Resilience of Finite Networks Against Simple and Combined Attack on Their Nodes

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Abstract

We compare the reliability of two finite networks with the same vertex degree and the same number of nodes; a regular 16x16 grid and a Poisson network. Both networks are subject to random removal of their nodes, and the network failure is defined as the reduction of the maximal component beyond some critical level α. The main tool for comparing the network resilience are the marginal cumulative D-spectra (signatures) of the net- works. It was demonstrated that the regular grid for small α is less reliable than the Poisson network. We study also the situation when multiple hits of the same node are allowed. We demonstrate that finite networks behave similar to infinite random network with regard to the fraction of nodes to be removed to create "similar" giant components containing the same fraction of network nodes. Finally, we consider a combined attack on net- work nodes by two-type of "shells" where the node fails only if it is hit by "shells" of both types. For this case, we derive a formula for determin- ing the minimal number of "shells" which destroy the network with given probability.

Key words:finite network, attack on network node, cumulative D-spectra (signature); combined attack on nodes.

1.Introduction and Preliminaries. D-spectrum

There are many works that study interaction between networks [1,3,4,5,10,11]. The following features are typical for probabilistic models of the network interaction: a) networks are assumed to be very large, formally infinite; b) network N_1 affects another network N_2 by creation of a random connections between their nodes in such a way that a node $a \in N_1$ **hits** a randomly chosen node $b \in N_2$. There are models of interaction in which the choice of node $b \in N_2$ is not random and "hubs" in N_2 have higher probability of being hit than "regular" nodes. Typically, in all these models, the particular structure of network N_2 is not specified, except for the node degree distribution.

In this paper we deal only with finite networks having well-defined structure. In other words, network N subject to an attack is given as a pair $N = (V, E)$, where V is the vertex or node set, $|V| = n$, and E is the edge or link set, $|E| = m$. We assume that if a node is attacked, then all links adjacent to it are erased, while the node remains untouched.

As a measure of network state in the process of node failures we consider the size of its largest component, i.e. the largest set of connected nodes in the network. It is more convenient to characterize the size of the largest component by the fraction α of nodes n in the network that belong to this component. For example, we will consider a 16x16 grid with $n = 256$. If its largest component has 200 nodes, we will say that $\alpha = 200/256 = 0.78$ We will also define several states of network degradation measured by these fractions $\alpha_9 > \alpha_8 > \ldots > \alpha_1$. We say that the network is in perfect state (State10) if its largest component has size $L \geq \alpha_9$, Generally, we say that the network is in state s if its largest component has size L, $\alpha_s > L \geq \alpha_{s-1}$, $s = 9,8,\ldots 2$. Finally, network state is 1 if $\alpha_1 > L$.

Our main goal is to analyze network probabilistic behavior when network nodes are subject to random node failures. The main tool for this analysis will be so called D-spectra technique.

Denote by $e_1, e_2, \ldots, e_n$ network components subject to failure (the nodes), and let π be a random permutation of components numbers,

$$\pi = (e_{i_1}, e_{i_2}, \ldots, e_{i_n}).$$

Suppose that all these components are up and we move along the permutation, from left to right, and turn each component from *up* to *down*. Suppose that network state is controlled after each step. Typically, we will observe exactly 9 occasions when network state has changed: first - from the perfect State10 to state $s = 9$, from $s = 9$ to $s = 8$, and so on, until the transition from $s = 2$ to $s = 1$.

Definition 1. (*The anchors*)

The ordinal number in the permutation π of the component whose turning down causes network state to change from $10 - k$ to $10 - (k + 1)$, $k = 0,1,\ldots,8$ is called the $(k + 1)$-st *anchor* and is denoted by $r_{s+1}(\pi)$. Each permutation has, therefore, k anchors. So, the first anchor signifies the transition $10 \Rightarrow 9$, the second - $9 \Rightarrow 8$,...,the ninth - $2 \Rightarrow 1$

Definition 2. *(Multidimensional D-spectrum)*

Assume that all $n!$ permutations are equally probable. The k-dimensional discrete density

$$f(\delta_1, \delta_2, \ldots, \delta_9) = P(r_i(\pi) = \delta_i, i = 1,2,\ldots,9) =$$

$$\frac{number\ \ of\ \ permutations\ \ with\ \ \ r_i(\pi)=\delta_i, i=1,2,\ldots,9}{n!} \tag{1}$$

for $1 \leq \delta_1 < \delta_2 < \cdots \leq \delta_9 \leq n$ is called network *multidimensional D-spectrum*.#

A few comments. Letter "D" for the spectrum signifies the process of destruction since we turned from *up* to *down* network components moving along the permutation. In literature, the multidimensional D-spectrum is termed also as a multidimensional signature, see [6,8].

Obviously,

$$\sum_{1 \leq \delta_1 < \delta_2 < \cdots < \delta_9 \leq n} f(\delta_1, \delta_2, \ldots, \delta_9) = 1.$$

It is important to stress that the D-spectrum is a *combinatorial parameter* of the network that depends only on network structure and its states definition. It does not depend on probabilistic characterization of the real random mechanism governing network component failures.

Our main interest will be in the probabilistic description of each particular anchor. Formally speaking, our main tool will be the distributions of the positions of each of the 9 anchors.

Definition 3. *The j-th marginal D-spectrum*

The distribution

$$f^{(j)} = (f_1^{(j)}, f_2^{(j)}, \ldots, f_n^{(j)})$$

of the position of the j-th anchor is called the j-th *marginal D-spectrum*.#

Here $f_i^{(j)} = P(\ the\ \ j - th\ \ anchor\ \ position\ \ is\ \ \ i)$.

Obviously,

$$f_i^{(j)} = P(r_j(\pi) = i) =$$

$$\sum_{1 \leq \delta_1 < \delta_2 < \ldots \delta_j = i < \ldots < \delta_9 \leq n} f(\delta_1, \delta_2, \ldots, \delta_j = i, \ldots, \alpha_9) \tag{2}$$

In what follows, it is more convenient is to operate with a so-called cumulative (marginal) D-spectra.

Definition 4. *The j-th cumulative D-spectrum*

The cumulative distribution function (cdf) $F^{(j)}(x)$ of the position of the j-th anchor in random permutation π is called the j-th *cumulative D-spectrum*:

$$F^{(j)}(x) = \sum_{i=1}^{x} f_i^{(j)}, x = 1,2,\dots,n. \# \quad (3)$$

Let us clarify the probabilistic meaning of $F^{(j)}(x)$. Divide all network states into two sets U and D. Let all states $J \geq (10 - j)$ belong to U, and all the remaining states - to D, $j = 0,1,2,\dots,9$. Denote by $Y_{(j)}$ the random number of components needed to turn down in the course of the destruction process to cause the transition from U to D. Note that $f_i^{(j)} = P(Y_{(j)} = i)$. Then

$$F^{(j)}(x) = P(Y_{(j)} \leq x).$$

In words: $F^{(j)}(x)$ is the cdf of the number of components to be destroyed to cause the transition from U to D

Remark 1

Suppose that the network has only two states: the UP state, if its largest component has size $L \geq \alpha = 0.7$, and the complementary state D. There will be only one anchor designating the position of the component whose destruction leads to the transition $U \Rightarrow D$. The corresponding D-spectrum is nothing but so-called *signature* introduced by Samaniego [15] and the cumulative D-spectrum is the so-called *cumulative signature*, see [15,16].#

Remark 2

Consider a star network with central node a and three peripheral nodes b, c, d that are connected to a by links $(a, b), (a, c), (a, d)$. If node a fails, the network disintegrates into four isolated components. If network state is defined according to the size of its largest component, we observe a jump from state 4 to state 1.

Formally speaking, it may happen that in the process of component destruction we may observe a transition from state $J - A$ to state $J - A - B$, $B > 1$. Suppose it happens after destructing component standing on the i-th position. Then we put $r_{A+1}(\pi) = \ldots = r_{A+B}(\pi) = i$ and therefore formally provide that all permutations have the same number of anchors.#

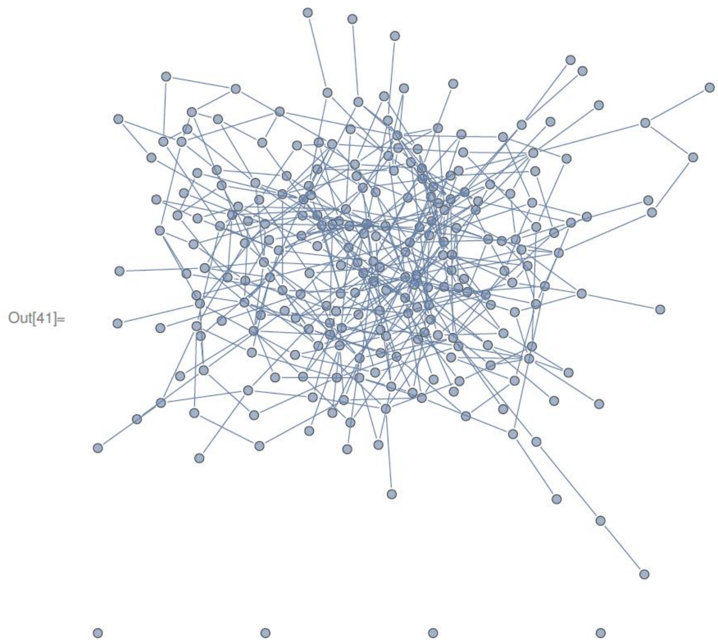

Figure 1: Random graph with $n = 252$ nodes and average node degree $d = 3.75$

2. **Resilience of a Regular Grid vs Random Graph**

In this section we compare the resilience of two finite networks having approximately the same number of nodes and the same average node degree. The first network (call it 'Grid') is a 16x16 regular grid with $n = 256$ nodes and 480 edges, see Figure 3. The average node degree is 960/256=3.75.

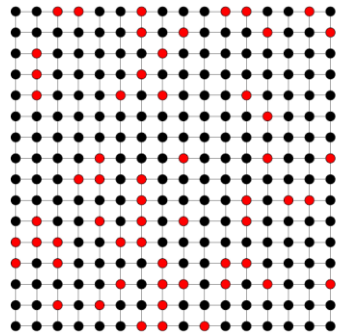

Figure 2: 16x16 Grid. Failed nodes are shown by red

The second network is a random Poisson graph (call it "Map") with 252 nodes and 473 edges thus having the same average node degree 3.75. Each of these networks was subject to random node removal. For Grid and Map we introduced several states according to the *fraction L* of all nodes in the maximal (connected) component. This component is an analogue of the giant component in an infinite network:

$$State10: L \geq 0.9; State9: 0.8 \leq L < 0.9; \ldots; State2: 0.1 \leq L < 0.2; State1: L < 0.1.$$

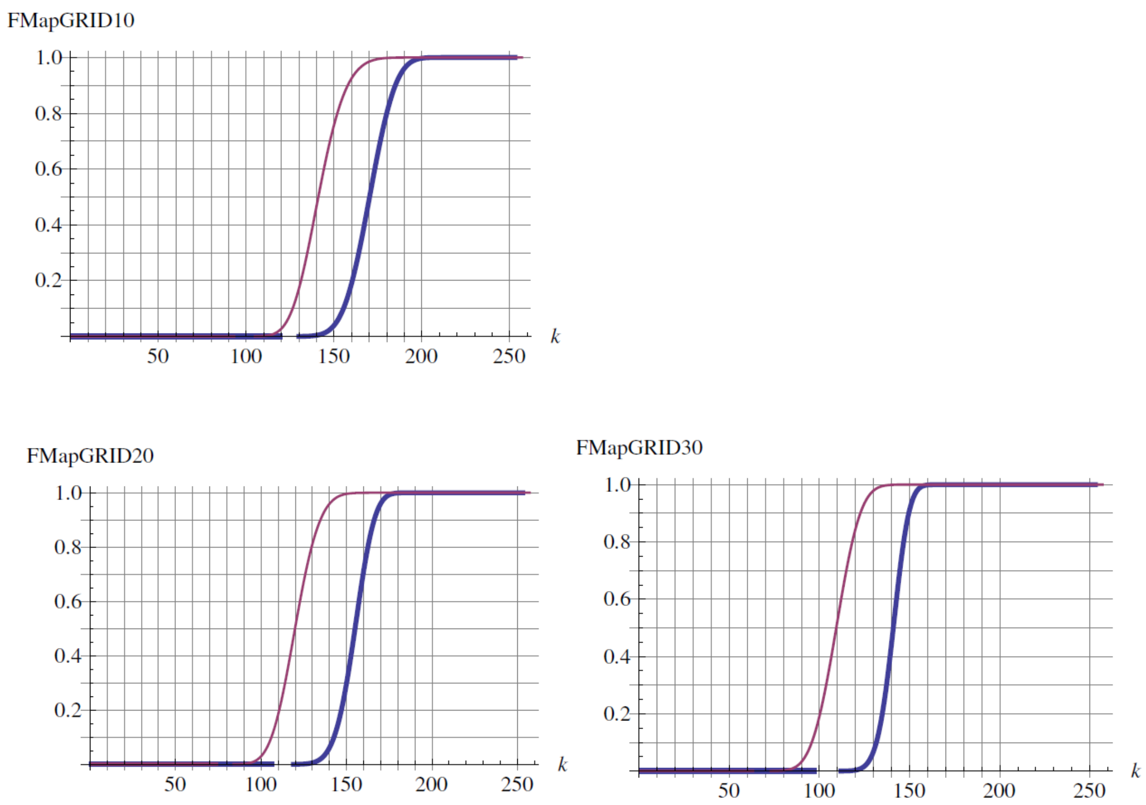

Figure 3: Cumulative D-spectra for Grid vs Map networks. Upper pair is for transition 2 ⇒ 1; left lower - for transition 3 ⇒ 2, right - for transition 4 ⇒ 3. In each pair, the right curve (blue) is for Map, the left - for Grid

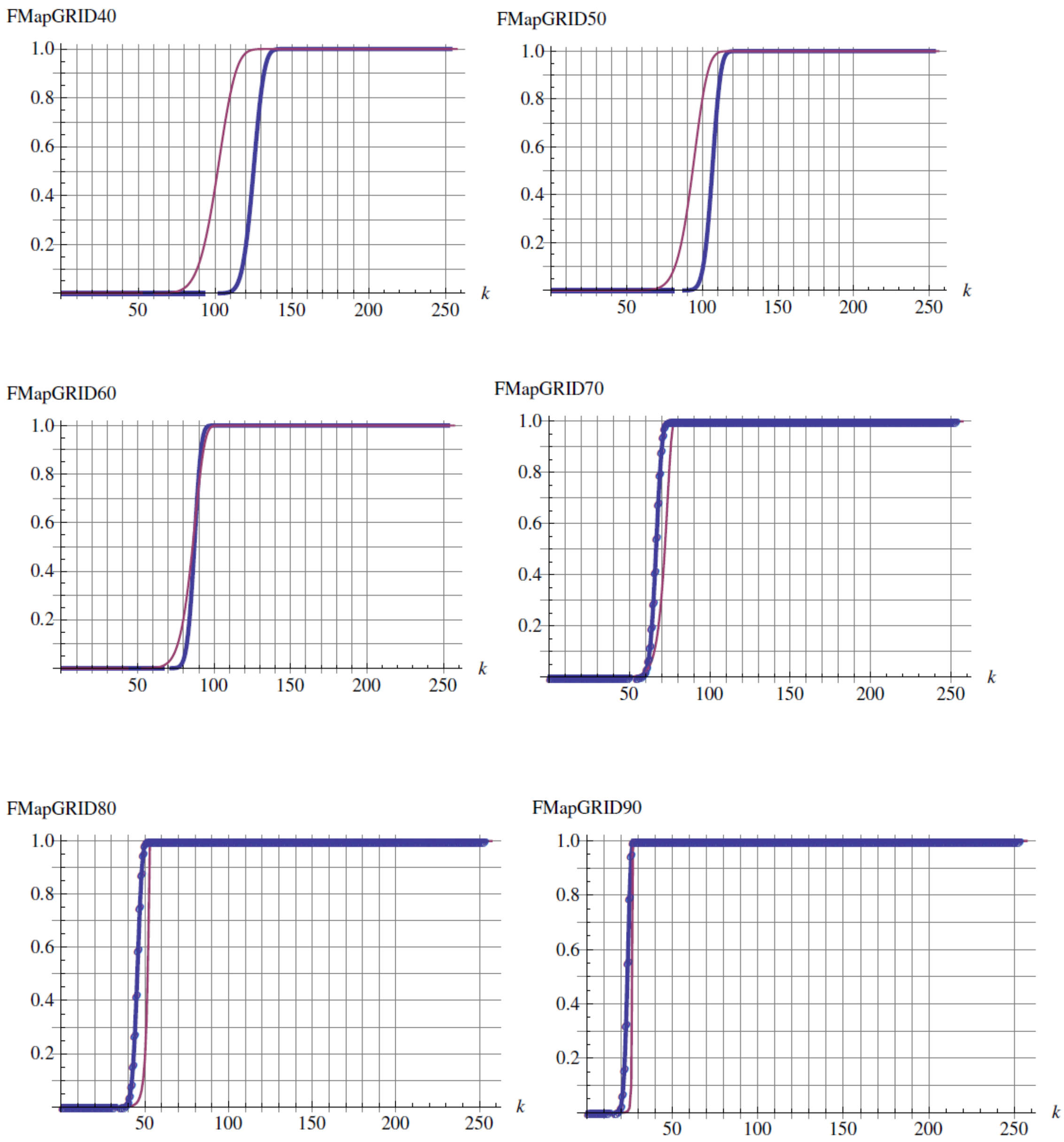

Figure 4: Cumulative D-spectra for Grid vs Map networks. Upper pair is for transition 5 ⇒ 4 (left) and 6 ⇒ 5 (right). The red curve is for Grid, blue - for Map. Here again Map is more resilient. Situation changes for the transitions 7 ⇒ 6,8 ⇒ 7,9 ⇒ 8 and 10 ⇒ 9, see the graphs in the middle and the bottom

.

Random permutation π has therefore nine anchors $r_1(\pi), r_2(\pi),\ldots, r_9(\pi)$ signifying the transition from *State10* to State9, from *State9* to State8, etc. The cumulative marginal D-spectra are shown on Figures 3 and 4. Figure 3 shows Grid vs Map marginal spectra for the transitions 2 ⇒ 1,3 ⇒ 2,4 ⇒ 3.

The spectrum for Grid is in red, for Map -in blue. The most surprising and not expected phenomenon is that the Grid marginal spectra are shifted to the left from the Map spectra. It means that Map is more resilient than the Grid! Let us examine the graph *FMapGrid30*. If about 110 nodes are destroyed, the largest component of Grid with probability about 0.5 has $0.3 \cdot 256$ nodes while Map has not suffered at all. To cause the transition 3 ⇒ 2 with probability 0.5 for Map, one has to destroy about 140 nodes !

Two upper graphs on Figure 3 present the Map vs Grid spectra for the transitions $5 \Rightarrow 4$ and $6 \Rightarrow 5$. Here again we see that blue curves (Map spectra) are on the right of the Grid spectra.The advantage in resilience of Map vs Grid vanishes when the fraction of nodes in the maximal component becomes ≥ 0.6, see the graphs in the middle row and in the bottom of Figure 4. When the maximal component has 0.9-0.7 faction of all nodes, Grid is slightly more resilient. For $\alpha = 0.6$ both spectra practically coincide, see middle graphs, on the left.

In [7], comparison was made of the resilience for small random networks ($n \leq 40$) vs regular networks, for node degrees $d = 3{,}4$ and 5. Network failure was defined as the decrease of component becomes below $0.3n$. It was observed that for $d = 5$, the regular network is more resilient, but its advantage over random network became very small when d was 4 or 3.

3. Multiple Hits

When an external source produces a hit on a randomly chosen node of a network that has N nodes, $N \to \infty$, the probability of multiple hits of the same node can be neglected. The situation changes drastically when the network subject to an external attack has a finite number of nodes n. Formally, we are in a situation well-studied in classical probability theory. Suppose that b balls are randomly placed into n boxes. We need to find the probability $p(k|b)$ that there will be exactly k boxes that will contain at least one ball. This problem is known is combinatorics as *occupancy* problem and its solution is given by the famous DeMoivre's formula [2], p 242:

$$p(k|b) = \frac{n!}{k!(n-k)!} \sum_{t=0}^{k} (-1)^t \frac{k!}{t!(k-t)!} \frac{(k-t)^b}{n^b}, k = 1, \ldots, min(n, b). \tag{4}$$

We are interested now in finding network $DOWN$ probability $P(DOWN; b)$ when it is hit by b "balls". (A node that receives more than one hit remains *down*). Suppose that the network entrance into the *DOWN* state is described by j-th marginal cumulative D-spectrum $F^{(j)}(x)$. Using the Total Probability formula, we obtain that

$$P(DOWN; b) = \sum_{k=0}^{\min(n,b)} p(k|b) \cdot F^{(j)}(k), \tag{5}$$

where $p(k|b)$ is given by (4).

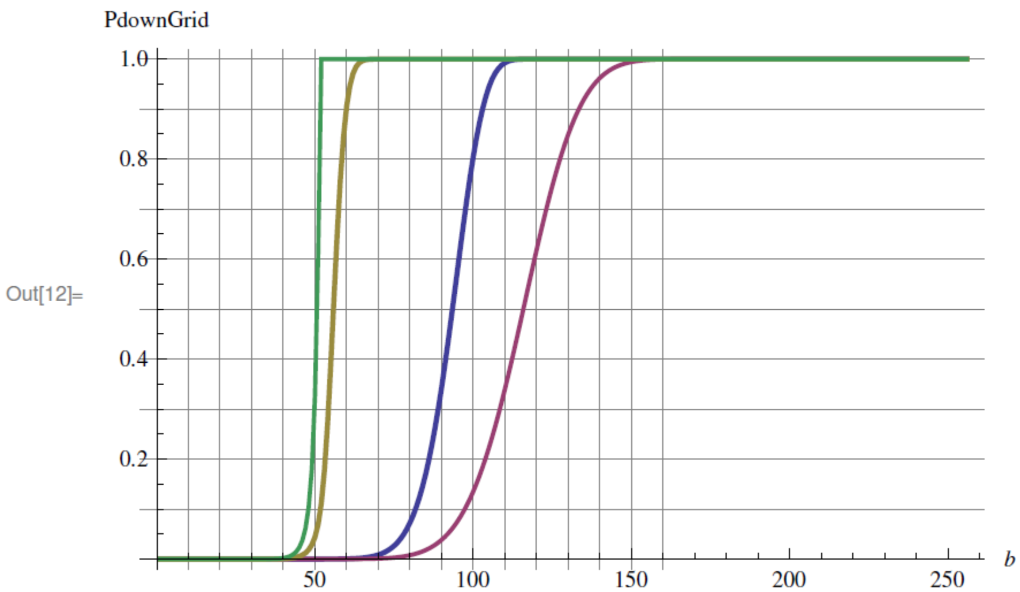

Figure 5: Comparison of $P(DOWN; b)$ (right curve in each pair) with $F^{(j)}(b)$ (left curve in each pair), for GRID. Left pair is for $j = 2$; right pair - for $j = 5$

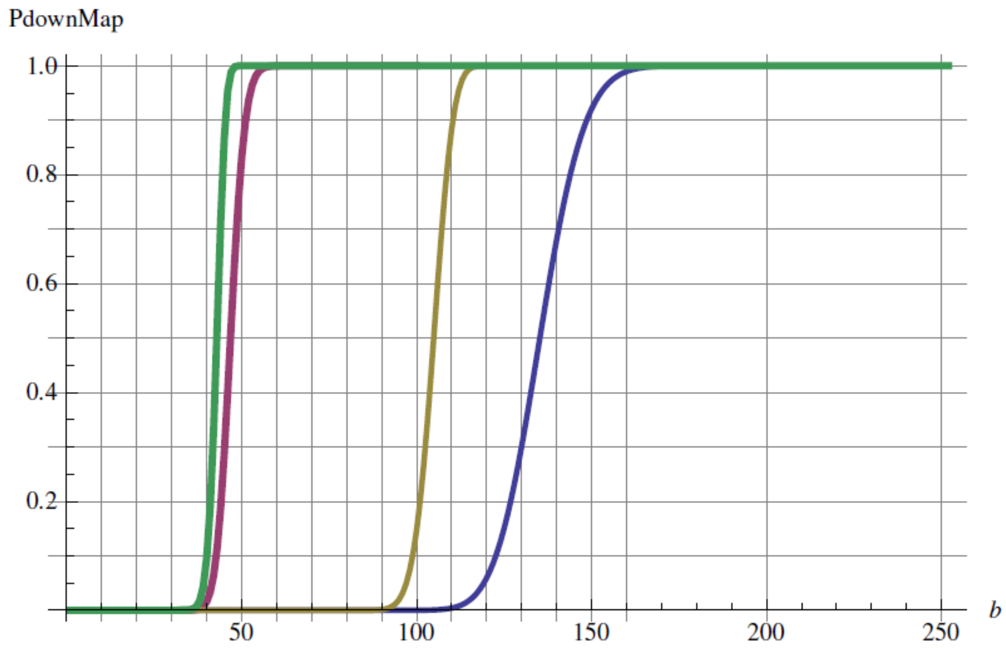

Figure 6: Comparison of $P(DOWN;b)$ (right curve in each pair) with $F^{(j)}(b)$ (left curve in each pair) for Map. Left pair is for $j=2$; right pair - for $j=5$

Examine, for example, the right pair of curves on Figure 5. The left one (in blue) is the cumulative marginal spectrum of GRID, for which the *DOWN* state is defined as the drop of largest component below $0.5n$. The red curve is $P(DOWN;b)$ as a function of the number of "balls" thrown on the nodes of GRID. We see that, e.g., for probability 0.8, the horizontal distance between the curves is about 25, which means that about 25 nodes (out of approximately 125) are hit more than once. The comparison of the green and yellow curve on this figure shows that here the number of nodes with multiple hits is much smaller because the transition $9 \Rightarrow 8$ takes place after considerably smaller number of damaged nodes (50-60). Figure 6 presents a similar picture for Map network.

4. Comparing Giant Component in Infinite Poisson Network With Maximal Component in a Finite Network

Let us consider first a Poisson random infinite network in which a fraction β of its nodes is randomly chosen and removed. Then the size of the giant component G can be found from the following equation, see [12] page 597:

$$G = (1-\beta)(1-e^{-d\cdot G}), \tag{6}$$

where d is the average node degree. Take $d = 3.75$. Let us take $G = 0.9(0.1)0.1$ fraction of all nodes and find out from (6) the corresponding values of β. These values are presented in the second row of Table 1.

Table 1: Giant component vs maximal component in the Map

G	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
β	0.068	0.203	0.245	0.329	0.409	0.485	0.556	0.621	.680
$q_{0.5}$	0.079	0.0171	0.258	0.341	0.420	0.496	0.559	0.614	0.680

Now let us compare the β-values with the number of nodes hit in the Map in order to provide with probability 0.5 that the maximal component is reduced to 0.9(0.1)0.1 fraction of 252 nodes. For this purpose we have to look at the values of $q_{0.5}$, the 0.5-quantiles, of the marginal cumulative D-spectra,

$$F^{(j)}(q_{0.5}) \approx 0.5.$$

The values of $q_{0.5}$ are given in the third row of Table 1. We can say that the average fraction of nodes to be removed to create a giant component in an infinite Poisson network is very close to the median fraction of nodes to be removed in finite Map network.

5. Network Attacked by Interacting "Shells"

This title means that the network nodes are hit ("bombarded" or "contaminated") by two different kind of substances ("shells"). The "substances" (it might be infection, computer viruses, explosives, etc.) have the property that they interact between themselves and make damage to a node only if this node is hit *by both substances* .

Using the neutral combinatorial language, M white balls and N black balls are randomly located in n boxes (representing the nodes) The node is hit (fails) if in the corresponding box are balls of different colors. For example, there are 5 boxes numbered 1, 2, 3, 4, 5, boxes 1 and 2 contain one white ball, box 5 has 2 white balls. Also, each box contains one black ball. So, boxes 1,2 and 5 contain balls of different colors and represent the nodes that are hit. Our task is to find out the number B of boxes containing balls of both colors.

Lemma 1.
The mean value of B equals

$$E(B) = n^2(1 - (1 - \tfrac{1}{n})^M)(1 - (1 - \tfrac{1}{n})^N). \# \quad (7)$$

Lemma 2.

$$E(B^2) = n(1 - (1 - 1/n)^M(1 - (1 - 1/n)^N) + n(n - 1)(1 - 2(1 - 1/n)^M + (1 - 2/n)^M) +$$

$$n(n - 1)(1 - 2(1 - 1/n)^N + (1 - 2/n)^N). \# \quad (8)$$

The proof of Lemma 1 and 2 are given in the Appendix [18].

The following theorem follows from Lemmas 1 and 2:
Theorem 1
If $M = \alpha n$, $N = \beta n$, $n \to \infty$ then

$$E(B) = n(1 - e^{-\alpha})(1 - e^{-\beta}), \quad (9)$$

and

$$Var(B) = n(1 - e^{-\alpha})(1 - e^{-\beta})(1 - (1 - e^{-\alpha})(1 - e^{-\beta})). \# \quad (10)$$

The following theorem was established by [14], see also [13], Section 3:
Theorem 2
If $n \to \infty$, and $0 < c_1 < \beta < c_2 < \infty$, and $0 < c_1 < \alpha < c_2 < \infty$, the random variable $Y = \frac{B-E(B)}{(Var(B))^{0.5}}$ is asymptotically normal $N(0,1)$.#

Denote by q_ε the ε-quantile of $N(0,1)$. Then we arrive at the following
Corollary
Suppose $M = N$.To guarantee that with probability $1 - \varepsilon$ that the number of nodes hit by balls of both colors is at least B_{min}, we have to take $N = \gamma \cdot n_0$ where n_0 is the number of nodes in the network and γ is the root of the following equation:

$$q_\varepsilon n_0^{0.5}((1 - e^{-\gamma})(1 - (1 - e^{-\gamma})^2)^{0.5} + n_0(1 - e^{-\gamma})^2 = B_{min}. \# \quad (11)$$

Example

Suppose that we want to guarantee with probability 0.9987 that the maximal component of Grid network will be less or equal 128 nodes, i.e. $\alpha = 0.5$. We see from Figure 4 that this will be provided if the number of failed nodes is at least $120 = B_{min}$.We find that $q_{0.0013} = -3$ and solve the equation (11):

$$-3(256^{0.5})(1-e^{-\gamma})(1-(1-e^{-\gamma})^2)^{0.5} + n(1-e^{-\gamma})^2 = 120,$$

Using *Mathematica* "FindRoot" operator [17], we find that the root equals $\gamma = 1.384$, which means that $M = N = 1.385 \cdot 256 = 354$.

6. Concluding remarks

We demonstrated that the resilience and survivability of finite networks under random attack on their nodes can be efficiently studied using marginal D-spectra techniques. Let us note without going into technical detail that the spectra can be efficiently estimated by well-developed Monte Carlo algorithms, with sufficient accuracy and in short CPU times, see [9].

Comparison between a regular grid and random graph having the same number of nodes and the same node degree reveals that the regular graphs are considerably less resilient for $\alpha \le 0.5$ and that their inferiority in reliability vanishes when the networks's largest components contain large fraction of the nodes ($\alpha \ge 0.6$)

We demonstrated how to compute network reliability by taking into consideration multiple hits of their nodes.

Our simulation revealed that there are certain similarities between creation of a giant component in infinite random network and the largest component in a finite random network.

Finally, we investigated the case of combined attack on a network nodes with two interacting "substances". In this attack, a node fails only if it is hit by two types of "shells". We showed how one can obtain an estimate of the number of "shells" of both types that guarantee network destruction with given probability.

Appendix [18]

1. Let n be the number of boxes (bins) and M be the number of red balls. Each ball is randomly allocated to one of the boxes. Let

$$R_i = \{\, box \;\; i \;\; contains \;\; at \;\; least \;\; one \;\; red \;\; ball \,\}$$

Denote by 1_{R_i} the 1/0 indicator variable of the event R_i. Obviously,

$$P(R_i) = E[1_{R_i}] = 1 - (((n-1)/n))^M. \qquad (12)$$

Let $X = \sum_1^n 1_{R_i}$. Obviously,

$$E[X] = \sum_1^n E[1_{R_i}] = nE[1_{R_i}] = n(((n-1)/n))^M. \qquad (13)$$

If $n, M \to \infty$ and $M = \gamma n$, then

$$E[X] = n(1-e^{-\gamma})$$

2. Suppose we have N white balls which, independently of the red balls,are located randomly into the same n boxes (bins). Denote by B the random number of boxes containing balls of both colors. Obviously,

$$B = \sum_{i=1}^n 1_{R_i} 1_{W_i}. \qquad (14)$$

From linearity of expectation and independence of events R_i and W_i,

$$E[B] = \sum_{i-1}^n E[1_{R_i}]E[1_{W_i}] = n(((n-1)/n))^M \cdot (((n-1)/n))^N. \qquad (15)$$

3. For deriving the expression for $Var[B]$ we need the following formula:

$$E[1_{R_1} \cdot 1_{R_2}] = 1 - 2(\tfrac{n-1}{n})^M + (\tfrac{n-2}{n})^M. \qquad (16)$$

Note that

$$E[1_{R_1} \cdot 1_{R_2}] = P(R_1 \wedge R_2) = 1 - P(\overline{R_1} \vee \overline{R_2}) =$$

$$P(\ box\ 1\ or\ box\ 2\ is\ empty\ ) = P(\overline{R_1}) + P(\overline{R_2}) - P(\overline{R_1} \wedge \overline{R_2}) =$$

$$2(\tfrac{n-1}{n})^M + (\tfrac{n-2}{n})^M,$$

and (16) follows.

4.Now we are ready to obtain the expression for $E[B^2]$.

$$E[B^2] = E(\textstyle\sum_1^n 1_{R_i} \cdot 1_{W_i})^2 = E(\sum_1^n 1_{R_i}^2 \cdot 1_{W_i}^2 + \sum_{i \neq j} 1_{R_i} 1_{R_j} 1_{W_i} 1_{W_j}) =$$

$$nE[1_{R_1} 1_{W_1}] + n(n-1)E[1_{R_1} 1_{W_1} 1_{R_2} 1_{W_2}] = n(\tfrac{n-1}{n})^M(\tfrac{n-2}{n})^N +,$$

$$n(n-1)(1 - 2(\tfrac{n-1}{n})^M + 2(\tfrac{n-2}{n}))^M \cdot n(n-1)(1 - 2(\tfrac{n-1}{n})^N + 2(\tfrac{n-2}{n}))^N.$$

Now $Var[B] = E[B^2] - (E[B])^2$. Substituting the expressions for $E[B^2]$ and $E[B]$ we obtain that

$$Var[B] = n(1 - x^M)(1 - x^N) + n(n-1)(1 - 2x^M + y^M)(1 - 2x^N + y^N) - (n(1 - x^M)(1 - x^N)))^2, \quad (17)$$

where $x = (n-1)/n$ and $y = (n-2)/n$.

Now assume that $M = N \to \infty$. Then $x \to e^{-\gamma}$ and $y \to e^{-2\gamma}$. After simple algebra we obtain that

$$E[B] = n(1 - e^{-\gamma})^2. \quad (18)$$

and

$$Var[B] = n(1 - e^{-\gamma})^2 \cdot (1 - (1 - e^{-\gamma})^2). \quad (19)$$

It is remarkable that the variance of B is asymptotically of order n, i.e. the st.deviation of B is of order $\sqrt{n}$.#

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