generated from $Ber(p)$ while each $p$ is generated from truncated Normal distribution with $\mu = 0.1$ and $\sigma = 0.1$. The results are summarized below.

- `cachet` was timed out after 2 days and was unable to deliver a solution. The lower bound of $3.439E + 09$ was supplied.
- `SampleSearch` failed to initialize, probably, because the problem is too big.
- The SIS Algorithm ($N = 100$) delivered $4.261 \times 10^{32}$ in 648.6 seconds. The RE is $4.813 \times 10^-2$

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

**Figure:** Histogram of 1,000 Importance Weights for Model 3.
A graph with $|V| = 1,000$ and $|E| = 249,870$. The graph was generated in the following way. We defined the number of vertexes to be 1,000 and each edge $(v_i, v_j)$ was generated from $Ber(p)$ while each $p$ is generated from truncated Normal distribution with $\mu = 0.5$ and $\sigma = 0.3$. The results are summarized below.

- *cachet* was timed out after 2 days and was unable to deliver a solution. The lower bound of $9.601E + 10$ was supplied.
- *SampleSearch* failed to initialize, probably, because the problem is too big.
- The SIS Algorithm ($N = 100$) delivered $2.773 \times 10^{11}$ in 1,718 seconds. The RE is $1.579 \times 10^{-2}$

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

**Figure:** Histogram of 1,000 Importance Weights for **Model 4**.
What next?

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Thank You!