On a Single Discrete Scale for Preventive Maintenance with Two Shock Processes Affecting a Complex System

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Abstract. A new approach to optimal maintenance of systems (networks) is suggested. It is applied to systems subject to two external independent shock processes. A system 'consists' of two parts and each shock process affects only its own part. A new notion of bivariate signature is suggested and used for obtaining survival characteristics of a system and further optimization of the preventive maintenance actions. The PM optimization is considered in the univariate discrete scale that counts the overall numbers of shocks of both types. An example of a transportation network is considered.

Keywords: Shock process; network; signature; preventive maintenance

1. Introduction

The goal of this paper is to introduce a new approach to optimal maintenance of systems (networks) subject to two deterioration processes via constructing a single (univariate) scale. We consider the case when this deterioration is induced by two external shock processes and, therefore, is discrete in nature. The univariate scale 'is counting' the overall number of shocks experienced by a system and thus can be considered as 'time free', similar to our previous work on univariate deterioration (see later). We see a remarkable potential in the suggested approach that can be applied in the future to other important in reliability practice settings.

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The literature on optimal preventive maintenance of technical systems consists of thousands of entries starting with the ground-breaking paper by Barlow and Hunter (1960) and followed by numerous papers and a number of monographs entirely devoted to this problem (e.g., Gertsbakh, 2002; Nakagawa, 2008; Wang and Pham, 2006). Most of the developed models deal with either periodic strategy, where preventive maintenance (PM) is conducted at deterministic or random instants of time or the age-based strategies, when the next PM is performed at the time defined by the last failure (repair) or by the state (condition) of an item. The latter is usually called the "condition-based maintenance" (see, e.g., Castro et al., 2013; Nicolai, 2008; Pham, 2012 and references therein). It is important to note, that to the large extent, the PM literature is devoted either to single unit systems or to simple multi-unit systems such as series systems. More complex structures such as networks usually do not allow for tractable PM solutions.

In Finkelstein and Gertsbakh (2015, 2016)) we were the first to use signatures of coherent systems (networks) for obtaining the corresponding optimal preventive maintenance strategy. We have considered the n-component systems subject to a process of shocks. It was assumed that each shock 'kills" at random one operating at this time component and that the shock process is the only source of system failures. Our main goal was to consider this problem in the time-free framework. It was shown that conventional analysis in the time scale requires the Poisson assumption on the process of shocks and even in this case derivations of the corresponding cost per unit of time function are rather cumbersome. Therefore, we have considered the problem of optimal PM in the time-free way employing the new alternative discrete scale. The main advantage of the developed approach is that the type of the process of shocks does not matter and that only the shocks counts are important. The optimal PM had to be performed after a shock with the obtained number, or on the failure of a system (repair), whichever comes first. We briefly review the details of these approach in the next section.

In the current paper we will consider the case when the system is subject to two independent shocks processes affecting its different parts. For instance, one process can affect only edges and the other only nodes of a network. The main question is: how to obtain the optimal strategy in the time-free approach for this 'bivariate' case? We will show that some practical and meaningful assumptions allow for reducing the bivariate problem to the univariate one.

A natural object for applying in the future the developed theory is, the so-called, multiplex network, (see for example Baxter et al. (2016)). These networks have several sets of edges and a single set of nodes. For convenience, the edges can be characterized by different colors. For example, in a multiplex network, the green edges can correspond to the water supply and red edges can correspond to the power supply. The edges can fail due to shocks from two different shock processes. The first shock process affects only the green edges, whereas the second one affects only the red edges. The network failure can be defined, for example, as an event when the number of nodes receiving water **and** power supply falls below some critical level. The detailed example of a simpler network is considered in Section 5. There are two types of nodes in this network and each shock process affects only its own part, whereas the edges are not affected by shocks.

The paper is organized as follows. In Section 2, we briefly review the univariate case (one shock process) and discuss an important for further presentation interpretation of the univariate setting. In the methodological Section 2, we consider the case of two 'parts' of our system in series with each shock process affecting only its own part. In Section 4, a more general case under additional assumptions is considered. An example is presented in Section 5. Finally, brief concluding remarks are given in Section 6.

2. **Preliminaries. One process of shocks**

Our previous work (see Finkelstein and Gertsbakh (2015, 2016)) was using signatures as a crucial tool of the developed approach to optimal PM, therefore, let us first briefly recall the basics of this approach. We have considered n-component systems subject to an orderly (without multiple occurrences) process of shocks (each shock 'kills" at random one operating at this time component and it is the only source of failures). In the next sections we will generalize the setting to the bivariate case when a system is subject to two point processes of shocks.

It is well known that the structure function of the coherent system with i.i.d. components can be effectively defined by the values of the discrete distribution $(f_1, f_2, ..., f_n)$ where f_i is the probability that the system failure takes place at the i-th consecutive failure of a system's component (Samaniego, 1985, 2007; Gertsbakh and Shpungin, 2011). The corresponding discrete distribution

$$F(x) = \sum_{i=1}^{x} f_i, \tag{1}$$

is sometimes also called the cumulative D-spectrum (Gertsbakh and Shpungin, 2009). It is clear that we can describe our setting by signatures as well. Thus, with the ith shock $1 \leq i \leq n$, the system fails with probability f_i.

When the system fails, all failed components are replaced and the system continues operation. For simplicity, all repair actions are assumed to be instantaneous. As with each survived shock, the number of failed components is increasing and it is maximal (n) at failure of the system, it can be cost-wise reasonable to perform the PM action replacing all failed components. As often in the PM problems we adopt the criterion of the minimal long-run expected costs per unit of 'time' for obtaining the optimal level of PM. The 'time' is now counted in the new discrete scale, which is the number of shocks experienced by a system. In accordance with the definition of the signature, the expected cost of a renewal of the failed system (to "as good as new state") is defined as

$$c_0(f_1 + 2f_2 + ... + nf_n) + c_{ER}, \tag{2}$$

where the first term is the expected cost of the failed components and c_{ER} is the additional cost for 'emergency repair'. As degradation in our model is monotone and is induced by shocks experienced by a system, we can consider the corresponding number of shocks as a new alternative scale (discrete). Therefore, we want to find an optimal k for the PM replacing all failed components that will minimize the corresponding long-run cost per unit of 'time'.

The mean length of the renewal cycle with the PM scheduled at (just after) the kth shock, $1 \leq k \leq n-1$,is (Finkelstein and Gertsbakh, 2015):

$$L(k) = \sum_{i=1}^{k} if_i + k(1 - F(k)) \tag{3}$$

and the average cost per cycle is

$$(1 - F(k))(kc_0 + c_{PM}) + \sum_{i=1}^{k}(c_0 i + c_{ER})f_i, \tag{4}$$

where the PM action is performed replacing k failed components with the additional PM cost c_{PM}. Then the average cost per unit of 'time' is $C(k) = D(k)/L(k)$ and the corresponding optimization problem is to find k^* that satisfies

$$C(k^*) = \min_k(C(k), k = 1, 2, ..., n).$$

In Finkelstein and Gertsbakh (2015) this optimization problem was analyzed and illustrated by the network with 34 links and 24 nodes and the optimal k^* was numerically obtained. In Finkelstein and Gertsbakh (2016) these results were extended to the case when each shock kills one of the components with probability p and it survived with probability $1 - p$.

We will now present another simple but meaningful for further discussion in this paper and for the future work interpretation of the above setting with one shock process. Consider an object (e.g., material, system) with a random strength described by the Cdf $F(x)$ with support in $[1, 2, ..., \infty]; F(0) = 0, F(\infty) = 1$. Denote the corresponding pmf by $(f_1, f_2, ...,), f_0 = 0$ and assume that each shock affecting our object decreases its strength by one unit.

Alternatively, we can describe the setting in terms of degradation, i.e., there is a random threshold for the accumulated degradation; each shock increases degradation on one unit and an object fails when the accumulated degradation reaches this threshold. Thus $F(x)$ in (1) gives now a probability that an object will be destroyed after x shocks. Note that $f_i \neq 0, i \geq 1$ in this setting. When the object fails, the object is perfectly repaired meaning that degradation is set to the initial level and it continues operation. For simplicity, all repair actions are assumed to be instantaneous. As previously, it can be cost-wise reasonable to perform the PM action that decreases degradation to initial level. Thus c_0 now defines the cost of decreasing degradation on one unit and c_{PM} and c_{ER} have the same meaning as previously. Obviously, the optimization problem is formulated in the same way and equations (2)-(4) hold with the slight difference that the support of our distribution is now $[0, \infty)$, but the finite upper bound can be also considered in this interpretation as well.

Another important distinction of this interpretation from the one considered in our previous work, is that now we can consider the imperfect PM, i.e., the degradation can be reduced not to the initial 'perfect' level but to the intermediate one. However, this topic needs further investigation.

3. **Two shock processes. Series system**

Assume first, that our system consists now of two independent parts in series and each shock process affects only its own part. For definiteness, assume that the survival of each part is described via the random strength concept as above. The example of a more general system (not a series one) will be considered in Section 4. The assumption of independence here is important, as it will allow us to formulate the optimization problem explicitly and to understand the meaning of the main role players for a general case, where the explicit in the defined sense formulation is not possible or too complex.

Assume that the shock processes are independent and they are the only causes of failures of a system. Denote by X and Y the random number of shocks till failure for the first and the second part, respectively. Let

$$F^1(x) = P[X \leq x] = \sum_0^x f_i^1, f_0^1 = 0, x \in [0, 1, 2..., \infty);$$

$$F^2(y) = P[Y \leq y] = \sum_0^y f_i^2, f_0^2 = 0, y \in [0, 1, 2, ..., \infty).$$

Then, obviously, due to independence, the corresponding bivariate Cdf and the survival functions are given as

$$F(x, y) = Pr[X \leq x, Y \leq y] = F^1(x)F^2(y) = \sum_{i=0}^{x} \sum_{j=0}^{y} f_i^1 f_j^2,$$

$$\overline{F}(x,y) = Pr[X > x, Y > y] = \overline{F}^1(x)\overline{F}^2(y) = (1 - \sum_0^x f_i^1)(1 - \sum_0^y f_y^2).$$

The survival function for our series system is the same as above, i.e., $\overline{F}_s(x,y) = \overline{F}(x,y)$, whereas the probability that a system will fail if it experiences not more than x and y shocks, respectively, is

$$F_s(x,y) = 1 - \overline{F}_s(x,y) = F^1(x) + F^2(y) - F^1(x)F^2(y). \tag{5}$$

Alternatively, we can also write the following representation,

$$F(x,y) = \sum_{i=0}^{x}\sum_{j=0}^{y} f_{ij}, f_{00} = 0, \tag{6}$$

where f_{ij} can be interpreted as the analogue of the univariate signature in the bivariate case: it is the probability that the system will fail on the $(i+j)$th shock, were i is the number of shocks of the first type and j is the number of shocks of the second type. As the shocks arrive consequently, obviously, in this case the system can fail only either on the ith shock of type 1 (after experiencing j shocks of type 2) or on the jth shock of type 2 (after experiencing i shocks of the first type). Note that summation starts from 0 as the failure can occur even if there are no shocks of one type. The corresponding matrix $||f_{ij}||_{i,j=0}^{\infty}$ can be considered as a definition for the *new notion* of the *bivariate signature* for the described specific case.

Due to the independence of the parts of our series system we can define f_{ij} in (6) explicitly via univariate signatures as

$$f_{ij} = f_i^1 + f_j^2 - f_i^1 f_j^2, \tag{7}$$

which is consistent with (5) and (6), as substituting (7) into (6) results in (5). It follows from our description, that distinct from the univariate case, f_{ij} is not the probability mass function for $F(x,y)$ as $Pr[X = i, Y = j] = f_i^1 f_j^2 \neq f_{ij}$.

After these preliminary considerations, we will discuss now how to reduce our setting to the one dimensional case in order to proceed with optimal PM of the system. Note that we observe two independent shock processes and the overall state of the system. Thus our information is: the numbers of shocks of each type experienced by the system and the event of a failure of a system if it occurs. Ideally, we would like to stop and perform the PM at some values of x or y or on failure of a system, whichever comes first (the bivariate setting). This can be hopefully done via the corresponding numerical bivariate optimization. However, it can be more practically efficient to reduce the problem to univariate optimization via *one alternative scale.* There can be various ways of choosing one equivalent scale. This topic was considered previously in the literature only for the continuous scales, e.g., time and mileage (see Kordonsky (1997), Finkelstein (2004,2008)), whereas we are dealing here with the discrete ones. For instance we can 'forget' about one shock process and count the shocks only from the other one and then solve the optimization problem for the univariate case described in the previous section. This, of course, can lead to substantial errors. On the other hand, we can count *all* shocks and define the overall shock number for the PM that minimizes expected costs per unit of 'time'. This approach seems reasonable but requires additional assumptions or knowledge of the relevant characteristics of the shock process. In what follows, we will deal with this reduction to a single univariate scale and leave the discussion of the formal bivariate (multivariate) settings for further studies.

Assume that we have additional information on the shock processes of the following nature. We assume or know that the resulting (univariate) shock process is the mixture of two independent shock processes: each event from this process is of type 1 with probability p and of type 2 with probability $1-p$. Thus, we will count the arriving shocks without distinguishing the types (sometimes in applications we even do not observe the type of a shock) and make our decision to perform the PM based on the number of observed shocks. This model takes into account different rates of the shock processes. Denote the rates of the overall processes by $r(t)$. Then, in accordance with our assumption it can be written as,

$$r(t) = r_1(t) + r_2(t) = pr(t) + (1-p)r(t),$$

where $r_1(t)$ and $r_2(t)$ are the rates of the shock processes affecting each part of the system, respectively. However, as we are not considering the problem in the time scale, we will need only the mixing probability p.

We must modify the procedure of the previous section for the described setting. We will need to obtain the equation for the univariate signatures for the system in this case via the univariate signature for the parts. We cannot use (7) for this as the mixing probability should be involved now. Therefore, (7) can be 'modified' to

$$f_{ij,p} = pf_i^1 + (1-p)f_j^2, i, j \geq 0, f_{00} = 0. \tag{8}$$

where the sub index "p" indicates that this variant of the signature already takes into account the specific shock process. The meaning of (8) is as follows. The system had experienced $i+j$ shocks. If the last shock was of the first type than the probability of failure is pf_i^1 and $(1-p)f_j^2$, otherwise. Let us call the corresponding matrix $||f_{ij,p}||_{i,j=0}^{\infty}$ the *bivariate p-signature* for our system. Obviously $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij,p} = 1$, which follows from (8) and the definitions of the univariate signatures.

In order to obtain the univariate signature, we will count now the overall number of shocks experienced by the system and will denote it also by i. Thus the probability that the system will fail on the i-th shock in the described mixed process, i.e., $f_{si,p}, i = 1, 2, ...,$, and the corresponding cumulative distribution can be obtained respectively as

$$f_{si,p} = \sum_{m=1}^{i} C_{m-1}^{i} p^{m-1}(1-p)^{i-m+1}(pf_m^1 + (1-p)f_{i-m}^2), F_{sp}(x) = \sum_{1}^{x} f_{si,p}. \quad (9)$$

where $(f_{s1,p}, f_{s2,p}, ..., f_{sn,p}, ...)$ can be interpreted as the *univariate* p-signature of our system and $C_k^n = n!/(k!(n-k)!)$ denotes the number of combinations of k-out-of n.

The structure of (9) is as follows: if, for instance, with probability p the last shock is of the first type, then the probability of the failure of the first part on this shock is f_m^1 (if it was the m-th shock of type 1) and we must multiply it by the corresponding binomial probability. The similar reasoning applies to the second term.

The PM decreases the accumulated degradation to a zero level for each part of the system. Then, similar to (3), the mean length of the renewal cycle with the PM on the k-th shock (or the renewal on the failure, whichever comes first) is

$$L_s(k) = \sum_{i=1}^{k} i f_{si,p} + k(1 - F_{sp}(k)). \quad (10)$$

Recall that each shock increases degradation on one unit, but generally these units are different for different parts of our system. If they are the same $c_{01} = c_{10} = c_0$, then the average cost on a cycle is similar to (4):

$$(1 - F_{sp}(k))(kc_0 + c_{PM}) + \sum_{i=1}^{k}(c_0 i + c_{ER}) f_{si,p}, \quad (11)$$

and we can proceed with optimization in the same way as in the previous section, just substituting f_i by $f_{si,p}$ obtained in (9). However, if they are different,

$$\begin{aligned} D_s(k) = (1 - F_{sp}(k)) \sum_{m=0}^{k} C_m^k (p^m (1-p)^{i-m} (c_{01}m + c_{02}(k-m) + c_{PM}) \\ + \sum_{i=1}^{k} \sum_{m=1}^{i} C_{m-1}^{i} (p^{m-1}(1-p)^{i-m+1}((pf_m^1(mc_{01} + c_{ER}) + \\ (1-p)f_{i-m}^2((i-m)c_{02} + c_{ER})). \quad (12) \end{aligned}$$

Indeed, the first term in the right hand side of (12) corresponds to the mean cost on the renewal cycle for the case when the system did not fail. When

$c_{01} = c_{02} = c_0$ it obviously reduces to the first term in the right hand side of (4). The structure of the second term is based on the univariate signature (9). Then the average cost per unit of 'time' is

$$C_s(k) = \frac{D_s(k)}{L_s(k)} \tag{13}$$

and the corresponding optimization problem is

$$C_s(k^*) = \min_k (C_s(k), k = 1, 2, ...).$$

In practice, $C(k)$ can be numerically obtained for each $k > 1$ and the minimal value chosen. It follows also from general considerations, that there should be an optimal solution for the sufficiently large $c_{ER} - c_{PM}$. The procedure and reasoning are the same for the finite number of shocks. For instance, this occurs when each part of a system contains the finite number of components and the shock of each kind kills with equal probabilities one operating component of the corresponding part (Finkelstein and Gertsbakh, 2015).

The reasoning of this section has an important methodological aspect. We define and discuss the notion of bivariate signatures for the simple series system. In the next section a more general case will be considered and later illustrated by a practical example.

4. Two shock processes. General system

Consider now a general, not necessarily series system with two parts or, better to say, with two types of components (edges and links, of a network, or weaker and stronger components, for example). The first shock process affects only the components of the first type, whereas the second shock process affects only the components of the second type. The shock processes are statistically independent. Can we generalize our approach with a single alternative scale of the described above type to this case? It turns out that the reasoning can be even simpler than in the previous section, however, we need stronger assumptions for implementing our approach.

Let, as previously, in the independent case, $||\widetilde{f}_{ij}||$ be the bivariate signature of our system: the probability of its failure either on the i-th shock of the first type or on the j-th shock of the second type (see the definition in the previous section). However, apart from the division into two parts, our system has a general, not specified structure. Therefore, we do not have explicit expressions similar to (7) and (8) and as a result, we cannot proceed in the same line similar to equations (9)-(12). Knowing the structure and the dependence structure between the parts, we probably can proceed for some special cases, but having in mind further PM optimization, the problem seems to be too complex.

Therefore, we will move in the following direction. Assume that for the specified mixed shock process (with its p and $1-p$) we can perform a simulation experiment for our system and obtain in this way the matrix $||\widetilde{f}_{ij,p}||$ directly (see the next section for the corresponding example). Alternatively, these values can be obtained from the 'historical data' if relevant. Thus the elements of this matrix, distinct from the previous section are already obtained for the given shock process, whereas in the previous section, (8), due to independence is defined via the corresponding univariate signatures.

It follows from our definition that:

$$\sum_{i=1,j=1}^{\infty} \widetilde{f}_{ij,p} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \widetilde{f}_{ij,p} = 1. \tag{14}$$

Similar to the previous section, we will perform now the reduction to the univariate scale and denote the probability that the system will fail on the i-th shock in this scale by $\widetilde{f}_{si,p}, i = 1, 2, ...$ Obviously, the discrete distribution $(\widetilde{f}_{s1,p}, \widetilde{f}_{s2,p}, ..., \widetilde{f}_{sn,p}, ...)$ can be interpreted as the corresponding univariate signature. Thus similar to (9), but for a more general case,

$$\widetilde{f}_{si,p} = \sum_{m=0}^{i} \widetilde{f}_{m,i-m,p}, \tag{15}$$

and

$$\widetilde{F}_{s,p}(x) = \sum_{i=1}^{x} \widetilde{f}_{si,p}. \tag{16}$$

It should be noted once again that in (9) we had an explicit expression for the univariate signature (series system) and in (15) (general system) the bivariate p-signature $||\widetilde{f}_{ij,p}||$ is obtained from the simulation experiment.

As previously, let the PM be performed just after the shock with the number k or after the failure, whichever comes first. The mean length of a cycle has the similar to (8) form:

$$\widetilde{L}_s(k) = \sum_{1}^{k} i\widetilde{f}_{si,p} + k(1 - \widetilde{F}_{sp}(k)), \tag{17}$$

where $\widetilde{f}_{sp}$ and $\widetilde{F}_{sp}(x)$ are defined now by (15) and (16), respectively. Due to methodological and practical reasons, let us make simplifications in the cost structure, as compared with the cost structure in the previous section. Let any PM (after a shock of an arbitrary number) has the cost $c_{PM} < c_{ER}$ which is similar to the classical PM problems for the continuous case (see, e.g., Barlow and Hunter (1960), Nakagawa (2008)). Note that c_{PM} and c_{ER} are now not the additional costs as in the previous sections but the overall costs of PM and emergency repair.

Then the expected cost on the cycle is (see, e.g., Barlow and Hunter (1960), Ross (1996), Wang (2002), Wang and Pham (2006), Nakagawa (2008) for the corresponding continuous case)

$$\widetilde{D_s}(k) = (1 - \widetilde{F}_{sp}(k))c_{PM} + c_{ER}\widetilde{F}_{sp}(k) \tag{18}$$

and the cost per 'unit of time' is defined also by

$$\widetilde{C}_s(k) = \frac{\widetilde{D_s}(k)}{\widetilde{L}_s(k)}.$$

Of course, when we suggest different costs of PM for both parts similar to what we have in the previous section, the situation is more complex and probabilities of the events of the shocks of the two kinds should be involved. In this case, the structure of the system should be employed.

The corresponding optimization problem for finding optimal k^* that minimizes this ratio is formulated as above:

$$\widetilde{C_s(k^*)} = \min_k(\widetilde{C}_s(k), k = 1, 2, ...).$$

Note, if our system can experience only finite number of shocks of both types (say n and l), then $f_{ij,p} = 0$ for $j \geq n$ and $j \geq l$, which means that if necessary the corresponding cells in the matrix should be populated by zeroes (see the example below).

5. **Example**

As a numerical example, we consider a transportation network with 34 nodes, and and 68 edges (Barabasi-Albert,1999). Five nodes of the network are 'declared' as terminals, whereas the remaining 29 nodes are subject to failures. The nodes are of two types: six "strong" nodes and 23 "weak" nodes. The network (nodes) is subject to a mixed shock process. Each arriving shock destroys one of the operable strong nodes with probability p=0.25 and with probability $1 - p = 0.75$ it destroys one of the operable weak nodes.

Network failure is defined as the loss of terminal connectivity, i.e., one or several terminals become isolated from other terminals. The described settings can be also interpreted in terms of the enemy attacks on some infrastructures.

For the sake of further notation in this example, let us re-denote f_{ij} as f_{xy} and omit the 'wave' over f_{xy}. We have simulated the shock process affecting our network and obtained the estimates of the f_{xy} values. We remind that f_{xy} is the probability that the network fails either on the xth failure of the weak node or on the yth failure of the strong node. Fig 1 presents the $||f_{xy}||$ matrix in the form of the map. The horizontal axes is for the weak nodes (x) and the vertical axes is for strong ones (y).

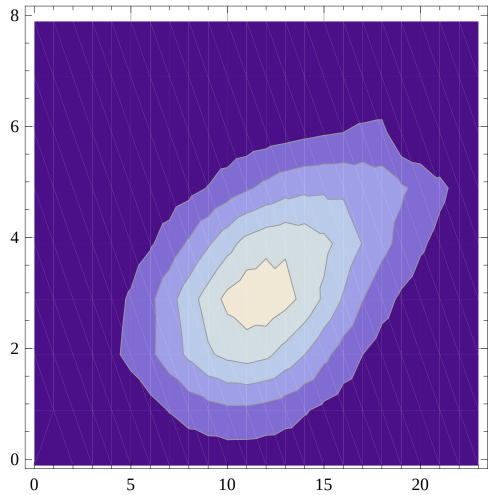

Figure 1: The map of $||f_{xy}||$. In the dark area, the cells f_{xy} are zeroes, and each change of the intensity of color marks the increase of f_{xy} by 0.05. The central area correspond to the values of f_{xy} above 0.03.

We have calculated $\widetilde{f}_{si,p}$ values following (15), and $\widetilde{F}_{sp}(i)$- following (16). The graph of $\widetilde{F}_{sp}(i)$ is presented in Fig 2, We see that the network failure can occur starting already with $i = 5$ shocks and it will be destroyed with probability 1 after 28 shocks.

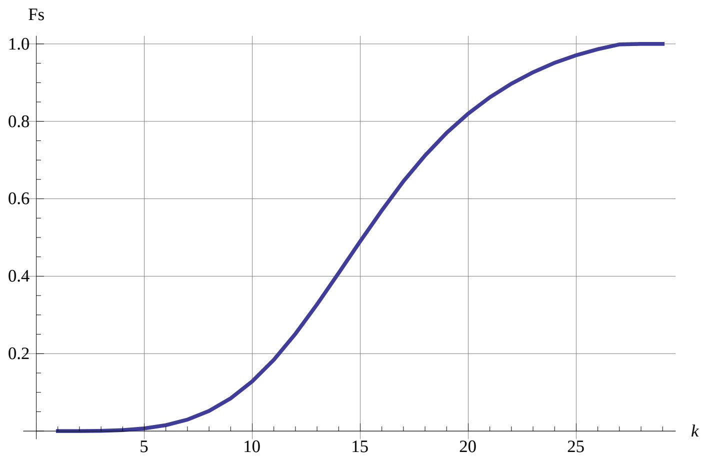

Figure 2: The graph of $\widetilde{F}_{sp}(k)$.

After obtaining the values of $\widetilde{f}_{si,p}$ and $\widetilde{F}_s(i)$ and using (17) and (18), we can obtain now the optimal value of k that defines the optimal maintenance strategy. Let $c_{ER} = 1$, and $c_{PM} = 0.1, 0.2, 0.3$. Fig 3 shows the corresponding curves for $\widetilde{C}_s(k)$ for these values of c_{PM}

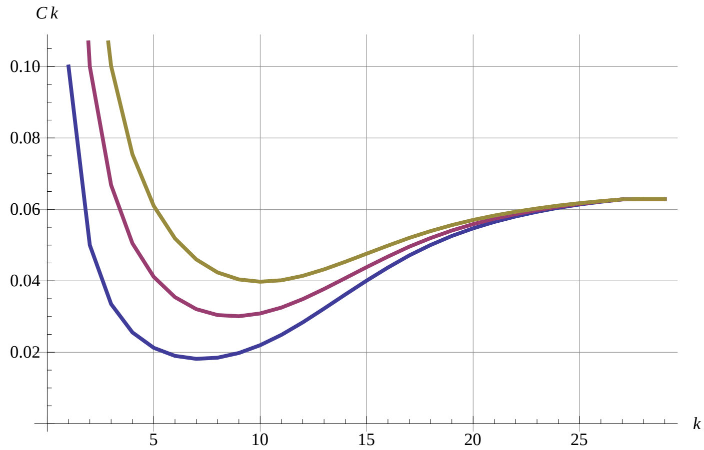

Figure 3: The $\widetilde{C}_s(k)$ curves for: $c_{PM} = 0.1$ (lower), $c_{PM} = 0.2$ (middle) and $c_{PM} = 0.3$ (upper).

It can be seen from these graphs that the optimal PM 'time' is on the 7th shock (for $c_{PM} = 0.1$), on the 8th shock for $c_{PM} = 0.2$, and after observing 10 shocks for $c_{PM} = 0.3$. The he optimal PM policy gives a significant cost reduction, as compared to "repair only at system failure" policy that corresponds to k=28 (approximately twice (!) for the middle curve).

6. **Concluding remarks.**

The conventional PM literature considers either single unit systems or simple multi-unit, e.g., parallel-series systems. More complex structures such as networks usually do not allow for tractable optimal PM solutions. Our approach deals with these systems for the specified setting.

The goal of this paper is to generalize our approach suggested in Finkelstein and Gertsbakh (2015, 2016) to the case when a system (network) is subject to two types of independent shock processes and each shock can affect only its own part of a system. This generalization appeared to be not straightforward at all and we had to come up with the new notion of the corresponding bivariate signature and the new procedure for the conversion of the problem from the bivariate to the univariate case. In order to do so, an alternative single discrete scale was suggested.

The essence of our approach is to use the overall number of shocks (of both types) experienced by a system as a discrete parameter for minimization of the long-run costs per unit of 'time'. Thus the optimal PM is performed after the shock with the obtained optimal number or the replacement of the whole failed system occurs, whichever comes first. For finding the optimal solution, we need only the bivariate signature of the system (and the corresponding basic costs), which dramatically differs from the conventional age-based or condition-based approaches to preventive maintenance.

We have demonstrated how our approach works for an example of a road (communication) network subject to a combined "attacks" on its nodes. It was shown that the optimal PM can decrease substantially the long-run costs.

The developed theory can be also applied in the future to the, so-called. multiplex networks (see Baxter etal 2016). These networks have several "layers" of edges connecting the same set of nodes. For convenience, the edges can be characterized by different colors. For example, a multiplex network with green and red edge sets is a natural representation of a supply network, where the green edges correspond to the water supply and the red edges correspond to the power supply. The edges can fail due to shocks generated by two external shock processes. The first shock process affects only the green edges, whereas the second one - only the red edges. The network failure can be defined, for example, as an event when the number of nodes receiving water **and** power supply falls below some critical level,

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