Using D-spectra for Calculating Network Residual Resilience After Random Attack on Its Nodes or Edges

Ilya B. Gertsbakh[a], Yoseph Shpungin[b], Radislav Vaisman[c]

[a]*Department of Mathematics, Ben-Gurion University , Beer-Sheva, 84105, Israel*
[b]*Software Engineering Department, Sami Shamoon College of Engineering, Beer-Sheva, 84100, Israel*
[c]*School of Mathematics and Physics, The University of Queensland, Australia*

Abstract

Network residual resilience is a reliability parameter of the network after some nodes or edges of the network have been randomly eliminated from the network as a result of random "attack". We consider networks which are monotone binary systems and demonstrate how the residual resilience can be calculated by means of cumulative signature (D-spectra), for three types of random attacks on network components (nodes or edges) and for several versions of network reliability criteria.

Keywords: Residual reliability, monotone systems, D-spectrum, signature, occupancy problem, connectivity, shocks

1. The network and failure mechanisms of its components

The network $N = (V, E)$ is a connected graph with node set V and edge set E. Edges are undirected. Nodes or edges are subject to "random attack" as a result of which some nodes or some edges are eliminated (fail). If a node $v \in V$ fails, all edges adjacent to v are erased, and v becomes and isolated node. If edge $e(a, b) \in E$ fails, then it is erased.

We will distinguish several random mechanisms of network components failures.

Email addresses: `elyager@bezeqint.net` (Ilya B. Gertsbakh), `yosefs@sce.ac.il` (Yoseph Shpungin), `r.vaisman@uq.edu.au` (Radislav Vaisman)

A. *Edge or node p-lottery.* Each edge or each node, independently of other nodes or edges, fails with probability p, $0 < p < 1$.

B. *Shock process.* A random number of nodes or edges is hit by some external shock process developing in time. Each new shock hits with equal probability any non failed node or edge.

C. *Balls and boxes.* Each component subject to failure is represented by a box. A certain amount R of balls is randomly allocated into the boxes. Each ball, independently of other balls, is put into one of n boxes with equal probabilities. It can happen that some boxes will contain more than one ball. Each box which contains at least one ball corresponds to a failed node or failed edge.

2. Residual resilience criteria

Suppose that the network is subject to some of the above failure mechanisms A,B or C, acting on network nodes or on network edges. Our network is a binary object which can be in two states *UP* or *DOWN.*

We consider in only monotone binary systems consisting of binary components. Formally, it means the following. System component state is described by a binary vector $\mathbf{x} = (x_1, x_2, ..., x_n)$, where $x_i = 1(0)$ if component i is *up (down).* System state is $\varphi(\mathbf{x}) = 1$ if the system is *UP* and 0 if the system is *DOWN.* In the network context, x_i denote the state of components subject to failure.

It is important to note that we consider in this paper only monotone networks, which means the following.
If state vector $\mathbf{x} = (x_1, ...x_n)$ is replaced by another state vector $\mathbf{x}^0 = (x_1^0, ..., x_n^0)$ where at least for one i, $x_i < x_i^0$ then $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{x}^0)$. In words: If the system is *DOWN* and at least one component in *up* goes *down*, then the network remains in *DOWN.*

We next introduce the following reliability criteria.

Criterion 1 Network is *UP* iff its largest component contains at least β-fraction of all network nodes n, $0 < \beta < 1$.

Criterion 2 Network is *UP* iff its largest component is greater or equal than the number of isolated nodes.

Suppose the network contains several special nodes which do not fail at all. We call them *terminals*. Denote by T the set of all terminals, $T \subseteq V$, Define a *cluster* as a connected subset of nodes which contains at least one terminal.

Criterion 3 The network is *UP* iff it has one or two clusters.

Criterion 4 The network is *UP* iff all its terminals are connected (this is a so-called terminal connectivity).

Suppose that a positive number d_e is assigned to edge $e \in E$ and that $T = s, t$.

Criterion 5 The network is *UP* if and only if the $s \Rightarrow t$ flow exceeds a given integer D.

Suppose now that one node (call it α) is declared "capital" an it never fails. Assign also a positive number l_e to an edge $e = (a, b)$. We interpret l_e as a length of edge e. For any node v its distance from the capital equals the length of the shortest path between v and α. Let us call all nodes whose distance from the capital does not exceed some constant D, the *central set* of the network.

Criterion 6 The network is *UP* iff its central set contains at least m nodes.

Obviously, that all the above criteria guarantee that the networks are monotone systems.

3. Cumulative signature (cumulative D-spectrum)

Suppose that the components of the network subject to failure are numbered by integers $1, 2, ..., n$ and let $\pi = (i_1, i_2, ..., i_n)$ be a random permutation of component numbers. Imagine the following process of sequential component destruction. Take any random permutation. Assume that initially all components are *up*. Start moving along the permutation from left to right and turn *down* one component after another. Check network state after each destruction. Let $\alpha(\pi)$ be the ordinal number of the component in π when we observe for the first time that the network is *DOWN*. This number will be called the *anchor* of the permutation. For example, suppose components

i_1, i_2, i_3, i_4 are turned *down* and the network is still *UP*, but after component i_5 is turned down, the network becomes *DOWN*. Then the anchor of the corresponding permutation will be equal 5, (see [1, 2, 3]).

Assume that all $n!$ permutation are equally probable. Denote by A_j the event

$$A_j = \{\text{anchor} \quad \alpha(\pi) = j\}$$

Obviously, the probability of this event

$$f_j = P(A_j) = \frac{\text{the number of permutations with anchor equal} \quad j}{n!}.$$

The vector $\mathbf{f} = (f_1, f_2, ..., f_n)$ is called D-spectrum of the network ("D" stands for "destruction"). Obviously, f is a discrete density function, $\sum_1^n f_j = 1$. The D-spectrum coincides numerically with the so-called *signature* introduced first by Samaniego [4, 5]. In [6] it was termed *ID - internal distribution* of the network.

For us most important will be so-called cumulative D-spectrum $F(k)$ defined as

$$F(k) = \sum_{r=1}^{r=k} f_r, k = 1, 2, ..., n.$$

Let Y be the random number of components failing in random order which cause system failure. Then obviously, $f_r = P(Y = r)$ and

$$F(k) = P(Y \leqslant k).$$

For each monotone network and for each *DOWN/UP* criterion we can calculate its own cumulative D-spectrum. For convenience, the D-spectrum related to criterion s, see above, will be denoted as $F^{(s)}(k)$.

Suppose we have a non monotone system. For example, we have a network with nodes subject to failures. An isolated node is considered as a single-node component.If a node fails, it erased, together with edges adjacent to it. Suppose that the network is *UP* if its nodes form one component. Clearly, this criterion makes the network non monotone. For example, consider a chain with five nodes a,b,c,d,e and four edges $(a, b), (b, c), (c, d)$ and (d, e). After node 1 fails, network is UP. After node 3 fails it goes $DOWN$. After node 2 fails, it again goes UP. Non monotone systems do not allow defining for them signatures or D-spectra, since some permutations may have more than one anchor.

4. Calculating Network Residual Reliability

In this section we present our main results regarding network $DOWN$ probability for various versions of node (edge) failure mechanisms.

Claim 4.1. *Let n be the number of components subject to failure. For random lottery model (A), and criterion $s, s = 1, ..., 6$ network $DOWN$ probability equals*

$$P(DOWN) = \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} F^{(s)}(i). \tag{1}$$

Proof: The proof is obvious: the multiple at $F^{(s)}(i)$ is the probability of network to be $DOWN$ if exactly i components have failed, Then $P(DOWN)$ for random lottery follows by the formula of Total Probability.

□

Let us assume now that network component failures occur as an action of an external shock process. Assume, without loss of generality, that shocks appear according to a homogeneous Poisson process (HPP) $\xi(t)$ with intensity $\lambda = 1$. In this case, the number of failed components during the interval $[0, t_0]$ equals $min(\xi(t_0), n)$. If $\xi(t_0) > n$, we assume that the network will be $DOWN$ with probability 1.

Claim 4.2. *In the external HPP shock process, the probability that network failure will occur on the interval $[0, t_0]$ equals*

$$P(DOWN; t_0) = \sum_{j=1}^{n} \frac{(\lambda t_0)^j e^{-\lambda t_0}}{j!} F^{(s)}(j) + P(\xi(t_0) > n). \tag{2}$$

Proof: The proof is obvious. □

Let us turn now to "boxes and balls" model. Denote by n the number of nodes or edges ("boxes") subject to failure and by R the number of balls. In this model, we have to take into account that one box can contain more than one ball, i.e. one node (edge) can receive more than one shock. Denote by $p(k \mid R)$ the probability that exactly k boxes will contain at least one ball.

Then obviously the probability that R balls will bring the network $DOWN$ equals

$$P(DOWN \mid R) = \sum_{k=1}^{min(n,R)} p(k \mid R) F^{(s)}(k). \tag{3}$$

.

Finding the probabilities $p(k \mid R)$ is a nontrivial problem. In combinatorics, it is known as "occupancy" problem. Its solution has been found by DeMoivre, see [6], p. 242:

$$p(k \mid R) = \frac{n!}{k!(n-k)!} \sum_{t=0}^{k} (-1)^t \frac{k!}{t!(k-t)!} \Big(\frac{k-t}{n}\Big)^R, k = 1, ..., min(n, R).$$

Substituting $p(k \mid R)$ into (3), we arrive at

Claim 4.3. *In "balls and boxes" model, the probability that the network will be $DOWN$ if it is "attacked" by R "balls" equals*

$$P(DOWN \mid R) = \sum_{k=1}^{min(n,R)} \frac{n!}{k!(n-k)!} \cdot \tag{4}$$
$$\cdot \sum_{t=0}^{k} (-1)^t \frac{k!}{t!(k-t)!} \Big(\frac{k-t}{n}\Big)^R F^{(s)}(k).$$

5. Example

As an example we consider a 11×11 rectangular grid with capital α in the center, see Figure 1. The central part of the grid consists of all nodes whose shortest distance from the capital does not exceed 6. Each edge has length 1. Components subject to failure are network nodes. Network is *(DOWN)* if the central part of it contains less than 24 nodes, see criterion 6 in Section 2.

To analyse the network residual resilience we need to estimate the D-spectrum $F^{(6)}(k)$. Figure 2 presents the graph of the spectrum.

Let us analyze network resilience for various mechanisms of node failure. If nodes fail in accord with the $p - lottery$ model see **A**, Section 1), we compute network $DOWN$ probability as a function of the value of p using formula (1). Table 1 presents numerical data on $P(DOWN)$ as a function of p.

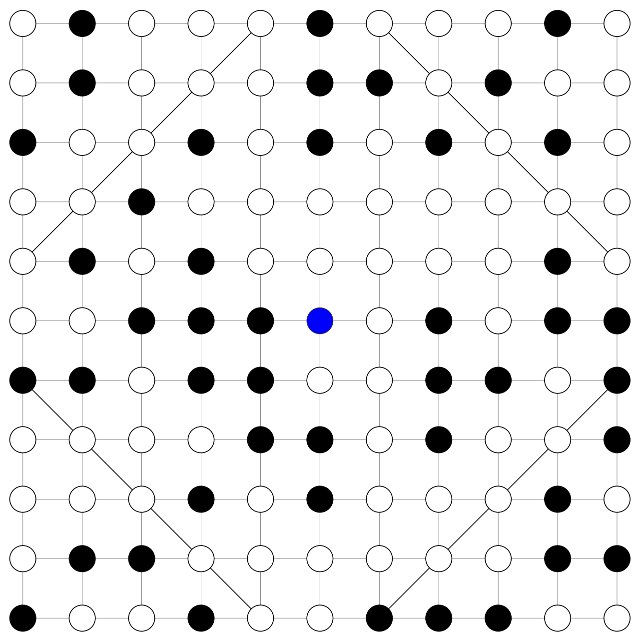

Figure 1: 11×11 grid, the capital node (blue) and the central area. Failed nodes are black

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$P(DOWN)$	0.000001	0.00224	0.02956	0.22035	0.65339	0.94761	0.99853
RE %	9	1	0.4	0.2	0.07	0.05	0.01

Table 1: $P(DOWN)$ as function of p, p-lottery model

These results were checked by Crude Monte Carlo (CMC) based on 1,000,000 replications of p-lottery scheme. The corresponding relative errors are presented in the bottom row of Table 1.

Suppose now that the grid is subject to an external Poisson shock process (see **B**, Section 1). Important is that each shock hits one of non failed nodes (there are no repeated hits of the same node). Assume that shocks appear in accordance with HPP with intensity $\lambda = 1$. So, we assume that in the interval $[0, 60]$ min, on the average, the grid receives 60 hits.

Table 2 presents the calculations results obtained by formula (2). It is seen, for example, that the grid is $DOWN$ with probability almost 1 if the shock process lasts 90 minutes. A short attack lasting 30 minutes destroys the network with probability near zero - 0.01012.

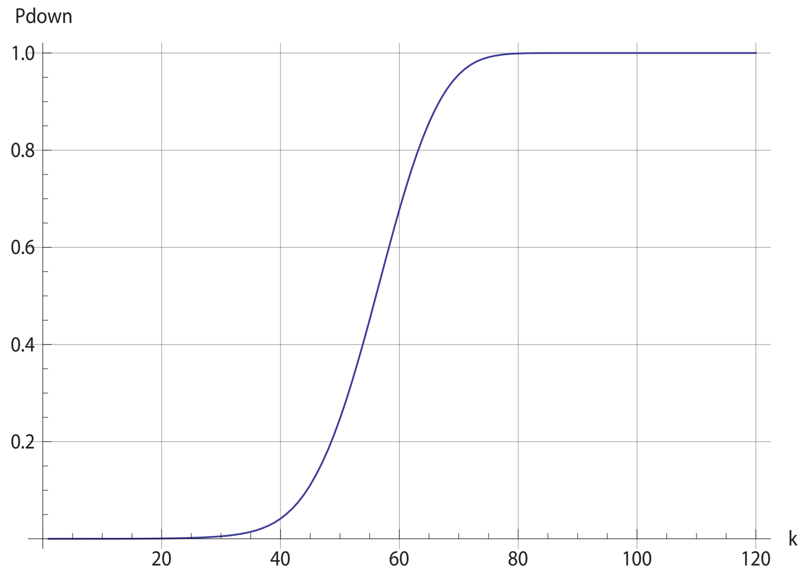

Figure 2: Cumulative D-spectrum $F^{(6)}(k)$ for 11×11 grid

t [min]	30	40	50	60	70	80	90
$P(DOWN;t)$	0.01012	0.0756	0.2978	0.6450	0.8825	0.9772	0.9973

Table 2: $P(DOWN;t)$ as function of t, shock model B

Let us turn now to the "boxes" and "balls" model. We have $n = 120$ node-boxes, and let us randomly allocate R balls into the boxes. Nonempty boxes correspond to failed nodes. Using formula (4) we can compute system $DOWN$ probability when the number of balls changes from 10 to 140, see Table 3. It is seen from it that in order to achieve that $P(DOWN \mid R) > 0.98$ the number of balls should be at least 120. If $R = 80$, $P(DOWN \mid 80) \approx 0.61$. Compare this with the graph of the D-spectrum on Figure (2). We see, that if 60 nodes fail, system fails with probability near 0.6. We see therefore that about 20 "hits" fell on the nodes which already were hit once.

6. Concluding remarks

Estimation of D-spectrum. Monte Carlo algorithms for estimation of cumulative spectra have been described in detail in [1], Chapter 8 and [2],

R	10	20	30	40	50	60	70
$P(DOWN \mid R)$	0.00003	0.00049	0.00279	0.1302	0.05616	0.17547	0.37864
R	80	90	100	110	120	130	140
$P(DOWN \mid R)$	0.60926	0.79596	0.91050	0.96649	0.98906	0.9968	0.9992

Table 3: $P(DOWN; R)$, "balls" and "boxes" model

Section 1.8. Basically, for all network DOWN criteria the algorithms work similarly and consists of repeating the following basic steps.

a. Generation of random permutation of component numbers;

b. finding the anchor of the permutation;

c. computing the cumulative D-spectrum.

d. The steps a,b and c are repeated M times.

For $n = 80 - 120$ sufficient accuracy is achieved for $M = 5 \cdot 10^5 - 10^6$. The most time consuming operation is b. because it involves checking whether the system made a transition $UP \Rightarrow DOWN$ after component was turned from *up* to *down*. Typically, a binary search of anchor position is carried out.

Estimation of the D-spectrum in the above Example was performed on Core i5 laptop processor with 4GB of RAM. Spectra calculation with $M = 10^6$ took 61 seconds.

Role of monotone property. Writing this paper was inspired by the work [5], in which the authors considered a connected network whose components (nodes or edges) fail according to the p-lottery random mechanism. Network is UP if and only if it remains connected after elimination of failed nodes or failed edges. It it important to note that with this $UP/DOWN$ definition the network is not a monotone system. Consider, for example that after some nodes have failed, the remaining network consists of two components C_1 and C_2. C_1 is an isolated node x, and C_2 has two nodes a, b and edge $e(a, b)$. Obviously, the network is $DOWN$. Suppose that node x fails. Now the network becomes connected, and therefore we observe a $DOWN \Rightarrow UP$ transition after component failure. Considering non monotone systems makes it impossible to use D-spectra (signature) technique for analyzing system residual resilience thus considerably complicating the algorithmic part of the network reliability analysis.

There are, however, some possibilities to use a permutation based algorithm to count failure sets for a non monotone system. This algorithm is considerably less efficient than the algorithm for finding the D-spectra. Besides, it becomes possible to compute network $DOWN$ probability only for p-lottery model. Because of these complications it has been decided to limit our present paper to a consideration of monotone systems only.

Acknowledgement

The work of Radislav Vaisman was supported by the Australian Research Council Centre of Excellence for Mathematical & Statistical Frontiers, under grant number CE140100049.

References

[1] I. B. Gertsbakh, Y. Shpungin, Models of network reliability: analysis, combinatorics, and Monte Carlo, CRC Press, Boca Raton, 2009.

[2] I. Gertsbakh, Y. Shpungin, Network Reliability and Resilience, SpringerBriefs in Electrical and Computer Engineering, Springer, Berlin, 2011.

[3] I. B. Gertsbakh, Y. Shpungin, Stochastic models of network survivability, Quality Technology and Quantitative Management 9 (1) (2012) 45–48.

[4] F. J. Samaniego, On closure of the ifr under formation of coherent systems, IEEE Transactions on Reliability 34 (1985) 69–72.

[5] F. J. Samaniego, System signatures and their application in engineering reliability, Springer, New York, 2007.

[6] T. Elperin, I. B. Gertsbakh, M. Lomonosov, Estimation of network reliability using graph evolution models, IEEE Transactions on Reliability 40 (5) (1991) 572–581.