Efficient Sequential Importance Sampling for Counting Vertex Covers via Stochastic Relaxation Technique

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(joint work with Zdravko Botev (UNSW) and
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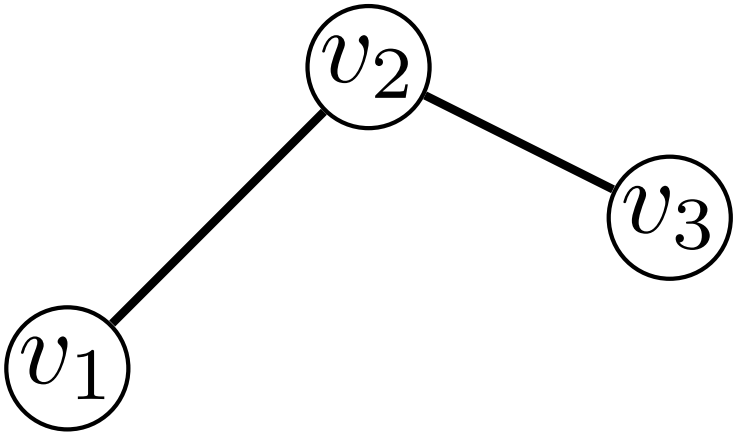
Overview

Vertex Cover

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set.



$$\mathrm{VC} = \{\{v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_2, v_3\}\}.$$

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- Proposed solution — Sequential Importance Sampling (SIS).

  R. Vaisman, Z. I. Botev and A. Ridder, Sequential Monte Carlo for Counting Vertex Covers in General Graphs, Statistics and Computing, 2014.

A basic SIS Algorithm

Given a graph $G = (V, E)$, $|V| = n$, execute the following steps.

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2. For each $i = 1, \ldots, n$, flip a $\tilde{p}_i - \tilde{q}_i$ coin.
   - If heads: take v_i to the cover ($\mathbf{C} = \mathbf{C} \cup \{v_i\}$), and set $X = \tilde{p}_i^{-1} X$,
   - else, set $X = \tilde{q}_i^{-1} X$,

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Theorem (Estimator unbiasedness)

For **any** $\tilde{p}_i$ *and* $\tilde{q}_i = 1 - \tilde{p}_i$ $(i = 1, \ldots, n)$, *it holds that:*

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What is the best way to choose the probability $\tilde{p}_i$ for $i = 1, \ldots, n$?

The Importance Sampling

Choose:

$$\tilde{p}_i^* = \frac{|\{\text{valid covers which contain } \mathbf{C} \cup \{v_i\}\}|}{|\{\text{valid covers which contain } \mathbf{C}\}|} \quad \forall i = 1, \ldots, n.$$

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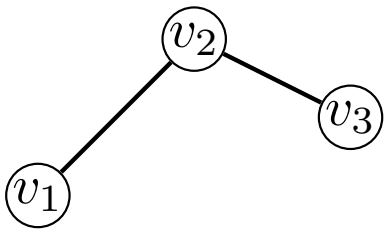

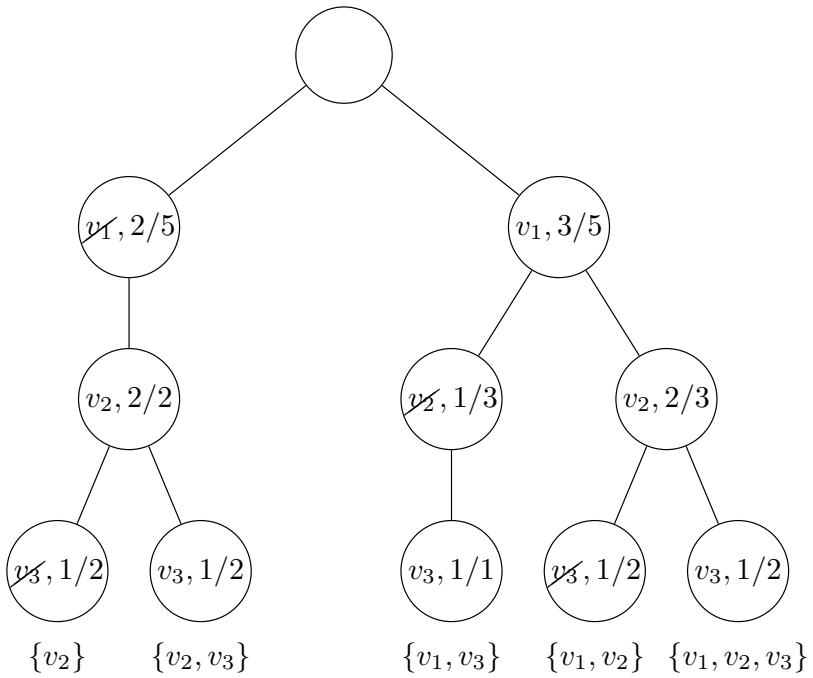

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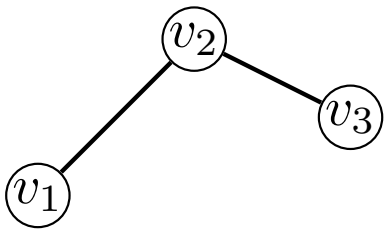

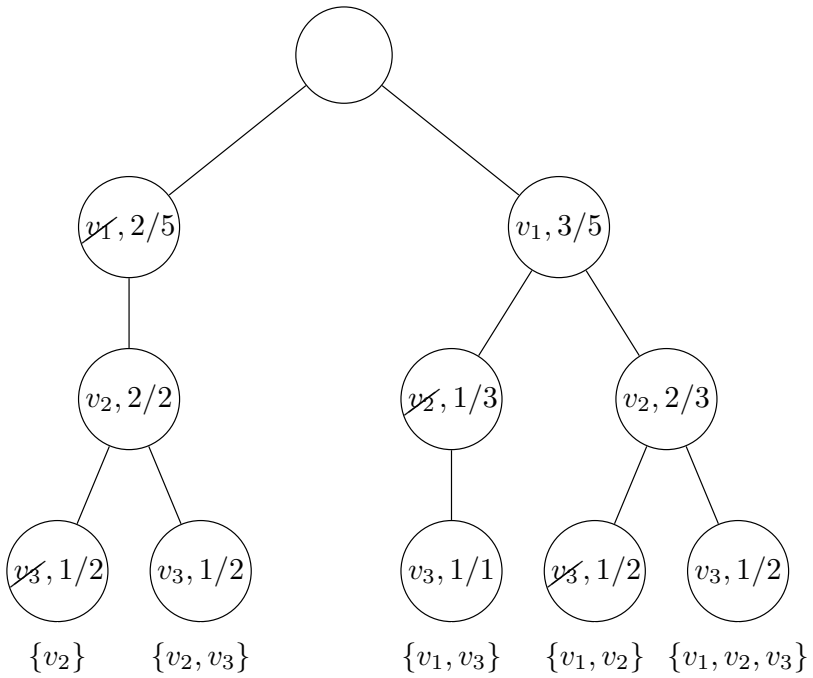

For example, suppose we generate $v_1 \to v_2 \to \cancel{v_3}$. In this case,

$$X = \tilde{p}_1^{-1} \cdot \tilde{p}_2^{-1} \cdot \tilde{q}_3^{-1} = \frac{5}{3} \cdot \frac{3}{2} \cdot \frac{2}{1} = 5.$$

Our problem summary

- Unfortunately, calculating

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- Can we choose $\tilde{p}_i = 1/2$? – Yes, **BUT** the estimator will generally result in high variance...
- This is a very important problem, and we will try to solve it using stochastic graph relaxation technique.

Stochastic Graph Relaxation (1)

- Let $G = (V, E)$ be a graph and let $|V| = n$. Introduce some vertex ordering $(v_1, v_2, \cdots v_n)$ and denote by $d_i = \{j \mid (v_i, v_j) \in E,\ j > i\}$ a neighbors set of v_i such that each neighbor v_j satisfies $j > i$.

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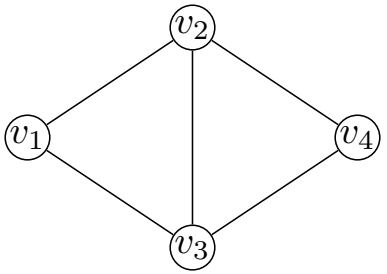
Definition (Induced probability vector)

A probability vector induced by G is given by

$$P = (p_1, \cdots, p_n) = \left(\frac{|d_1|}{n-1}, \frac{|d_2|}{n-2}, \cdots, \frac{|d_{n-1}|}{1}, 0 \right).$$



$$(v_1, v_2, v_3, v_4) \quad \Rightarrow \quad P = \left(\tfrac{2}{3}, \tfrac{2}{2}, \tfrac{1}{1}, 0\right).$$

Stochastic Graph Relaxation (2)

Consider a probability space Ω_G of all random graphs $G' = (V', E')$ where the set of vertexes remains the same as in G i.e $V' = V$ but each edge (v_i, v_j) $i < j$ is present with probability $p_i = \frac{|d_i|}{n-i}$. In particular, the follwing graphs can be generated from $P = \left(\frac{2}{3}, \frac{2}{2}, \frac{1}{1}, 0\right)$.

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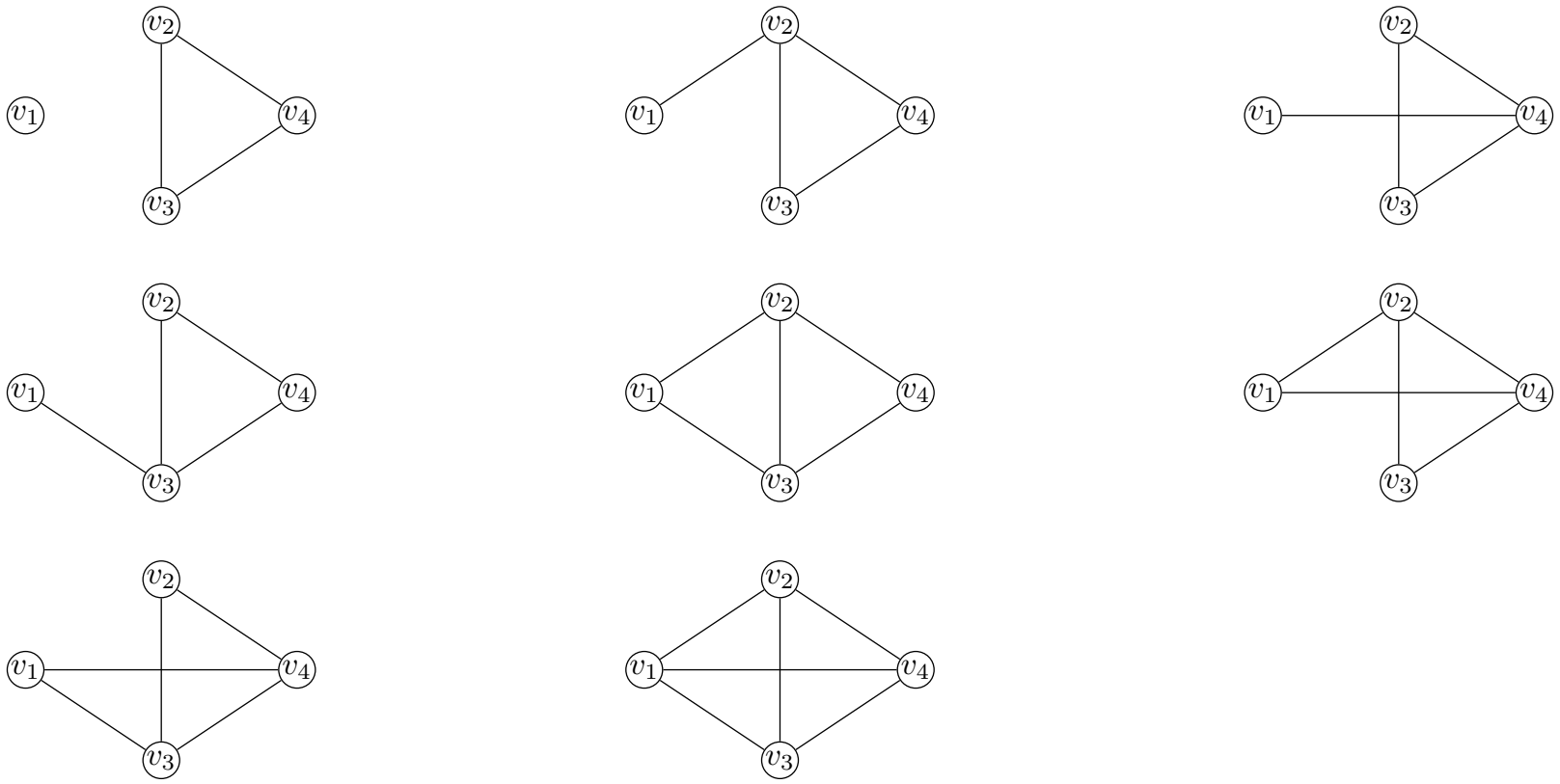

Stochastic Graph Relaxation (3)

Let G' be a random variable such that $G' \in \Omega_G$, and note that $\mathbb{P}(G' \in \Omega_G)$ is well defined by probability vector P we can write the expected number of vertex covers under Ω_G as

$$\mathbb{E}_{\Omega_G}\left(|\mathrm{VC}(G')|\right) = \sum_{G'' \in \Omega_G} \mathbb{P}(G'')|\mathrm{VC}(G')|.$$

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Proposed approximation to the number of covers in G

We propose to use the $\mathbb{E}_{\Omega_G}\left(|\mathrm{VC}(G')|\right)$ value, as an approximation to the real number of covers in the original graph G.

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Theorem

There exists a deterministic polynomial time Dynamic Programming Algorithm that calculates $\mathbb{E}_{\Omega_G}\left(|\mathrm{VC}(G')|\right)$ *analytically.*

A nice feature — an exact vertex cover computation

Lemma

Given that an instance $G = (V, E)$ induce a probability vector $P = (p_1, \cdots, p_n)$ where each p_i satisfies $p_i \in \{0, 1\}$, the Dynamic Programming Algorithm provides the exact answer to the number of vertex covers, i.e $|\mathrm{VC}(G)| = \mathbb{E}_{\Omega_G}[|\mathrm{VC}(G')|]$.

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Proof.

Notice that there is only one graph in Ω_G when G induce a vector P where $p_i \in \{0, 1\}$, this follows immediately from the construction process of random graph under this particular Ω_G. In other words, we have that $\forall G'(V, E') \in \Omega_G,\ G'(V, E') = G(V, E)$ so $|\mathrm{VC}(G')| = \mathbb{E}_{\Omega_G}(|\mathrm{VC}(G')|)$.

$\square$

Random Graph $G(n, 1/2)$

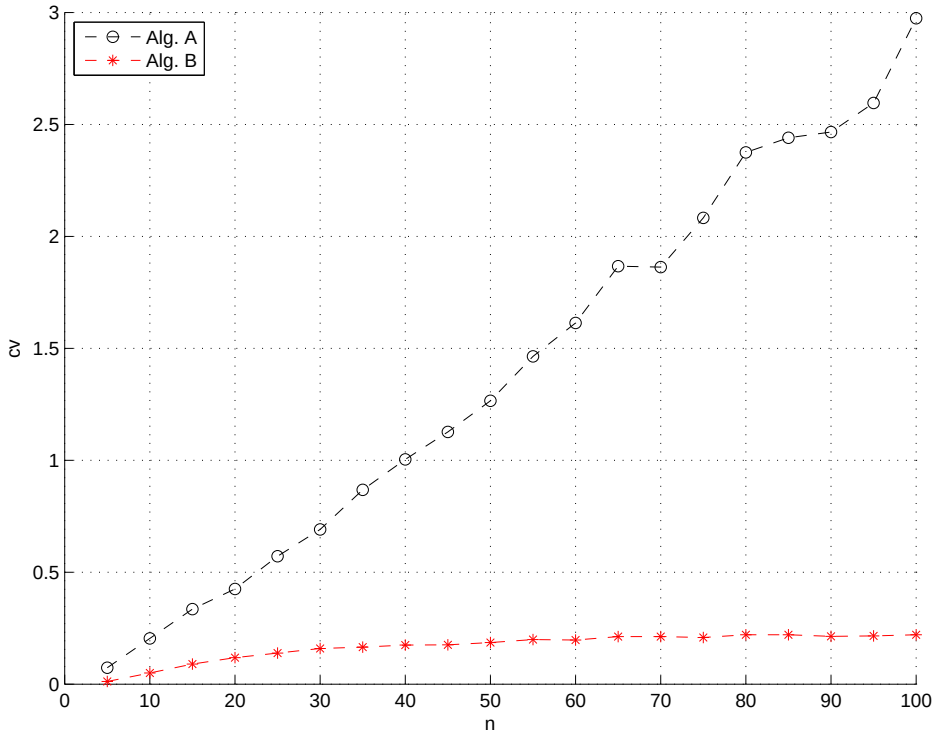

Figure: Algorithm A — Rassmussen Fully Polynomial Randomized Approximation Scheme. Algorithm B — SIS.

Model 1

Model 1 A graph with $|V| = 100$ and $|E| = 2,432$. The graph was generated in the following way. We defined the number of vertexes to be 100 and each edge (v_i, v_j) was generated with probability $Ber(p)$ while p is also a random variable such that $p \sim Uni(0, 1)$.

- *cachet* delivers an exact solution of $244,941$ in 0.75 seconds.
- We ran *SampleSearch* for 10 times and it provides an average of $192,251.25$ using 60 seconds time limit.
- The SIS Algorithm $(N = 100)$ delivered 2.440×10^5 in 1.698 seconds. The RE is 1.614×10^{-2}

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

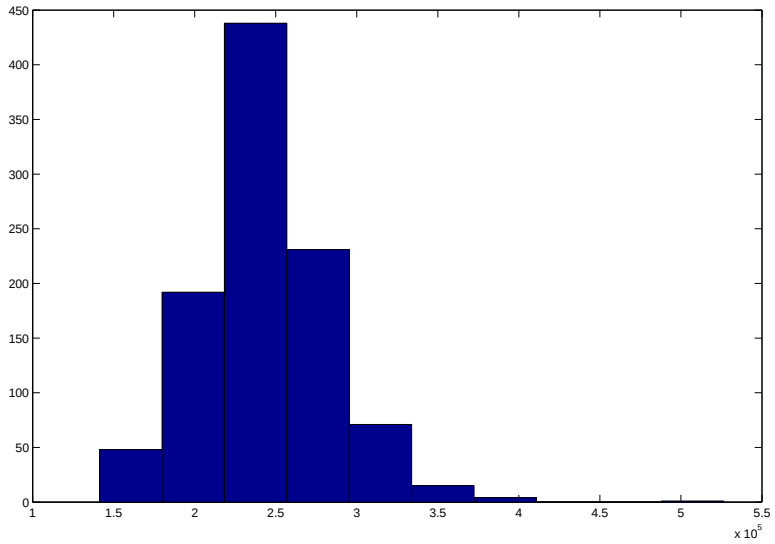

Figure: Histogram of 1,000 Importance Weights for **Model 1**.

Model 2

A graph with $|V| = 300$ and $|E| = 21,094$. The graph was generated in the following way. We defined the number of vertexes to be 300 and each edge (v_i, v_j) was generated with probability $Ber(p_i)$ while p_i is also a random variable such that $p \sim Uni(0,1)$. The results are self explanatory.

- *cachet* delivers an exact solution of 1.306×10^{14} in about 17 minutes.
- We ran *SampleSearch* for 10 times and it provides an average of 6.001×10^{13} using $1,200$ seconds time limit.
- The SIS Algorithm ($N = 100$) delivered 1.387×10^{14} in 56.64 seconds. The RE is 4.171×10^{-2}

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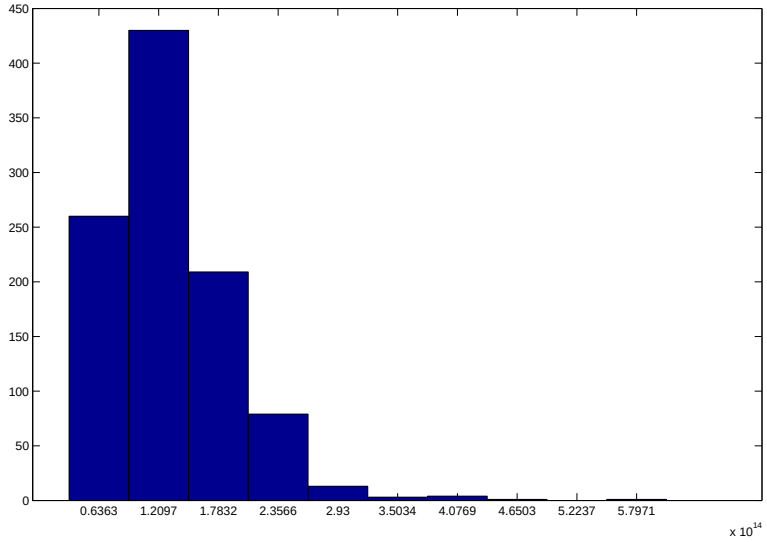

Figure: Histogram of 1,000 Importance Weights for **Model 2**.

Model 3

A graph with $|V| = 1,000$ and $|E| = 64,251$. The graph was generated in the following way. We defined the number of vertexes to be 1,000 and each edge (v_i, v_j) was generated from $Ber(p)$ while each p is generated from truncated Normal distribution with $\mu = 0.1$ and $\sigma = 0.1$ The results are summarized bellow.

- *cachet* was timed out after 2 days and was unable to deliver a solution. The lower bound of $3.439E + 09$ was supplied.
- *SampleSearch* failed to initialize, probably, because the problem is too big.
- The SIS Algorithm ($N = 100$) delivered 4.261×10^{32} in 648.6 seconds. The RE is 4.813×10^{-2}

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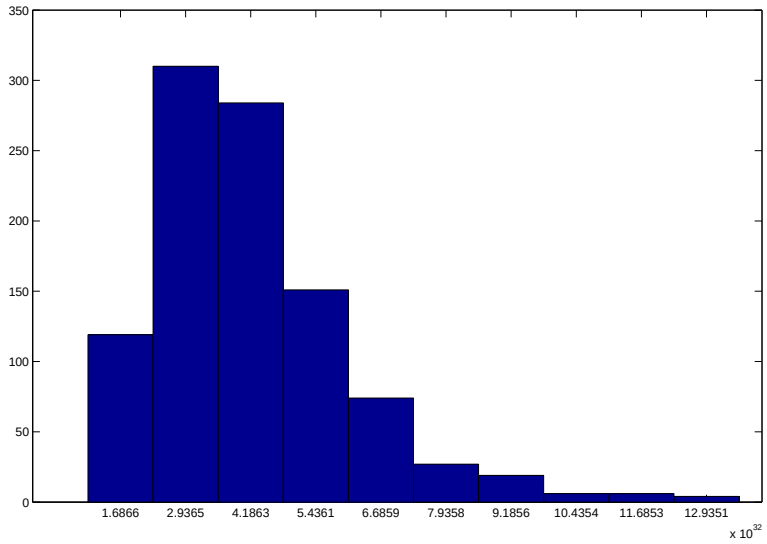

Figure: Histogram of 1,000 Importance Weights for **Model 3**.

Model 4

A graph with $|V| = 1,000$ and $|E| = 249,870$. The graph was generated in the following way. We defined the number of vertexes to be 1,000 and each edge (v_i, v_j) was generated from $Ber(p)$ while each p is generated from truncated Normal distribution with $\mu = 0.5$ and $\sigma = 0.3$ The results are summarized bellow.

- *cachet* was timed out after 2 days and was unable to deliver a solution. The lower bound of $9.601E + 10$ was supplied.
- *SampleSearch* failed to initialize, probably, because the problem is too big.
- The SIS Algorithm ($N = 100$) delivered 2.773×10^{11} in 1,718 seconds. The RE is 1.579×10^{-2}

The following figure provides a typical Histogram of the Importance Weights obtained in a single run of SIS Algorithm.

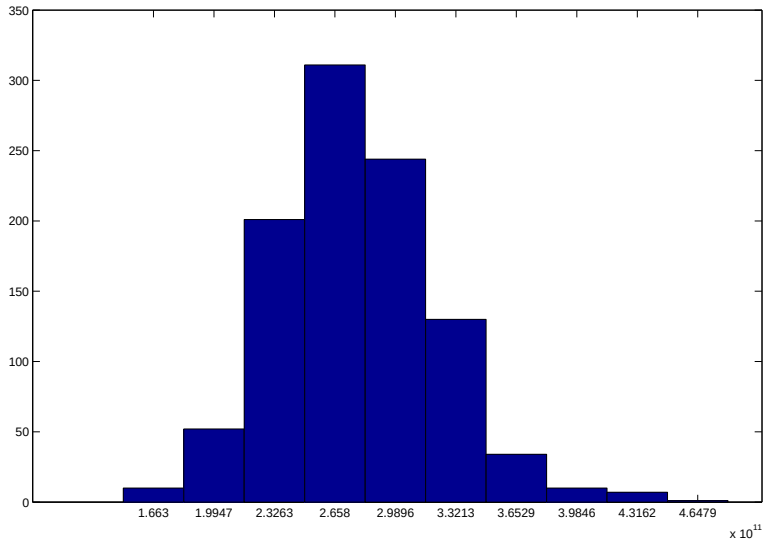

Figure: Histogram of 1,000 Importance Weights for **Model 4**.

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Thank You!